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Study of the Renormalization Group for Small Coupling Constants*

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A renormalizable quantum field theory is said to be stagnant if it is asymptotically free. We study the renormalization group for small coupling constants. In particular the pseudoscalar-fermion theory with an internal-symmetry group G and in which the coupling matrix furnishes a representation of G is considered. A representation is said to be stagnant if the associated theory is stagnant. We show that Cartan's four families A, B, C, D and the exceptional algebra G_2 possess no stagnant representation. On the basis of this result we conjecture that there are no asymptotically free quantum field theories in four dimensions. Some possible asymptotic behaviors of field theories are also described.

I. INTRODUCTION

Perhaps the most remarkable feature of renormalizable quantum field theory is the fact that the asymptotic behavior of such a field theory is determined by the zeros of a certain function of the coupling constants. In recent years, this point has been most forcefully emphasized by Wilson.[1] The origin of this assertion lies in the renormalization-group[2] equation, which for the case of a $\lambda\phi^4$ theory reads as follows[3]:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}-n\gamma(\lambda)\right]\Gamma_{AS}^{(n)}(p_1\cdots p_n)=0\ . \quad (1.1)$$

Here $\Gamma_{AS}^{(n)}(p_1\cdots p_n)$ denotes[4] the leading asymptotic part of one-particle-irreducible renormalized Green's functions. Following Coleman[4] we find it convenient to think of the hydrodynamical analog of Eq. (1.1). With the identification $t\rightarrow\frac{1}{2}\ln(\mu^2/s)$, $\lambda\rightarrow x$, and $\beta(\lambda)\rightarrow v(x)$, Eq. (1.1) may be interpreted as a description of a population of "bacteria" moving with a fluid along a one-dimensional pipe with $\Gamma_{AS}^{(n)}$ = bacterial density. Now if there is a stagnant point, namely a value x_E such that $v(x_E)=0$ and such that the velocity field in a neighborhood of x_E points toward x_E, then the bacteria will accumulate there if one waits long enough. Hence the asymptotic property of $\Gamma_{AS}^{(n)}$ is dictated by the zeros of $\beta(\lambda)$.

Less picturesquely, the general solution[4] of Eq. (1.1) is

$$\Gamma_{AS}^{(n)}=s^{(4-n)/2}f^{(n)}\left(\lambda'\left(\lambda,\tfrac{1}{2}\ln\frac{s}{\mu^2}\right),\frac{p_ip_j}{s}\right)\times\exp\left[-n\int_0^{(1/2)\ln(s/\mu^2)}d\tau\,\gamma(\lambda'(\lambda,\tau))\right], \quad (1.2)$$

where $f^{(n)}$ is some function and $\lambda'(\lambda,t)$ is the solution of

$$\frac{d\lambda'(\lambda,t)}{dt}=\beta(\lambda') \quad (1.3)$$

such that $\lambda'(\lambda,0)=\lambda$. Clearly if $\beta(\lambda_E)=0$, then $\lambda'(\lambda,\infty)=\lambda_E$ and the asymptotic behavior of Eq. (1.2) simplifies dramatically.

An intensive search for λ_E [and its analog in quantum electrodynamics (QED)] has been mounted in the last couple of years,[5] but so far the search has been in vain, at least for theories in four-dimensional space-time. On the other hand, $\beta(0)=0$, as is obvious from the definition of $\beta(\lambda)$. Thus the origin in the coupling-constant plane is a possible stagnant point. It is known, however, that the fluid flows away from the origin in renormalizable theories with one coupling constant, namely $\lambda\phi^4$ and QED. [In other words $\beta(\alpha)=a\alpha$ for QED and $\beta(\lambda)=b\lambda^2$ for $\lambda\phi^4$, where a and b are positive numerical constants whose precise values are of no import.] In this paper we address ourselves to the following question: Are there renormalizable theories in which the origin in the coupling-constant plane is a stagnant point? We are forced to study theories with more than one coupling constant. For each coupling constant λ_i there corresponds a $\beta_i(\lambda_1,\ldots,\lambda_n)$ [which describes the flow of a fluid in n-dimensional space with fluid velocity $\vec{v}(\vec{x})$].

Clearly, one motivation for considering this problem is that the neighborhood of the origin is completely explorable by renormalized perturbation theory. Thus we *assume* that a neighborhood of the origin exists such that inside this neighborhood $\vec{\beta}(\vec{\lambda})$ is adequately given by, say, the lowest-order term in a perturbative expansion. This neighborhood may be rather small; indeed, if Adler's conjecture[5] about QED turns out to be correct, then the neighborhood referred to here in the case of QED would be described by $|\alpha|\ll\frac{1}{137}$.

While we certainly do not mean to suggest that the present perturbative discussion may be applied to strong interactions, it is tempting to think that a possible explanation of deep-inelastic scaling is afforded by the existence of a stagnant point at the origin. In that case $\vec{\lambda}'(\vec{\lambda},\infty)=0$ and the theory would be asymptotically free. Another amusing possibility is the following: The origin is not a stagnant point, but a given bacterium may flow by close to the origin at some time. It will eventually flow away of course. However, quantum field theory is such that $\vec{v}(\vec{x})\to 0$ as $\vec{x}\to 0$ and thus this bacterium will spend a long time milling around the origin. In other words, it may happen that the scaling phenomenon we are seeing presently at SLAC would persist for a logarithmically large energy range and then eventually disappear at some superhigh energy.

We will now leave all idle speculations aside. The bulk of this paper is devoted to a study of the pseudoscalar-fermion theory $ig\bar{\psi}_\beta\gamma_5\psi_\alpha\phi_c\Gamma^c_{\beta\alpha}$ with a given symmetry group G. We make the restrictive assumption that Γ^c furnishes an irreducible representation of G. Γ^c is said to be stagnant if the corresponding theory is stagnant. The problem is solved by an exercise in the theory of Lie algebra. We find no stagnant representation for the four Cartan families and G_2.

The group-theoretic structure of the renormalization-group equations is of course independent of space-time dimension. Hence the discussion here may also be useful in $4-\epsilon$ calculations.[6] Also, the same type of group-theoretic structure governs various "bootstrap"-like equations,[7] and so the group-theoretic analysis given here may be helpful in other contexts.

We start by reviewing the situation in theories with one coupling constant (Sec. II). The group theory is discussed in Sec. IV and Appendix A. In Appendix B we discuss some possible asymptotic behavior of quantum field theory.

II. THEORIES WITH ONE COUPLING CONSTANT

In QED, the renormalized coupling constant $\alpha = Z_1^{-2}Z_2^{\,2}Z_3\alpha_0 = Z_3\alpha_0$, since gauge invariance implies $Z_1=Z_2$. (The subscript zero will always denote bare quantities.) Z_3, being a wave-function renormalization, satisfies the Källén-Lehmann[8] bound $0\le Z_3\le 1$, which implies that in the perturbative expansion $Z_3 = 1 - a\alpha\ln(\Lambda/m_0)+\cdots$ with a = positive number and so

$$\beta(\alpha) = m_0\frac{\partial\alpha}{\partial m_0}\bigg|_{\Lambda,\,\alpha_0} = a\alpha\ .$$

A heuristic and more physical explanation is as follows: Z_3 describes the shielding of a point charge by pair creation. As m_0 increases it becomes more difficult to create pairs, and hence $\partial Z_3/\partial m_0$ is positive. Gauge invariance thus underlies the fact that the origin is not a stagnant point in QED. The situation for massive boson theory[9] is clearly the same.

For the theory $\mathcal{L}_I = -(\lambda/4!)\,\phi^4$, one easily calculates $\beta(\lambda) = (3/32\pi^2)\lambda^2+\cdots$. The familiar formal argument of Baym[10] and others indicates that λ must[11] be positive. Hence the fluid again flows away from the origin.

III. THEORIES WITH MORE THAN ONE COUPLING CONSTANT

It is clear that the origin is not stagnant for the theory

$$\mathcal{L} = \tfrac{1}{2}\sum_i\left[(\partial_\mu\phi_{0i})^2 - \mu_0^{\,2}\phi_{0i}^{\,2}\right] - \sum_{ijkl}\lambda_0^{ijkl}\phi_{0i}\phi_{0j}\phi_{0k}\phi_{0l}\ , \qquad (3.1)$$

since the β corresponding to λ^{iiii} is equal to $\sum_{kl}(\lambda^{iikl})^2$. The only known renormalizable theories involving many massive vector bosons and a non-Abelian group invoke the Higgs mechanism.[12] We will not here attempt a discussion of the possible stagnancy of the origin for these gauge theories, although such a study would be interesting.

Thus we are left with the pseudoscalar-fermion theory to which we now turn. (The scalar-fermion theory will be briefly mentioned later.) The theory has two coupling constants, viz.,

$$\mathcal{L}_I = ig_0\bar{\psi}_0\gamma_5\psi_0\phi_0 - \frac{\lambda_0}{4!}\phi_0^{\,4}\ .$$

The existence of two coupling constants complicates a general discussion of the renormalization group. (See for example Ref. 13.) Fortunately, matters simplify for our lowest-order calculations. One finds

$$\beta_g = \left(2\mu_0^{\,2}\frac{\partial g}{\partial\mu_0^{\,2}} + m_0\frac{\partial g}{\partial m_0}\right)[1+O(g,\lambda)]\,,$$

$$\beta_\lambda = \left(2\mu_0^{\,2}\frac{\partial\lambda}{\partial\mu_0^{\,2}} + m_0\frac{\partial\lambda}{\partial m_0}\right)[1+O(g,\lambda)]\,,$$

$$\left(2\mu^2\frac{\partial}{\partial\mu^2} + m\frac{\partial}{\partial m} + \beta_g\frac{\partial}{\partial g} + \beta_\lambda\frac{\partial}{\partial\lambda} + \cdots\right)\Gamma_{\rm AS} = 0.$$

(We do not bother to write down the terms proportional to γ.) Now $g = g_0Z_1^{-1}Z_2Z_3^{1/2}$. The wave-function renormalization factors satisfy the Källén-Lehmann bound $Z_2\le 1$ and $Z_3\le 1$, and hence give positive contributions to β_g by the argument given in Sec. II. The author knows of no *a priori* argument for the sign of the contribution of Z_1^{-1}, in contrast to QED. A straightforward computation shows that it is positive and that

$$\beta_g = \frac{5}{16\pi^2} g^3 = \frac{2+1+2}{16\pi^2} g^3 \; . \tag{3.2}$$

The next-order terms omitted are of order g^5 and λg^3. We have indicated in the second equality the origin of the factor of 5: Z_1^{-1} and $Z_3^{1/2}$ contribute 2 each, while Z_2 contributes 1, to make up a total of 5. This separation will be important later. The lowest-order contributions to β_λ are of order λ^2, g^4, and λg^2, coming from the "bubble" diagram, the square-box diagram, and the wave-function renormalization Z_3, respectively. A simple computation gives

$$\beta_\lambda = a\lambda^2 - bg^4 + c\lambda g^2 \; . \tag{3.3}$$

a, b, and c are positive numerical constants whose precise value does not concern us. That a and c are positive follows from the foregoing discussion. (Here the omitted terms begin with order λ^3, $\lambda^2 g^2$, λg^4, and g^6.) The flow of the fluid near the origin is then described by [according to Eq. (1.3)]

$$\frac{d\lambda}{dt} = a\lambda^2 - bg^4 + c\lambda g^2 \; , \tag{3.4}$$

$$\frac{dg}{dt} = dg^3 \tag{3.5}$$

(a, b, c, d all positive). By inspection the origin is not a stagnant point, since $|g|$ always increases for small g.

We thus have to turn to more complicated theories. Let us consider the pseudoscalar-fermion theory with an internal-symmetry group G:

$$\mathcal{L} = \bar{\psi}_{0\beta}(i\gamma\partial - m_0)\psi_{0\beta} + \tfrac{1}{2}(\partial_\mu \phi_0^a)^2 - \tfrac{1}{2}\mu_0^{\,2}(\phi_0^a)^2 + ig\bar{\psi}_{0\beta}\gamma_5\psi_{0\alpha}\phi_{0a}\,\Gamma^a_{\beta\alpha} + \text{a } \phi^4 \text{ term} \; , \tag{3.6}$$

where Γ^a is a Hermitian matrix. Since the theory of representations is much better developed than the theory of Clebsch-Gordan coefficients, we will restrict ourselves to the case in which ϕ_a belongs to the adjoint representation and in which the matrices $\Gamma^a_{\beta\alpha}$ furnish an irreducible representation of the Lie algebra A of G. (We consider compact groups only.)

We will prove in Sec. IV that there exist three purely group-theoretic numbers s_1, s_2, and s_3 such that

$$\Gamma^a\Gamma^b\Gamma^a = s_1\Gamma^b \; , \tag{3.7}$$

$$\Gamma^a\Gamma^a = s_2 1 \; , \tag{3.8}$$

$$\text{tr}\,\Gamma^a\Gamma^b = s_3\delta^{ab} \tag{3.9}$$

(repeated indices are summed over). For the theory described in Eq. (3.6) we clearly have [cf. Eq. (3.2)]

$$\beta_g = (2s_1 + s_2 + 2s_3)\frac{g^3}{16\pi^2} \tag{3.10}$$

to lowest order in perturbation theory. Thus the origin is a stagnant point if $(2s_1 + s_2 + s_3) < 0$. Now evidently $s_2 > 0$ and $s_3 > 0$. Thus the problem is reduced to the question of existence of representations with s_1 large and negative.

Let us compose two definitions:

(a) A representation is said to be dangerous if $s_1 < 0$.

(b) A representation is said to be stagnant if $2s_1 + s_2 + 2s_3 < 0$.

Clearly a stagnant representation is necessarily dangerous, but not vice versa.

We will now show that the only dangerous representations are of low dimensionality and may be completely enumerated, and that there are no stagnant representations for Cartan's four families and the exceptional algebra G_2.

IV. AN EXERCISE IN THE REPRESENTATION THEORY OF LIE ALGEBRAS

We have attempted to make this section self-contained by reviewing some of the basic concepts of Lie algebras. For further information the reader may consult for example Refs. 14–16. A Lie algebra is characterized by $[X_\sigma, X_\rho] = c_{\sigma\rho}{}^\tau X_\tau$. Out of the structure constants the all-important metric tensor may be constructed:

$$g_{\sigma\rho} = c_{\sigma\mu}{}^\nu c_{\rho\nu}{}^\mu \; . \tag{4.1}$$

We restrict ourselves to simple groups in which case the inverse of $g_{\sigma\rho}$ exists and is called $g^{\sigma\rho}$.

Define the matrix $K_\rho{}^\sigma \equiv \text{tr}X_\rho X^\sigma$. The cyclicity of the trace implies

$$\begin{aligned} 0 &= \text{tr}[X_\tau, X_\rho X^\sigma] \\ &= c_{\tau\rho}{}^\mu\,\text{tr}X_\mu X^\sigma + \text{tr}X_\rho X_\mu c_\tau{}^{\sigma\mu} \end{aligned} \tag{4.2}$$

and thus

$$[\mathcal{C}_\tau, K] = 0 \; , \tag{4.3}$$

where $(\mathcal{C}_\tau)_\rho{}^\sigma = -c_{\tau\rho}{}^\sigma$ furnishes the adjoint representation. It is easy to show that the adjoint representation is irreducible for simple groups. Hence by Schur's lemma

$$\text{tr}X_\rho X^\sigma = s_3\delta_\rho{}^\sigma \tag{4.4}$$

for some numerical constant s_3.

We next consider $g^{\rho\sigma}X_\rho X_\sigma$. This is Casimir's first operator, of course, and commutes with X_τ for all τ, as is easily shown. Applying Schur's lemma again, one finds

$$g^{\rho\sigma}X_\rho X_\sigma = s_2 \; . \tag{4.5}$$

Finally, let us show that $X_\sigma X_\rho X^\sigma = s_1 X_\rho$ for some constant s_1. We simply compute as follows:

$$\begin{aligned} X_\sigma X_\rho X^\sigma &= c_{\sigma\rho}{}^\tau X_\tau X^\sigma + X_\rho X_\sigma X^\sigma \\ &= \tfrac{1}{2} c_{\sigma\rho\tau}[X^\tau, X^\sigma] + s_2 X_\rho \\ &= -\tfrac{1}{2} c_{\rho\tau}{}^\sigma c_{\nu\sigma}{}^\tau X^\nu + s_2 X_\rho \\ &= -\tfrac{1}{2} g_{\rho\nu} X^\nu + s_2 X_\rho \\ &= (s_2 - \tfrac{1}{2}) X_\rho . \end{aligned} \quad (4.6)$$

Indeed, one obtains the relation

$$s_1 = s_2 - \tfrac{1}{2} . \quad (4.7)$$

s_2 and s_3 are obviously related as follows:

$$d s_2 = r s_3 , \quad (4.8)$$

where d = dimension of the representation and r = order of the algebra (= the number of generators). Thus a representation is dangerous if and only if $s_2 < \frac{1}{2}$. A sufficient condition for the representation to be *not* stagnant is $s_2 \geq \frac{1}{3}$.

It is convenient to use the Cartan-Weyl basis in which the set X_ρ is divided into two classes: H_i, $i = 1, \ldots, l$ (l = rank) and E_α, such that

$$[H_i, H_j] = 0 , \quad (4.9)$$

$$[H_i, E_\alpha] = \alpha_i E_\alpha , \quad (4.10)$$

$$[E_\alpha, E_{-\alpha}] = \alpha^i H_i , \quad (4.11)$$

$$[E_\alpha, E_\beta] = N_{\alpha\beta} E_{\alpha+\beta} \quad (4.12)$$

when $\alpha + \beta$ is a nonvanishing root. The l-component vectors $\vec{\alpha} = (\alpha_1, \ldots, \alpha_l)$ are the roots of the algebra. The conventional normalization $g_{\alpha\,-\alpha} = 1$ has been used. Also $g_{ik} = \sum_\alpha \alpha_i \alpha_k$.

A change of basis is effected by the transformation $Y_\sigma = S_\sigma{}^\rho X_\rho$. It is easy to see that the values of s_1, s_2, and s_3 are invariant under a change of basis. In particular it is always possible to change the basis to one in which $g_{\rho\sigma}$ is the identity matrix $\delta_{\rho\sigma}$. This is the basis used in Sec. III spanned by Γ^a with $\Gamma^a = \Gamma_a$.

An l-component vector $\vec{m} = (m_1, \ldots, m_l)$ is associated with each state $|\psi\rangle$ in a representation as follows:

$$H_i |\psi\rangle = m_i |\psi\rangle . \quad (4.13)$$

A weight (or a root) is said to be positive if its first nonvanishing component is positive (for some definite choice of labeling the H_i's). A weight $\vec{m}$ is higher than another weight $\vec{m}'$ if $\vec{m} - \vec{m}'$ is positive. A representation is uniquely characterized by its highest weight (denoted by $\vec{L}$). For each algebra of rank l there exist l fundamental weights $\vec{L}^{(i)}$ ($i = 1, \ldots, l$), such that for any representation $\vec{L} = \sum_{i=1}^{l} K_i \vec{L}^{(i)}$ with K_i non-negative integers.

To evaluate s_2 one applies Casimir's operator $X^\rho X_\rho$ to the state with highest weight:

$$\begin{aligned} X^\rho X_\rho |\vec{L}\rangle &= (g^{ik} H_i H_k + \sum_\alpha g^{\alpha\,-\alpha} E_\alpha E_{-\alpha}) |\vec{L}\rangle \\ &= (g^{ik} L_i L_k + \sum_+ [E_\alpha, E_{-\alpha}]) |\vec{L}\rangle \\ &= (g^{ik} L_i L_k + \sum_+ \alpha^i L_i) |L\rangle , \end{aligned} \quad (4.14)$$

where the summation $\sum_+$ is over the positive roots only. It is conventional to define $2\vec{R} = \sum_+ \vec{\alpha}$. Then

$$s_2 = \vec{L} \cdot \vec{L} + 2\vec{R} \cdot \vec{L} . \quad (4.15)$$

By applying the operator $X^\rho X_\sigma X_\rho$ to $|\vec{L}\rangle$ one may readily check Eq. (4.7).

We are now ready to solve our problem. Let us illustrate the procedure for Cartan's family B_l ($l = 2, \ldots, \infty$) (B_l corresponds to the orthogonal group in $2l+1$ dimension). The roots are[14] $\pm\vec{e}_i$, $\pm\vec{e}_i \pm \vec{e}_j$, $1 \leq i < j \leq l$, where $\vec{e}_i$ are l orthonormal vectors in l-dimensional space. Hence the metric is

$$g^{ij} = \frac{1}{2(2l-1)} \delta^{ij}$$

and

$$\begin{aligned} 2\vec{R} &= \sum_{i=1}^{l} \vec{e}_i (2l + 1 - 2i) \\ &= (2l-1, 2l-3, \ldots, 1). \end{aligned}$$

The l fundamental weights are

$$\begin{aligned} \vec{L}^{(1)} &= (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}) , \\ \vec{L}^{(2)} &= (1, 0, \ldots, 0) , \\ \vec{L}^{(3)} &= (1, 1, 0, \ldots, 0) , \\ &\cdots \\ \vec{L}^{(l)} &= (1, 1, 1, \ldots, 1, 0) . \end{aligned}$$

Let $s_2^{(i)}$ be the value of s_2 for the representation characterized by $\vec{L}^{(i)}$. Using Eq. (4.15) we find $s_2^{(1)} = \frac{1}{8} l(2l+1)/(2l-1)$, which is dangerous for $l = 2$ but not stagnant for any l. Also, $s_2^{(2)} = l/(2l-1)$ is not dangerous for any l. Furthermore, $s_2^{(j)} > s_2^{(2)}$ for $j \geq 3$. For any representation $\vec{L}$ we have (since $\vec{L}^{(i)} \cdot \vec{L}^{(j)} > 0$)

$$\begin{aligned} s_2 &= \sum_{i,j} K_i K_j \vec{L}^{(i)} \cdot \vec{L}^{(j)} + 2\vec{R} \cdot \sum_1 K_i \vec{L}^{(i)} \\ &\geq s_2^{(i)} \quad (\text{any } i). \end{aligned}$$

Hence the family B_l has no stagnant representation and only one dangerous representation.

The analysis for the other families and G_2 is given in Appendix A. The result shows that Cartan's four families and G_2 possess no stagnant representation. The reader is invited to work out the situation for the four remaining exceptional algebras using the method outlined here.

V. DISCUSSIONS

Scalar-fermion theory also involves a trilinear coupling constant, and so the fluid flow is in three dimensions. By a simple application of asymptotic γ_5 invariance, one sees that β_g in scalar-fermion theory is the same as β_g in pseudoscalar-fermion theory. Hence our discussion applies to scalar-fermion theory as well.

We have also examined a few simple cases in which $\Gamma^a_{\beta\alpha}$ is not a representation matrix. Just to give some examples we mention that an SU(2) theory with $\phi \in I=2$ and $\psi \in I=1$ is not stagnant. One may also study an SU(3) theory with ϕ and ψ in octet representation. This theory has three coupling constants in general and is clearly not stagnant.[17] In the case of a Yang-Mills gauge theory the coupling matrix does furnish representation of Lie algebras. Our group-theoretic analysis should be useful in this case.

On the basis of the analysis in Sec. IV, one is tempted to conjecture that no quantum field theory is asymptotically free. The numbers 2, 1, and 2 in Eq. (3.10) and the number $\frac{1}{3}$ in the stagnancy condition ($s_2 \geq \frac{1}{3}$) are determined purely by the structure of quantum field theory and the dimension of space-time. One would think that there is no *a priori* reason why the Lie algebras should "respect" these numbers. In any case it would be interesting to attempt to prove or to disprove this conjecture.

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APPENDIX A

Cartan's family C_l ($l=2, \ldots, \infty$) corresponds to the symplectic groups and is characterized by the roots[15] $\pm\vec{e}_i \pm \vec{e}_k$, $\pm 2\vec{e}_i$, $1 \leq i < k \leq l$. The metric is

$$g^{ij} = \frac{1}{4(l+1)}\,\delta^{ij}$$

and

$$2\vec{R} = 2(l, l-1, l-2, \ldots, 2, 1).$$

The l fundamental weights are[15]

$$\vec{L}^{(1)} = (1, 0, \ldots, 0),$$
$$\vec{L}^{(2)} = (1, 1, 0, \ldots, 0),$$
$$\cdots$$
$$\vec{L}^{(l)} = (1, 1, \ldots, 1).$$

One evaluates Eq. (4.15) to find

$$s_2^{(1)} = \frac{2l+1}{4(l+1)},$$

which is dangerous for all l but not stagnant for any l. Clearly, for any representation, $s_2 \geq s_2^{(1)}$ and there exists no stagnant representation in C_l.

The family D_l ($l \geq 3$) corresponds to the orthogonal group in $2l$ dimensions. The roots[15] are $\pm\vec{e}_i \pm \vec{e}_j$, $1 \leq i < j \leq l$. One finds

$$2\vec{R} = 2(l-1, l-2, \ldots, 1, 0)$$

and

$$g^{ij} = \frac{1}{4(l-1)}\,\delta_{ij}.$$

The l fundamental weights are

$$\vec{L}^{(1)} = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{1}{2}),$$
$$\vec{L}^{(2)} = (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}, \tfrac{-1}{2}),$$
$$\vec{L}^{(3)} = (1, 0, \ldots, 0),$$
$$\vec{L}^{(4)} = (1, 1, 0, \ldots, 0),$$
$$\cdots$$
$$\vec{L}^{(l)} = (1, 1, 1, \ldots, 1, 0).$$

Evaluating Eq. (4.15) one finds

$$s_2^{(1)} = \frac{l(2l-1)}{16(l-1)} = s_2^{(2)},$$

which is not stagnant for any l and is dangerous only for $l=3$. Also

$$s_2^{(3)} = \frac{(2l-1)}{4(l-1)}$$

is not dangerous for any l. Since $\vec{L}^{(i)} \cdot \vec{L}^{(j)} > 0$, B_l has no stagnant representations.

The family A_l ($l \geq 1$) corresponds to SU($l+1$). The root vectors[14] are $\vec{e}_i - \vec{e}_j$, $i, j = 1, \ldots, l+1$. $\vec{e}_i$ are $(l+1)$ orthonormal vectors in $(l+1)$-dimensional space. The roots, as well as the weights, lie in an l-dimensional subspace consisting of all vectors orthogonal to $\sum_{i=1}^{l} \vec{e}_i$. The l fundamental weights are

$$\vec{L}^{(1)} = \frac{1}{l+1}(l, -1, -1, \ldots, -1),$$

$$\vec{L}^{(2)} = \frac{1}{l+1}(l-1, l-1, -2, \ldots, -2),$$

$$\cdots$$

$$\vec{L}^{(l)} = \frac{1}{l+1}(1, 1, \ldots, 1, -l).$$

The metric

$$g_{ij} = \sum_{\alpha} \alpha_i \alpha_j = 2(l+1) P_{ij},$$

where $P^2 = P$ is a projection operator into the l-dimensional subspace defined above. Clearly g_{ij} has no inverse. However, g^{ij} may be defined by $g^{ij} g_{jk} = P_{ik}$. Thus $g^{ij} = [1/2(l+1)] P_{ij}$. We find

$$2\vec{R} = (l, l-2, l-4, \ldots, -l+2, -l)$$

and

$$s_2^{(j)} = \frac{(l+2)}{2(l+1)^2} j(l+1-j).$$

Clearly $s_2^{(j)} = s_2^{(l+1-j)}$ and $s_2^{(j)}$ reaches a maximum value for j = the integer(s) closest to $\frac{1}{2}(l+1)$; $s_2^{(j)}$ is smallest for $j=1$ or l.

$$s_2^{(1)} = \frac{l(l+2)}{2(l+1)^2}$$

is dangerous for all l but not stagnant for any l. Since $\vec{L}^{(i)} \vec{L}^{(j)} > 0$, it follows that the family A_l has no stagnant representations.

The roots and the two fundamental weights of the exceptional algebra G_2 are given in Ref. (16). One finds $s_2^{(1)} = \frac{1}{2}$ and $s_2^{(2)} = \frac{13}{16}$. The algebra G_2 has no dangerous representations.

APPENDIX B

We wish to study the flow pattern in the λ-g plane described by Eqs. (3.4) and (3.5). Since these equations are valid only in a neighborhood around the origin in the λ-g plane, the asymptotic behavior they exhibit must be discounted. Nevertheless we will now forget that these equations are grounded in lowest-order perturbative calculation and study them as a model of the type of equations one would encounter in quantum field theories. In other words, the asymptotic behavior exhibited by Eqs. (3.4) and (3.5), though assuredly not quantitatively descriptive of pseudoscalar fermion theory, may yet provide a qualitative clue to the behavior possible in field theory. Wilson[1] has discussed fixed point and limit cycle behavior. Equations (3.4) and (3.5) exhibit another type of behavior.

As it happens these equations are exactly soluble. Equation (3.5) gives $g^{-2} = g_p^{-2} - 2dt$, where $g_p = g(t=0)$. Eliminating t from Eq. (3.4), we find

$$\frac{d\lambda}{dg} = \alpha \frac{\lambda^2}{g^3} - \beta g + \gamma \frac{\lambda}{g}, \tag{B1}$$

where $\alpha \equiv a/d$, $\beta \equiv b/d$, and $\gamma \equiv c/d$ are three positive numbers. Equation (B1) may be recognized[18] as a generalized equation of the Riccati type.[19] It is convenient to use the parametrization $\lambda = \xi g^2 + z$, with

$$\xi \equiv \frac{-(\gamma-2) + [(\gamma-2)^2 + 4\alpha\beta]^{1/2}}{2\alpha}. \tag{B2}$$

Then the exact solution is

$$z^{-1} = M\left(\frac{g_p}{g}\right)^m - N\left(\frac{g_p}{g}\right)^2 \tag{B3}$$

where

$$N \equiv \frac{\alpha g_p^{-2}}{2\alpha\xi + \gamma - 2}, \quad M \equiv N + z_p^{-1}, \quad m \equiv 2\alpha\xi + \gamma. \tag{B4}$$

We note that $N > 0$ and $m > 2$. The initial condition is $g(t=0) = g_p$ and $\lambda(t=0) = \xi g_p^2 + z_p$. It is easy to see that the parabola defined by $\lambda = \xi g^2$ is a domain line of instability in the sense that if $z_p > 0$ then $\lambda \to +\infty$ as $t \to +\infty$, and that if $z_p < 0$ then $\lambda \to -\infty$ as $t \to +\infty$. Indeed, inspection of Eq. (B3) reveals that if $z_p < 0$, then

$$\lambda \to -\left\{\frac{(\gamma-2) + [(\gamma-2)^2 + 4\alpha\beta]^{1/2}}{2\alpha}\right\} g^2 \quad \text{as } t \to +\infty. \tag{B5}$$

On the other hand, if $z_p > 0$, as $t \to +\infty$ then

$$g \to g_{\text{critical}} = g_p (M/N)^{1/(m-2)}, \tag{B6}$$

and when $g = g_{\text{critical}}$, $\lambda = +\infty$.

If d were negative in Eq. (3.5) (this is what would have happened for stagnant representations), then the solution again has the form $\lambda = \xi g^2 + z$, but with

$$\xi = -\{(\gamma-2) + [(\gamma-2)^2 + 4\alpha\beta]^{1/2}\}/2\alpha.$$

Now if $z_p < 0$ then $(\lambda, g) \to (0, 0)$ as $t \to +\infty$. If $z_p > 0$ then $(\lambda, g) \to (+\infty, g_{\text{critical}})$, where $g_{\text{critical}} = g_p (N/M)^{1/(2-m)}$.

The existence of g_{critical} implies that under some circumstances the asymptotic behavior of a two-coupling-constant theory may be determined by a theory in which one of the coupling constants is infinite, while the other has a finite value which may be quite small.[20] Equation (B5) shows that under other circumstances coupling constants may be related at asymptotic energies. One may speculate that entirely new types of symmetry may emerge in the deep Euclidean region.[21]

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