Possible Strengthening of the Interpretative Rules of Quantum Mechanics*

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The usual interpretative rules of quantum mechanics are presented and are shown to be too weak. The reason is that they do not include the intuitive requirement of randomness of the outcome sequence obtained from an infinite repetition of measuring an observable on a state. A strengthening of the rules is proposed which includes, in essence, a precise definition of randomness. The resultant rule is seen to be intuitively more satisfying than the usual rules and to include the expectation value rule and to include essentially all of the spectrum rule of the usual rules. It also suggests that the relationship between the foundations of mathematics and quantum mechanics may be quite deep and complex.

I. INTRODUCTION

The purpose of this paper is to suggest a strengthening of the interpretative rules of quantum mechanics which is more precise and intuitively satisfying than the usual rules. The strengthening suggests that the relationship between the foundations of mathematics and quantum mechanics may be quite deep and complex. The reason for this is that the proposed strengthened rules involve mathematical logical concepts in an essential way.

The standard rules are given and criticized in Sec. II. It is seen that the expectation value rule is too weak in that there are many other tests the outcome sequence must satisfy besides a comparison between the limit mean and expectation value. In essence, the intuitive requirement that the outcome sequence must be random is not included.

It is proposed that the rules be strengthened by including, in essence, a precise definition of randomness. After a brief review of such definitions, a strengthened rule, rule $(3)_\tau$, is proposed in Sec. III as a replacement for both the usual expectation value rule and the spectrum rule. It is then proved that rule $(3)_\tau$ includes the expectation value rule and includes essentially all of the spectrum rule.

In Sec. IV the rule is discussed further. It is seen that the strengthened rule implies that the outcome sequence of the infinite repetition of the measurement of an observable on a system in some state is τ-random. The possible relationship between mathematical logic and quantum mechanics which arises through the use of mathematical logical concepts in rule $(3)_\tau$ is then discussed. It is also seen that rule $(3)_\tau$ can be applied to generalizations of quantum mechanics. Finally a possibly unsatisfactory aspect of rule $(3)_\tau$ is discussed and possible ways to remove this feature are noted.

II. THE INTERPRETATIVE RULES OF QUANTUM MECHANICS

Let $\mathcal{O}$ and $\mathcal{S}$ denote the respective collections of observation procedures and state preparation procedures. Let $\mathcal{A}$ be a von Neumann subalgebra of $B(\mathcal{H})$, the set of all bounded linear operators over some Hilbert space $\mathcal{H}$. Let $S(\mathcal{A})$ be the set of all states over $\mathcal{A}$ which are implementable as normalized density operators over $\mathcal{H}$.

A standard version of the usual interpretative rules of nonrelativistic quantum mechanics is as follows:

(1) Each state preparation procedure s in $\mathcal{S}$ corresponds to a density operator ρ_s in $S(\mathcal{A})$.

(2) Each observation procedure α in $\mathcal{O}$ corresponds to a self-adjoint operator A_α in $\mathcal{A}$.

(3) For each observation procedure α, the set S^α of possible outcomes satisfies $S^\alpha = \sigma(A_\alpha)$, where $\sigma(A_\alpha)$ is the spectrum of A_α.

(4) For each s and α, the limit mean $\overline{M}\psi_{s\alpha}$ of the outcome sequence $\psi_{s\alpha}$ obtained from an infinite repetition of measuring A_α on ρ_s must satisfy

$$\overline{M}\psi_{s\alpha} = \mathrm{Tr}\rho_s A_\alpha .$$

These rules are stated in a way which allows for the possibility that not every state has a corresponding preparation procedure and not every self-adjoint operator in $\mathcal{A}$ has a corresponding measurement procedure.[1]

In these rules an infinite repetition of measurements of A_α on ρ_s means an infinite repetition of the following operations: Prepare system in ρ_s, measure A_α, observe and record outcome, and discard system. By rule (3), $\psi_{s\alpha}$ is an element of R^ω, the set of all countably infinite sequences of real numbers.

It might be objected that we have stated rule (4) in terms of infinite repetitions of measurements whereas the most one ever carries out are finite sequences of measurements. Since this has been discussed in more detail elsewhere[2] the arguments will not be repeated here. Suffice it to say that replacing "infinite repetition" by "finite repetition" in rule (4) would make it much more imprecise and difficult to interpret. One would have to state exactly how many repetitions is a finite number and replace equality by some probabilistic statement which includes the requirement of convergence. The interpretation of this statement is either in terms of infinite repetitions, which brings one back to the original rules, or in terms of subjective probabilities whose meaning is quite imprecise.

Rule (3) is stated in terms of the spectrum rather than the eigenvalues to allow for the possibility that operators such as the momentum and position [truncated so as to make them elements of $B(\mathcal{H})$] are measurable. Also it can be shown[3] that the spectrum rule as stated cannot be correct in general. However, the differences between rule (3) and a corrected version[4] are small and for our purposes one obtains the same conclusion for both versions. Thus we shall work with rule (3) as stated.

The main criticism is with rule (4). The main problem is that rule (4) is incomplete. There are many other intuitive requirements on $\psi_{s\alpha}$ which are not expressed. For example, every experimenter, in carrying out an infinite repetition of a measurement of A_α on ρ_s, would discard the measurement sequence as being invalid if he found that the mean $\overline{M}_n\psi_{s\alpha}$ of the first n elements of the outcome sequence continually decreased as n increased. Why? Rule (4) says nothing about discarding such a sequence as such a sequence can still give $\overline{M}\psi_{s\alpha} = \mathrm{Tr}\rho_s A_\alpha$. As another example, let α be such that P_α is a projection operator and s be such that $\mathrm{Tr}\rho_s P_\alpha = \frac{1}{2}$. If $\psi_{s\alpha}$ consisted of an infinite alternating sequence of 0's and 1's, 010101..., the experimenter would discard the sequence of measurements as being incorrect even though the outcome sequence, $\psi_{s\alpha}$, satisfied rule (4) in that $\overline{M}\psi_{s\alpha} = \frac{1}{2} = \mathrm{Tr}\rho_s P_\alpha$.

These intuitive requirements can all be collected together into the one requirement that $\psi_{s\alpha}$ be a *random* sequence. Now it is clear that the rule (4) says almost nothing about this. That is, one would like to be able to prove that the outcome sequence $\psi_{s\alpha}$ obtained from an infinite repetition of doing s and α is random. Yet it is clear that such a proof is not forthcoming from rule (4).

The reason one cannot give such a proof is that an essential connection between probability measures and outcome sequences is missing. That is, rule (4) says almost nothing about what it means for a probability measure to be "correct" for an outcome sequence. The requirement that an infinite sequence of measurements be an *infinite repetition* of measurements of an A_α on some ρ_s implies that the probability measure assigned to the sample space of the infinite measurement sequence is a product measure generated from ρ_s and A_α. It says nothing about properties of $\psi_{s\alpha}$ and does not provide this connection. Rule (4) gives only one small part of the connection, i.e., that $\overline{M}\psi_{s\alpha} = \mathrm{Tr}\rho_s A_\alpha$, but leaves out all the others.

These arguments suggest that one strengthen the interpretative rules by putting a precise definition of randomness for sequences in R^ω into the rules, and this is in essence what will be done. However, before doing so it is worthwhile to review briefly the history of definitions of randomness.

The history of attempts to define random sequences can be characterized as a sequence of giving a definition, then later having it found invalid, and then replacing it with another definition. Either the definitions were too restrictive (such as that of Von Mises[6]) in that no sequences existed which satisfied the definition or the definitions were too weak (such as that of Church[7]) in that sequences which were random according to the definition had properties which, intuitively, a random sequence should not have. These and other definitions are discussed in more detail elsewhere.[2] At present one has a definition which is neither too restrictive nor too weak in the above sense.

The present definition of randomness as τ-randomness is an extension to sequences in R^ω of definitions given[8,9] for natural number sequences. These definitions, which say in essence that a sequence ψ is τ-random if there exists a product probability measure which is τ-correct for ψ, make essential use of the mathematical logical concept of τ-definability, or definability relative to a mathematical theory $\boldsymbol{\tau}$.

A mathematical theory τ consists of a set of formulas, terms, variables, constants, and relation and function symbols (the language of τ) together with a designated set of sentences (formulas with no free variables) as the axioms of the theory, and logical deduction rules. The formulas are built up inductively by means of the logical connectives from the atomic formulas which are formed from the terms and relation symbols. Similarly the terms are built up inductively from the variables, constants, and function symbols.

A structure[10] for τ consists of a universe U of elements and sets R and O of relations and operations on U such that $\langle U, R, O\rangle$ is an interpretation of τ. A model for τ is any structure for τ in which the axioms are true. A subset E of U is τ-definable from an element a of U if there is a formula $Q(\varphi, \underline{\alpha})$ in τ ($\underline{\alpha}$ is the name of α added to the language of τ) with one free variable φ such that $E = [\varphi \mid Q(\varphi, \alpha)$ true in $U]$. In particular, for our case, a subset E of R^ω is τ-definable from a probability measure P if there is a formula $Q(\varphi, \underline{P})$ ($\underline{P}$ is the name of P added to the language of τ) with one free R^ω variable such that $E = [\varphi \mid Q(\varphi, P)$ true with $\varphi \in R^\omega]$.[11]

III. THE STRENGTHENED INTERPRETATIVE RULES

It is proposed here to strengthen the usual interpretative rules of quantum mechanics by the replacement of both rule (4) *and rule (3)* by the following rule:

$(3)_\tau$ For each s in $\mathcal{S}$ and α in $\mathcal{O}$ the outcome sequence $\psi_{s\alpha}$ obtained from an infinite repetition of carrying out s and α must satisfy the following: Each property of elements of R^ω (the set of all infinite sequences of real numbers) which is τ-definable from $\tilde{P}_{\rho_s A_\alpha}$ and which is true $\tilde{P}_{\rho_s A_\alpha}$ almost everywhere on R^ω must be true for $\psi_{s\alpha}$. $\tilde{P}_{\rho_s A_\alpha}$ is the product probability measure on $\mathcal{B}(R^\omega)$ (the set of all Borel subsets of R^ω) which satisfies

$$\tilde{P}_{\rho_s A_\alpha} E_{Fj} = \mathrm{Tr}\rho_s \mathcal{E}^{A_\alpha}(F) = P_{\rho_s A_\alpha}(F) \tag{1}$$

for each $j = 0, 1, \ldots$ and each $F \in \mathcal{B}(R)$. $E_{Fj} = [\varphi \mid \varphi(j) \in F]$ and $\mathcal{E}^{A_\alpha}$ is the spectral measure of A_α.

In rule $(3)_\tau$ the right-hand equality of Eq. (1) merely defines the measure $P_{\rho_s A_\alpha}$ which is used later on. Each property p corresponds to a set E_p consisting of all and only those elements of R^ω which have property p. To say that p is true $\tilde{P}_{\rho_s A_\alpha}$ almost everywhere means that $\tilde{P}_{\rho_s A_\alpha} E_p = 1$ [and $E_p \in \mathcal{B}(R^\omega)$].

There are two general requirements on τ. One of them is that τ be restricted to be a theory with at most countably infinitely many formulas.[12] In this case one can prove that rule $(3)_\tau$ is not empty. That is, one can prove that for any P the set of all sequences which possesses every property which is τ-definable from P and true P almost everywhere is a set of P measure 1. The other requirement is that τ be strong enough to include probability theory on R^ω. Clearly this is necessary if rule $(3)_\tau$ is to have any strength.

Here we choose τ to be Zermelo-Frankel set theory. This choice, which satisfies both the requirements and is sufficiently strong to include essentially all mathematics, will be discussed more later on.

We show now that rule (4) of the usual interpretative rules is included in $(3)_\tau$. To carry this out it must be shown that for each s and α, $\overline{M}(\varphi) = \mathrm{Tr}\rho_s A_\alpha$ is τ-definable from $\tilde{P}_{\rho_s A_\alpha}$ and is true $\tilde{P}_{\rho_s A_\alpha}$ almost everywhere on R^ω. Let $f: R^\omega \to R$ be the random variable defined by

$$f = \int_R r\, dI_{E_{<r,0}} \tag{2}$$

with $E_{<r,0} = [\varphi \mid \varphi(0) < r] \subset R^\omega$ and $I_{E_{<r,0}}$ the characteristic function for $E_{<r,0}$. Let T be the one-sided shift operator, $T: R^\omega \to R^\omega$ defined by $(T\varphi)(j) = \varphi(j+1)$ for $j = 0, 1 \ldots$, and define $\bar{f}^T$ by

$$\begin{aligned} \bar{f}^T &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f(T^j(-)) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \int_R r\, dI_{E_{<r,j}} \end{aligned} \tag{3}$$

if the limit exists.

Now, if the limit exists, $\bar{f}^T$ is τ-definable from $\tilde{P}_{\rho_s A_\alpha}$ as the relevant concepts used are τ-definable from $\tilde{P}_{\rho_s A_\alpha}$ [f and $\bar{f}^T$ are equivalence classes of Borel functions modulo sets of $\tilde{P}_{\rho_s A_\alpha}$ measure zero] and thus

$$\begin{aligned} \bar{f}^T(\varphi) &= \int r\, d\,\mathrm{Tr}\rho_s \mathcal{E}^{A_\alpha}((-\infty, r]) \\ &= \mathrm{Tr}\rho_s A_\alpha \end{aligned} \tag{4}$$

is τ-definable from $\tilde{P}_{\rho A}$. [The right-hand side is τ-definable from $\tilde{P}_{\rho_s A_\alpha}$ as it is obviously τ-definable from $P_{\rho_s A_\alpha}$ and τ-definability from $P_{\rho_s A_\alpha}$ implies τ-definability from $\tilde{P}_{\rho_s A_\alpha}$.]

Since $\tilde{P}_{\rho_s A_\alpha}$ is a product measure generated by $P_{\rho_s A_\alpha}$ [Eq. (1)], $\tilde{P}_{\rho_s A_\alpha}$ is T-invariant and the a.e. ergodic and indecomposability theorems of probability theory[13] give the result that $\bar{f}^T$ exists $\tilde{P}_{\rho_s A_\alpha}$ almost everywhere. Since $f(T^j\varphi) = \varphi(j)$ for each j and φ, one has $\bar{f}^T = \overline{M}$ and rule $(3)_\tau$ then gives that $\overline{M}(\psi_{s\alpha})$ is well defined and $\overline{M}(\psi_{s\alpha}) = \mathrm{Tr}\rho_s A_\alpha$, which is just the statement of rule (4).

We now examine what rule $(3)_\tau$ says about the spectrum rule. While it is not possible to completely derive the rule from rule $(3)_\tau$, one can come very close to it. Let S^A denote the outcome set of an A measurement procedure, $\sigma_d(A)$ the discrete spectrum of A, and $\sigma_c(A)$ the continuous spectrum of A. We shall show that $\sigma_d(A) \subseteq S^A$ (S^A contains all eigenvalues of A) and the set of all points in S^A which are also in $\sigma_c(A)$ is a dense subset of $\sigma_c(A)$.

We first show that $\sigma_d(A) \subseteq S^A$. To this end let $f(n): n = 1, 2, \ldots$ be an enumeration of $\sigma_d(A)$.[14] Since $\mathcal{E}^A(\{f(n)\}) > 0$ for each n, there are states ρ_n lying entirely within $\mathcal{E}^A(\{f(n)\})\mathcal{H}$ for $n = 1, 2, \ldots$. For each n the point $f(n)$ is τ-definable from $P_{\rho_n A}$ as $[r \mid \mathrm{Tr}\rho_n \mathcal{E}^A(\{r\}) = 1]$ contains exactly one point, namely $f(n)$.

Let $\psi_{\rho_n A}$ denote the outcome sequence obtained from an infinite repetition of measuring A on ρ_n.

By the definition of ρ_n, $P_{\rho_n A}(\{f(n)\})=1$. Thus $\tilde{P}_{\rho_n A}E_{\{f(n)\},j}=1$ for each j and rule $(3)_\tau$ gives the result that $\psi_{\rho_n A}(j)=f(n)$ for each j. Since each outcome in $\psi_{\rho_n A}$ must be in S^A one has that

$$\{f(n)\}\subset S^A . \qquad (5)$$

Since this holds for each n and $\cup_n\{f(n)\}=\sigma_d(A)$ one has

$$\sigma_d(A)\subseteq S^A . \qquad (6)$$

To show that $S^A\cap(\sigma_c(A))$ is dense in $\sigma_c(A)$, let r be any point in $\sigma_c(A)$. Then by the definition[15] of continuous spectrum, one has $\mathcal{E}^A((a,b))>0$ for each open interval (a,b) with rational end points with $a<r<b$. Let $\rho_{a,b}$ be a state in $\mathcal{E}^A((a,b))\mathcal{K}$. Then $\mathrm{Tr}[\rho_{ab}\mathcal{E}^A((a,b))]=1$ and rule $(3)_\tau$ gives ($\mathrm{Rng}\psi_{\rho_{ab}A}$ = range set of $\psi_{\rho_{ab}A}$)

$$\mathrm{Rng}\psi_{\rho_{ab}A}\subset(a,b) \qquad (7)$$

for each (a,b) with $a<r<b$. But this means that

$$\mathrm{Rng}\psi_{\rho_{ab}A}\subseteq S^A\cap(a,b) . \qquad (8)$$

Since $\mathrm{Rng}\psi_{\rho_{ab}A}$ is not empty (in fact $\mathrm{Rng}\psi_{\rho_{ab}A}$ is countably infinite), and $b-a$ can be arbitrarily small, one has that $S^A\cap\sigma_c(A)$ is dense in $\sigma_c(A)$.

It should be noted that the validity of this proof depends on the weak condition that there be sufficiently many preparation procedures in the following sense: For each A, if there is a procedure for measuring A then for each eigenvalue r of A and for each open interval (a,b) with rational endpoints which includes a point of $\sigma_c(A)$ there must exist preparable states within $\mathcal{E}^A(\{r\})\mathcal{K}$ and within $\mathcal{E}^A(a,b)\mathcal{K}$. This condition can be made even weaker at the expense of a more complicated proof.

Thus one sees that rule $(3)_\tau$ implies the spectrum rule to the extent that one can show that the outcome set of an A measuring procedure must contain all the eigenvalues of A and for any point r which is in the continuous spectrum of A, and for each pair of rationals a, b with $a<r<b$ there are a countable infinity of points in S^A whose distance from r is less than $b-a$.

It is clear from this that rule $(3)_\tau$ comes very close to implying the spectrum rule. Note further that $(3)_\tau$ does not specify which dense subset of $\sigma_c(A)$ must be in S^A. In particular, for no real number r in $\sigma_c(A)$ can one show by rule $(3)_\tau$ that $r\in S^A$. This point will be returned to later on.

IV. DISCUSSION

There are several aspects of the suggested strengthening of the interpretative rules of quantum mechanics which are worth discussing.

It has been shown that rule $(3)_\tau$ implies rule (4) of the old rules. However, nothing has been said so far about the other conditions which an outcome sequence $\psi_{s\alpha}$ must satisfy. In particular one requires that $\psi_{s\alpha}$ be random (provided that ρ_s does not lie entirely in one eigenspace of A_α).

To see what rule $(3)_\tau$ says about this consider the following definition of τ-randomness:

A sequence ψ in R^ω is τ-random if there exists a product probability measure $\tilde{P}$ on $\mathcal{B}(R^\omega)$ generated from a nontrivial measure P on $\mathcal{B}(R)$ such that every property of sequences in R^ω which is τ-definable from $\tilde{P}$ and which is true $\tilde{P}$ almost everywhere is true for ψ. P is nontrivial if for no r in R does one have $P(\{r\})=1$.

As noted earlier, this definition is a direct extension to R^ω of earlier definitions given with various τ for sequences of natural numbers.[8,9,2]

It is immediately clear that rule $(3)_\tau$ implies τ-randomness. More precisely, one has that for all procedures α and s, if ρ_s does not lie entirely in an eigenspace of A_α, rule $(3)_\tau$ implies that the outcome sequence $\psi_{s\alpha}$, obtained from an infinite repetition of measuring A_α on a system in ρ_s, is τ-random.

The following question arises: If one uses rule $(3)_\tau$ as the correct interpretative rule, then $\psi_{s\alpha}$ is τ-random. But is $\psi_{s\alpha}$ random? An answer to this question is much more difficult. On intuitive grounds one would want to choose τ to be sufficiently strong so that any property which is intuitively required of a random sequence is included in the definition of τ-randomness. Also, τ must be strong enough so that the proofs given before are valid proofs, i.e., all the properties of various sets used in the proofs and which are required by the proofs to be τ-definable from $\tilde{P}_{\rho_s A_\alpha}$ must in fact be so.

These two requirements place a floor under the strength of τ in that τ must be sufficiently strong to satisfy them. For these reasons we have chosen τ to encompass most mathematics, i.e., Zermelo-Frankel set theory. This choice clearly satisfies the second requirement and is a choice made in other work.[9]

An argument based on the first requirement and which suggests that τ must be at least as strong as set theory with respect to definable singleton subsets of R^ω is the following: If τ is weaker than set theory in this respect, then there exists some singleton subset $\{\psi\}$ of R^ω which is not τ-definable, but is definable in set theory. It is then possible that $\psi=\psi_{s\alpha}$ for some s and α and thus that $\psi_{s\alpha}$ is definable in set theory. Now $\psi_{s\alpha}$ is supposed to be random, and, intuitively, it seems reasonable to require that a random sequence should not be definable even in set theory.

Thus to avoid a possible contradiction, it follows that τ must be at least as strong as set theory with respect to definable singleton subsets of R^{ω}. (From a result obtained elsewhere,[2] if $\psi_{s\alpha}$ is τ-random, then $\psi_{s\alpha}$ is not τ-definable from $\tilde{P}_{\rho_s A_\alpha}$.)

The possibility of a deep and close relationship between the foundations of mathematics and physics has been suggested, often indirectly, by other authors.[16] The satisfactory aspects of the strengthened interpretative rules of quantum mechanics proposed here also support this possibility. The reason is that the concept of definability relative to a mathematical theory enters into the rules in an essential way. Furthermore, it appears that the mathematical theory should encompass most mathematics.

The possibility of this relationship can be seen in another way as follows: The method of deciding among different physical theories is to compare the predictions of the different theories with experiment. In particular, for statistical theories one compares limit mean properties of outcome sequences with the expectations computed from the various theories and accepts the theory which agrees with experiment and rejects the others.

Now the concept of randomness enters here in an essential way. The reason is that the limit mean properties of an outcome sequence obtained from a sequence of measurements can be quite different for a nonrandom sequence than for a random one. Thus the choice between different physical theories of which ones agree with experiment depends critically on the requirement of and definition of randomness.

The definitions of randomness so far given, which appear to be most satisfactory, are those given in terms of the mathematical logical concept of definability relative to a mathematical theory τ which must be quite strong. Thus this suggests that such concepts may ultimately be closely related to the choice of which physical theory agrees with experiment. In particular, such a relationship is especially important when one is concerned with comprehensive physical theories such as quantum mechanics as questions concerning the nature of physical reality then may become relevant.

It is important to note that an alternative way to regard the rule $(3)_\tau$ is as a condition on maps between empirical procedures and mathematical objects. Thus in the above, rule $(3)_\tau$ becomes a condition which the maps $\mathcal{O} \to \mathcal{A}$ and $\mathcal{S} \to S(\mathcal{A})$ must satisfy. In such an approach rule $(3)_\tau$ becomes in essence a definition of τ-validity or τ-agreement between quantum mechanics and experiment. This approach, which has much to recommend it, was used elsewhere[17] as a condition on the above maps for quantum theories. There the restriction of the definition of "τ-validity$_2$" to the standard case of the infinite repetition of the measurement of an observable on a system in some state corresponds to rule $(3)_\tau$.

Also it must be stressed that the strengthening of the interpretative rules proposed here applies not only to quantum mechanics but to other statistical theories as well. Furthermore, it can be applied to processes which are more general than infinite repetitions. In fact, one can regard this paper as the specialization of the general theory[2] (easily extendible to R^{ω} from N^{ω}) to quantum mechanics and to infinite repetitions of single measurements.

However, it appears that this strengthening may play a more important role in quantum mechanics than in other theories (quantum mechanics is at present the basic microscopic physical theory and is essentially more statistical than classical mechanics) and may be essentially interrelated to quantum mechanics itself.

Rule $(3)_\tau$ can also be applied to generalizations of quantum mechanics. Consider for example the work of Ekstein[18] where external fields are included in one scheme. In Ekstein's setup one has for each external field an expectation E and a pair (Ψ_E, Φ_E) of maps where $E: \mathcal{S}\times\mathcal{O} \to R$, $\Psi_E: \mathcal{S} \to S(\mathcal{A})$, and $\Phi_E: \mathcal{O} \to \mathcal{A}$. $\mathcal{S}$ and $\mathcal{O}$ are sets of state preparation and observation procedures, $\mathcal{A}$ is the set of self-adjoint operators in a von Neumann algebra of operators on some Hilbert space, and $S(\mathcal{A})$ is the set of states over $\mathcal{A}$. For each external field one defines the corresponding E by $E(s,\alpha) = \overline{M}\psi_{s\alpha}$ where $\psi_{s\alpha}$ is the outcome sequence obtained from an infinite repetition of carrying out the state preparation procedure s and the observation procedure α in the external field.

In this case rule $(3)_\tau$ or its equivalent as the definition[17] of τ-validity (given in Ref. 17 as τ-validity$_2$) is a condition which must be satisfied by (Ψ_E, Φ_E) for each E. That is, for each $s \in \mathcal{S}$ and $\alpha \in \mathcal{O}$ and for each external field with corresponding E, rule $(3)_\tau$ applies as given with $\Psi_E(s)$ and $\Phi_E(\alpha)$ replacing ρ_s and A_α, respectively, and $\Psi_E(s)(\mathcal{E}^{\Phi_E(\alpha)}(F))$ replacing $\mathrm{Tr}\rho_s\mathcal{E}^{A_\alpha}(F)$ in Eq. (1). Note that in this case a *consequence* of rule $(3)_\tau$ is that $E(s,\alpha)$ is well defined and

$$E(s,\alpha) = \Psi_E(s)(\Phi_E(\alpha)) \tag{9}$$

hold for each E, s, α. On the other hand, Ekstein[18] gives Eq. (9) as the *definition* of Ψ_E and Φ_E and assumes without proof that the defining limit of $E(s,\alpha)$ exists.

Finally we want to note an unsatisfactory feature of rule $(3)_\tau$ and suggest possible ways to cure the problem. In the discussion of rule $(3)_\tau$ and the spectrum rule it was seen that for no real number r in $\sigma_c(A)$ can one prove that rule $(3)_\tau$ implies that r must be in S^A. [This does not contradict rule $(3)_\tau$ also saying that $S^A \cap \sigma_c(A)$ must be dense in $\sigma_c(A)$.]

However, there are also real numbers in $\sigma_c(A)$ which rule $(3)_\tau$ says must not be in S^A. More precisely, let r be any real number in $\sigma_c(A)$ which is τ-definable from $P_{\rho A}$ and let $\psi_{\rho A}$ be an outcome sequence obtained from an infinite repetition of measuring A on ρ. Then $\mathrm{Tr}\rho\mathcal{E}^A(\{r\})=0$ and rule $(3)_\tau$ says that $\mathrm{Rng}\psi_{\rho A} \subset R-\{r\}$ or conversely that r is not a possible outcome of a measurement of A on a system in state ρ. Now the intersection of all sets of real numbers τ-definable from some probability measure P contains the set of all τ-definable real numbers. Thus rule $(3)_\tau$ gives the unsatisfactory result that no real number in $\sigma_c(A)$ which is τ-definable is a possible outcome of any measurement of A.

It is best to illustrate what this means and does not mean by means of an example. Let A be the position operator for the line segment $[0, 10]$. That is, A gives the position in $[0, 10]$ and is zero outside. Now rule $(3)_\tau$ says that for no state ρ are the real numbers $1, 2, 3, \ldots, 10$ possible outcomes of a measurement of A (0 is not excluded as it is an eigenvalue of A). Note that the real number n $=n.00\cdots$ with an infinite number of zeros. Furthermore, for any interval (a, b) with a, b rational and $0<a<b\leq 10$, the set of real numbers in (a, b) excluded by rule $(3)_\tau$ is dense in (a, b). (The rationals are τ-definable.) However, there are also uncountably many real numbers in (a, b) which are allowed as outcomes (the τ-undefinable reals). Thus although $3.00\cdots$ is not a possible outcome in the example there are uncountably many real numbers arbitrarily close to $3.00\cdots$ which are possible outcomes.

This selecting out of the τ-definable reals in $\sigma_c(A)$ as not being possible outcomes of an A-measurement causes no difficulty with the probabilistic aspects of the theory. However, there is no reason why this particular (countable) subset of reals should be selected out for such special treatment. Just because a number can be "pointed at" or defined would not seem to be a reason to exclude it from $\sigma_c(A)$ as a possible outcome.

One possible way out of this difficulty is the following: It is clear that this problem occurs for only those A which have a nonempty continuous spectrum. For any such A the spectrum rule implies that any A measuring procedure has an uncountable infinity of possible outcomes.

Now it can be argued that any measurement procedure which one can carry out has at most a countable infinity of outcomes and, as a result, any A for which there exists a measurement procedure must be discrete. Since the problem for rule $(3)_\tau$ does not exist for discrete A, this argument obviously solves the difficulty.

This argument is helped by the fact that for each self-adjoint operator A there exists a sequence A_n, $n=1,2,\ldots,$ of discrete self-adjoint operators where[19] $A=\text{uniform}\lim_n A_n$. As a result one can have measurement procedures for the operators A_n without the unsatisfactory feature imparted by rule $(3)_\tau$ and which are arbitrarily close to A without having a measurement procedure for A itself.

Other possible ways out consist of adding further requirements into rule $(3)_\tau$ besides that requirement that a property be τ-definable from $\tilde{P}_{\rho_s A_\alpha}$ and true $\tilde{P}_{\rho_s A_\alpha}$ almost everywhere. For example, one can require that the property also be a limit mean or average property with respect to some noncyclic ($T^n \neq T$ for each n) transformation $T: R^\omega \to R^\omega$, which is τ-definable from $\tilde{P}_{\rho_s A_\alpha}$. However, we leave the discussion of such restrictions to future work.

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[1]E. P. Wigner, Am. J. Phys. 31, 6 (1963); J. L. Park and H. Margenau, in *Perspectives in Quantum Theory, Essays in Honor of Alfred Lande,* edited by W. Yourgrau and A. Van der Merwe (MIT Press, Cambridge, Mass., 1971), pp. 37–70; R. Giles, J. Math. Phys. 11, 2139 (1970).

[2]P. A. Benioff, J. Math, Phys. 11, 2553 (1970); 12, 361 (1971).

[3]Let $P_n: n=0,1,\ldots$ be a complete orthogonal family of projections on $\mathcal{H}$ and define A_0 by $A_0=\sum_n r(n)P_n$, where $r(n)$ with $n=0,1,\ldots$ is an enumeration of all the rational numbers in the interval $[0, 1]$. The discrete and continuous spectra, $\sigma_d(A_0)$ and $\sigma_c(A_0)$, of A_0 are equal to the respective sets of rational and irrational numbers in $[0, 1]$. But no point in $\sigma_c(A_0)$ can possibly be an outcome of a measurement of A_0 and thus S^α $=\sigma(A_\alpha)=\sigma_d(A_\alpha)\cup\sigma_c(A_\alpha)$ may not be correct for all α.

[4]One can correct rule (3) by replacing $S^\alpha=\sigma(A_\alpha)$ with $S^\alpha=\sigma_d(A_\alpha)\cup\sigma(A_{\alpha c})$. $A_{\alpha c}$, the continuous part of A_α, is defined as follows (Ref. 5): Let $P_D=\sum_{r\in\sigma_d(A_\alpha)}P_r$, where P_r is an eigenprojector of A_α, be the projection operator onto the discrete subspace of A_α on $\mathcal{H}$. Define $A_{\alpha c}$ to be the restriction of A_α to $(1-P_D)\mathcal{H}$. The spectrum of $A_{\alpha c}$ is pure continuous. One has $\sigma(A_\alpha)=\sigma(A_{\alpha d})\cup\sigma(A_{\alpha c})$; $\sigma(A_{\alpha d})$ and $\sigma(A_{\alpha c})$ are not necessarily disjoint; and $\sigma_d(A_\alpha)=\sigma_d(A_{\alpha d})\subseteq\sigma(A_{\alpha d})$. For the example A_0, P_D

=1, and thus $\sigma(A_{0c})$ is empty as A_0 has no continuous part.

[5]F. Riesz and B. Sz. Nagy, *Functional Analysis* (Ungar, New York, 1955), Sec. 132.

[6]R. Von Mises, *Mathematical Theory of Probability and Statistics,* edited by H. Geiringer (Academic, New York, 1964), Chap. I.

[7]A. Church, Bull. Am. Math. Soc. 46, 130 (1940).

[8]P. Martin Löf, Proceedings of Symposium on Proof and Theory and Intuitionism, Buffalo, edited by J. Myhill, A. Kino, and R. Vesley (North-Holland, Amsterdam, 1970).

[9]A. H. Kruse, Z. Math. Logik Grundlagen Math. 13, 299 (1967).

[10]J. Shoenfield, *Mathematical Logic* (Addison-Wesley, Reading, Mass., 1967), Chap. 2.

[11]The definition of definability used here is that of semantic definability. There is a more restrictive concept of syntactic definability (which avoids the model theoretic concept of truth) which can also be used in defining rule $(3)_T$. Since we cannot at present decide which of the two concepts is more appropriate in $(3)_T$, the semantic concept has been arbitrarily chosen.

[12]Relaxing this requirement means allowing theories with expressions of infinite length or uncountably many constants, etc. Since it is at least questionable whether one needs to consider such extended theories and the consequence of such a relaxation are quite nontrivial, the restriction is quite reasonable.

[13]M. Loeve, *Probability Theory,* third edition (Van Nostrand, Princeton, New Jersey, 1963), Secs. 30–32.

[14]It is assumed here that $\mathcal{H}$ is separable and thus $\sigma_d(A)$ is at most countable. The proof can also be extended to show that $\sigma_d(A) \subseteq S^A$ for nonseparable $\mathcal{H}$.

[15]M. H. Stone, in *Linear Transformations in Hilbert Space* (Am. Math. Society Colloquium Publication, 1932), Vol. XV, Chap. V, Sec. 5.

[16]P. J. Davis, Am. Math. Monthly 79, 252 (1972); D. Finkelstein, Phys. Rev. D 5, 230 (1972); 5, 2922 (1972); M. N. Hack, Nuovo Cimento 54B, 147 (1968); H. Putnam, in *Boston Studies in the Philosophy of Science,* edited by M. Wartofsky and R. Cohen (Humanities Press, New York, 1969), Vol. 5, pp. 179–204.

[17]P. A. Benioff, Found. Phys. (to be published). The spectrum rule appears separately in the definitions of τ-validity in this reference, as it was not realized at the time how much of the rule is implied by rule $(3)_T$.

[18]H. Ekstein, Phys. Rev. 184, 1315 (1969).

[19]Reference 5, Secs. 134 and 135.

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Fine Structure and Perihelion Advance*

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The evaluation of field-theoretical corrections to the properties of bound states can be greatly simplified by basing it on our relativistic Lippmann-Schwinger equation rather than the Bethe-Salpeter or Schrödinger equations. We have evaluated the fine structure to order α^4 for a system of two scalar particles bound by the potential that arises from exchanging massless quanta of spin 0, 1, or 2. In each case fine structure is clearly related to seagull and/or graviton self-interaction diagrams, while neither the simple exchange diagram nor the box diagrams make any contribution. The simplification with respect to earlier calculations is particularly striking in the case of gravity. Also presented here is a calculation of the advance of the perihelion of Mercury within the framework of a recently developed classical-relativistic mechanics of two interacting point particles. The calculation is explicitly covariant.

I. INTRODUCTION

The three calculations presented in this paper are concerned with scalar particles that are bound by the potential that arises from the exchange of massless quanta with spin 0, 1, or 2. The spin-1 quanta are photons, and the calculation is thus based on quantum electrodynamics. For spin 2 we use the flat-space formulation of Einstein's theory of gravitation. No special difficulties are encountered in the order of perturbation theory considered.

To extract information from field theory we start with the expansion of the two-body scattering matrix: $T=T_1+T_2+\cdots$. Here T_1 consists of all second-order diagrams – there is only one of any consequence – and T_2 includes all fourth-order diagrams. If K^{-1} is a convenient Green's function, we can define a potential $W=W_1+W_2+\cdots$ by means of the equation

$$T=-W+W\frac{1}{K}\,T\,. \tag{1.1}$$

Inserting the expansions of T and W one finds that

$$W_1=-T_1\,,\qquad W_2=-T_2+W_1\frac{1}{K}\,T_1\cdots\,. \tag{1.2}$$