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PHYSICAL REVIEW D VOLUME 7, NUMBER 12 15 JUNE 1973

Vector-Metric Theory of Gravity*

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(Received 15 February 1973)

A theory in which gravity is produced by a massless vector field in addition to the usual metric field is presented and found to be compatible with present solar-system experiments and cosmological expansion. A special case predicts the same first post-Newtonian gravitational experimental results as general relativity.

I. INTRODUCTION

In 1961, Brans and Dicke[1] proposed a theory in which gravitation was produced by two fields—a tensor metric field and an auxiliary scalar field. The field equations were derived from the Lagrangian action[2]

$$A=\int(-g)^{1/2}\left(16\pi L_m+\phi R+\frac{\omega}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right)d^4x\,,$$

where $L_m=L_m(g_{\mu\nu},\text{ matter variables})$ is the matter Lagrangian, ϕ is the scalar field, and ω is a dimensionless parameter. Variation of the matter variables (position, velocity, etc.) will produce equations of motion involving only $g_{\mu\nu}$, not ϕ. Matter "sees" only the metric field and free test bodies follow geodesics. Theories where matter exhibits this behavior are termed "metric theories."

If L_m contains additional gravitational scalar, vector, or tensor fields, matter will not in general follow geodesics. An analysis by Dicke[3] has shown that the high-precision null experiments—Eötvös experiments, Hughs-Drevor experiment, etc.—rule out the existence of vector or additional second-rank-tensor fields coupling directly to matter. As pointed out by Will and Nordtvedt,[4] however, such vector and tensor fields may exist along with the metric field as long as the additional fields do not couple directly to matter (i.e., do not enter L_m). While the additional fields in these metric theories do not affect the null experiments they will, in general, produce observable effects in light deflection and retardation experiments, planetary perihelion advance, orbiting gyroscope precession, nonsecular terms in planetary and satellite orbits, and geophysical phenomena. Will and Nordtvedt[4] have summarized these observational effects and their relationship to metric theories of gravity, showing that such effects are simply calculable from a parametrized post-Newtonian (PPN) metric, which exists for all metric theories.

In particular, it has been found that some metric theories predict observable effects due to the motion of the solar system relative to a preferred frame (such as the mean rest frame of the universe). These "preferred frame" or "ether" effects can occur in Lagrangian-based theories containing vector or higher-rank-tensor fields (the Brans-Dicke scalar theory exhibits no such effects). In this paper we present a metric theory of gravity containing a massless vector field in addition to the metric field. Committed to the spirit as well as to the law of general covariance in physics, we introduce no *a priori* fields or frames into the theory. We require a Lagrangian subject to the following conditions:

(1) The Lagrangian density is a four-scalar density.

(2) It generates positive-definite free-field energies for both the metric and the vector fields.

(3) It produces a "metric theory."

(4) It generates field equations containing no higher than second derivatives of the fields.

Such a Lagrangian is

$$A=\int(-g)^{1/2}(16\pi G_0L_m+R-F_{\mu\nu}F^{\mu\nu}+\omega K_\mu K^\mu R+\eta K^\mu K^\nu R_{\mu\nu})d^4x\,,\quad(1)$$

where $L_m=L_m(g_{\mu\nu},\text{ matter variables})$, $F_{\mu\nu}=K_{\mu|\nu}-K_{\nu|\mu}$ in analogy with electrodynamics, ω and η

are dimensionless parameters, and G_0 is an *a priori* or "bare" gravitational constant.

In Sec. II, we obtain the PPN metric for this theory and find a renormalized effective gravitational constant dependent on the cosmological strength of the vector field. Consequently, there is a time dependence of G coupled to the evolution of the universe. The PPN parameters will have weak "preferred frame" terms compatible with observational limits, or, for a special case, will reproduce exactly the PPN parameters of general relativity.

This last point is of particular importance, since it means that no solar-system experiment presently envisioned can differentiate between this theory and general relativity. Another theory in this category is a "prior geometry" theory due to Ni,[5] which uses an *a priori* preferred inertial frame. It is evident that experiments which are able to choose between these theories must involve post-post-Newtonian levels, such as occur in (i) gravitational radiation, (ii) cosmology, or (iii) extremely precise solar-system experiments. It is not clear which type of experiment offers the best possibility, but theories such as these should stand as a challenge to the gravitational experimenter to devise new and better ways to measure the extremely small effects of post-Newtonian gravity. In preparation for such experiments we have calculated here the post-post-Newtonian static term in g_{ss}, and in Sec. III we have generated cosmological solutions of the field equations. Both results differ from the results predicted by general relativity. The important question of gravitational radiation will be analyzed in a future paper.

II. THE PPN METRIC PARAMETERS

Will and Nordtvedt[4] have arrived at a general form for the first post-Newtonian metric valid in any inertial coordinate frame:

$$g_{00}=1-2\sum_i \frac{GM_i}{r_i}+2\beta\Big(\sum_i \frac{GM_i}{r_i}\Big)^2-(2\gamma+1+\alpha_3+\zeta_1)\sum_i \frac{GM_i v_i{}^2}{r_i}$$
$$-2(1-2\beta+\zeta_2)\sum_i \frac{GM_i}{r_i}\sum_{j\neq i}\frac{GM_j}{r_{ij}}+\zeta_1\sum_i \frac{GM_i}{r_i{}^3}(\vec{v}_i\cdot\vec{r}_i)^2$$
$$+\alpha_2\sum_i \frac{GM_i}{r_i{}^3}(\vec{W}\cdot\vec{r}_i)^2+(\alpha_1-\alpha_2-\alpha_3)\sum_i \frac{GM_i}{r_i}W^2+(\alpha_1-2\alpha_3)\sum_i \frac{GM_i}{r_i}\vec{W}\cdot\vec{v}_i\,,$$

$$g_{0k}=\tfrac{1}{2}(4\gamma+3+\alpha_1-\alpha_2+\zeta_1)\sum_i \frac{GM_i}{r_i}v_i^k+\tfrac{1}{2}(1+\alpha_2-\zeta_1)\sum_i \frac{GM_i}{r_i{}^3}(\vec{v}_i\cdot\vec{r}_i)x_i^k$$
$$+(\tfrac{1}{2}\alpha_1-\alpha_2)\sum_i \frac{GM_i}{r_i}W^k+\alpha_2\sum_i \frac{GM_i}{r_i{}^3}(\vec{W}\cdot\vec{r}_i)x_i^k\,,$$

$$g_{lm}=-\delta_{lm}\Big(1+2\gamma\sum_i \frac{GM_i}{r_i}\Big)\,,$$

where x_i^k are Cartesian components of the ith-source-to-field-point vector, v_i^k are Cartesian velocity components of the ith source ($v^k=-dx^k/dt$), W^k are Cartesian components of the velocity of the inertial coordinate system relative to the universe rest frame,

$$r_i=\Big[\sum_{k=1}^{3}(x_i^k)^2\Big]^{1/2},$$

$$r_{ij}=\Big[\sum_{k=1}^{3}(x_i^k-x_j^k)^2\Big]^{1/2},$$

M_i is the gravitational mass of the ith body, and G is the effective gravitational constant. It should be appreciated that this form is based on very few assumptions (see Nordtvedt[6]). The parameters $\beta, \gamma, \alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2$ in the metric are theory-dependent and may depend on cosmological factors through the influence of cosmological fields. In general relativity the PPN parameters have the value

$$\gamma=\beta=1\,,$$

$$\alpha_1=\alpha_2=\alpha_3=\zeta_1=\zeta_2=0\,,$$

and in the Brans-Dicke scalar-metric theory

$$\gamma=\frac{1+\omega}{2+\omega}\,,$$

$$\beta=1\,,$$

$$\alpha_1=\alpha_2=\alpha_3=\zeta_1=\zeta_2=0\,.$$

Inspection of the PPN metric shows that one can obtain all the PPN parameters by calculating in the universe rest frame ($W^k=0$), and that one can obtain all but ζ_2 by considering a single source. For most of our work we therefore use

$$g_{00}=1-2G\frac{M}{r}+2\beta\frac{G^2M^2}{r^2}-(2\gamma+1+\alpha_3+\zeta_1)\frac{GM}{r}v^2+\zeta_1\frac{GM}{r^3}(\vec{r}\cdot\vec{v})^2,$$

$$g_{0k}=\tfrac{1}{2}(4\gamma+3+\alpha_1-\alpha_2+\zeta_1)\frac{GM}{r}v^k+\tfrac{1}{2}(1+\alpha_2-\zeta_1)\frac{GM}{r^3}(\vec{r}\cdot\vec{v})x^k, \tag{2}$$

$$g_{lm}=-\delta_{lm}\left(1+2\gamma\frac{GM}{r}+2\delta\frac{G^2M^2}{r^2}\right),$$

where it is noted that a post-post-Newtonian term is added to g_{lm}.

In addition to PPN metric expansion, we will need a similar expansion for K_μ:

$$K_0=(\phi_0)^{1/2}\left[1+a_1\frac{GM}{r}+b\frac{G^2M^2}{r^2}+a_2\frac{GM}{r}v^2+a_3\frac{GM}{r^3}(\vec{r}\cdot\vec{v})^2+f\frac{GM}{r}(\vec{r}\cdot\vec{a})\right], \tag{3}$$

$$K_l=(\phi_0)^{1/2}\left[d\frac{GM}{r}v^l+d'\frac{GM}{r^3}(\vec{r}\cdot\vec{v})x^l\right].$$

The field equations are calculated by requiring that the action [Eq. (1)] be stationary under independent variation of the fields. Variation of $g_{\mu\nu}$ gives the equation

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+\omega(K_\mu K_\nu R+\phi R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}\phi R+\phi_{|\mu\nu}-g_{\mu\nu}\Box^2\phi)+\eta K^\alpha K^\beta(g_{\mu\alpha}R_{\nu\beta}+g_{\nu\beta}R_{\mu\alpha}-\tfrac{1}{2}g_{\mu\nu}R_{\alpha\beta})$$
$$-\tfrac{1}{2}\eta[g_{\mu\nu}(K^\alpha K^\beta)_{|\alpha\beta}+\Box^2(K_\mu K_\nu)-(K_\mu K^\alpha)_{|\nu\alpha}-(K_\nu K^\alpha)_{|\mu\alpha}]+2F_{\mu\alpha}F^\alpha{}_\nu+\tfrac{1}{2}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}=-8\pi G_0T_{\mu\nu}, \tag{4}$$

where $\phi\equiv K_\alpha K^\alpha$,

$\Box^2(\;)\equiv(\;)_{|\alpha\beta}g^{\alpha\beta}$.

The contraction of Eq. (4) is

$$R+(3\omega+\tfrac{1}{2}\eta)\Box^2\phi+\eta(K^\alpha K^\beta)_{|\alpha\beta}=8\pi G_0T. \tag{5}$$

Variation of K_μ in Eq. (1) gives

$$\omega RK^\mu+\eta R^{\mu\nu}K_\nu+2F^{\mu\nu}{}_{|\nu}=0. \tag{6}$$

The details of the solution of these equations are given in the Appendix. The linearized equations are first solved for a static source, giving for γ:

$$\gamma=1+\omega\phi_0\frac{2\omega-\eta-2}{1-\omega\phi_0(4\omega-1)}.$$

Light deflection and retardation experiments[7] show that $\gamma\approx1$, so we specialize to that case. There are three ways this condition can be realized. First,

$$\phi_0\ll1.$$

This weak-cosmological-field condition reduces all results arbitrarily close to general relativity. For that reason, it is the least interesting. There are two other conditions which give $\gamma\approx1$. These are

$$\text{(Case I)}\quad \omega=\tfrac{1}{2}\eta+1,$$

and (7)

$$\text{(Case II)}\quad \omega=0.$$

Proceeding with the solutions for each case, it is found that

Case I	*Case II*
$\gamma=1$	$\gamma=1$
$a_1=-1$	$a_1=\frac{1}{2}\eta$

and the gravitational constant in each case is renormalized:

Case I	*Case II*
$G=\dfrac{G_0}{1+\frac{1}{2}\eta\phi_0}$	$G=\dfrac{G_0}{1+\eta\phi_0+\frac{1}{4}\eta^2\phi_0}$.

(8)

This means that G, the effective gravitational constant which enters the metric and determines the strength of gravitational behavior in the solar system, depends on the strength of K_μ far from the solar system. In Sec. III, it is seen from cosmological considerations that this produces a time development of G, coupled to the evolution of the universe; and that the extreme weakness of G in the solar system can be viewed as being due to a renormalization of G_0 (~1) by a large cosmological $\eta\phi_0$.

The parameters β and δ come from the static solution of the field equations to second order. For the two cases we get

(Case I) $\beta = 1$,

which agrees with the value calculated in general relativity, and

$$\delta = 1 + \frac{1-4\omega}{2\phi_0^{-1} + 2\omega(1-4\omega)}, \tag{9}$$

which does not agree, general relativity having $\delta = 1$. Unfortunately, this parameter does not affect existing experiments to any measurable degree. In the other case,

$$\text{(Case II)} \quad \beta = 1 + \frac{\frac{1}{4}\eta(\eta+2)(\eta+4)\phi_0}{4+\eta\phi_0(\eta+4)},$$
$$\delta = 1 - \tfrac{1}{2}\eta\phi_0(\eta+4)\frac{(\eta+3) - \frac{1}{4}\eta\phi_0(\eta+4)}{4+\eta\phi_0(\eta+4)}. \tag{10}$$

The experimental result $\beta \approx 1$ requires $\eta \approx 0$, -2, or -4 and the unobservability of δ effects in present solar-system experiments imposes the limits $\eta = -4$, or $\phi_0 \sim 1$.

Solving the total dynamic linearized field equations gives the additional PPN parameters:

$$\text{(Case I)} \quad \zeta_1 = \alpha_3 = 0,$$
$$\tfrac{1}{2}\alpha_1 = \alpha_2 = \frac{4\eta}{4\phi_0^{-1} + 4 + 6\eta + \eta^2}, \tag{11}$$

$$\text{(Case II)} \quad \zeta_1 = \alpha_3 = 0,$$
$$\alpha_1 = \frac{2\eta\phi_0(3+\eta)}{1+\eta\phi_0 + \frac{1}{4}\eta^2\phi_0}, \tag{12}$$
$$\alpha_2 = \tfrac{1}{2}\alpha_1 - \frac{1}{2}\,\frac{3\eta\phi_0(\eta+2)}{2+4\eta\phi_0+\eta^2\phi_0}.$$

An examination of the configuration of a point source inside a massive spherical shell yields the second-order two-mass PPN parameter

$$\zeta_2 = 0$$

for both cases.

The ζ parameters measure 4-momentum nonconservation and are expected to be zero in theories derived from Lagrangian action principles.[4] The α parameters measure the existence of "preferred frame" or "ether" effects in gravitation. Except for the case $\eta = 0$, both cases predict such preferred frame effects. Nordtvedt and Will[8] have analyzed various geophysical and planetary orbital effects to arrive at upper limits on the α parameters. The restriction is

$$\alpha_1 < 0.1,$$
$$\alpha_2 < 0.1 \quad \text{(see Ref. 9)}.$$

There are two possibilities in Case I:

$$\eta \gtrsim 34,$$
$$\eta \lesssim 0.1. \tag{13}$$

Of special interest is the case where $\eta = 0$ ($\omega = 1$) in Eq. (11); then α_1 and α_2 are strictly zero. In this case, renormalization of G_0 is lost and the total set of PPN parameters is identical to those of general relativity. However, as can be seen by setting $\omega = 1$ in Eq. (9), δ still differs:

$$\delta = 1 + \frac{1}{2}\,\frac{3\phi_0}{3\phi_0 - 1}.$$

In Case II, the α restrictions give

$$\eta\phi_0 \lesssim \tfrac{1}{60}.$$

Case II already had $\omega \approx 0$, so this additional restriction on the magnitude of $\eta\phi_0$ shows this case to approach general relativity as a limiting form of the theory.

III. COSMOLOGY

To investigate cosmological solutions of the theory, we use a homogeneous dust model for matter and a Robertson-Walker cosmological metric:

$$g_{00} = 1,$$
$$g_{ij} = -\frac{S(t)^2}{(1+\frac{1}{4}\kappa r^2)^2},$$

from which we can calculate

$$R_{00} = 3\ddot{S}/S,$$
$$R = \frac{6}{S^2}(S\ddot{S} + \dot{S}^2 + \kappa),$$
$$(K^\alpha K^\beta)_{|\alpha\beta} = \ddot{\phi} + 3\phi\ddot{S}/S + 6\phi\dot{S}^2/S^2 + 6\dot{\phi}\dot{S}/S,$$
$$\Box^2\phi = \ddot{\phi} + 3\dot{\phi}\dot{S}/S, \tag{14}$$
$$\phi_{|00} = \ddot{\phi},$$
$$\Box^2(K_0K_0) = \ddot{\phi} + 3\dot{\phi}\dot{S}/S - 6\phi\dot{S}^2/S^2,$$
$$(K^\alpha K_0)_{|0\alpha} = \ddot{\phi} + 3\dot{\phi}\dot{S}/S - 3\phi\dot{S}^2/S^2,$$

where $\phi \equiv K_\mu K^\mu = K_0K^0$, and the dot denotes time differentiation. Substituting these into Eq. (6) gives

$$(2\omega+\eta)\frac{\ddot{S}}{S} + 2\omega\frac{\dot{S}^2+\kappa}{S^2} = 0, \tag{15}$$

and in Eq. (4) there results

$$(1-\omega\phi)\frac{\dot S^2+\kappa}{S^2}+\eta\phi\frac{\dot S^2}{S^2}-(2\omega+\eta)\phi\frac{\ddot S}{S}+(2\omega+\eta)\tfrac{1}{2}\dot\phi\frac{\dot S}{S}=\tfrac{8}{3}\pi G_0T_{00}\,.$$

Using (15) to eliminate second derivatives of S yields

$$(1+\omega\phi)\frac{\dot S^2+\kappa}{S^2}+\eta\phi\frac{\dot S^2}{S^2}+(2\omega+\eta)\tfrac{1}{2}\dot\phi\frac{\dot S}{S}=\tfrac{8}{3}\pi G_0T_{00}\,. \qquad (16)$$

The K_l and g_{0l} equations vanish identically, and the g_{ij} equation is consistent.

Equation (15) integrates to give

$$\dot S^2+\kappa=\left(\frac{S_0}{S}\right)^p, \qquad (17)$$

where $S_0^{\,p}$ is a constant of integration,

$$p\equiv\frac{2\omega}{\omega+\frac{1}{2}\eta},$$

and for future use we define

$$q\equiv\frac{\eta}{\omega+\frac{1}{2}\eta}\,.$$

Equation (17) integrates again, yielding

$$t=\int\frac{S^{p/2}dS}{(S_0^{\,p}-\kappa S^p)^{1/2}}\,. \qquad (18)$$

The time development of S depends only on the scale constant S_0 and on the parameters ω and η; it is independent of $T_{\mu\nu}$.

Equation (16) is made first order in $d\phi/dS$ as follows: Using Eq. (17) to eliminate $\dot S$, and dividing by $\omega+\frac{1}{2}\eta$, Eq. (16) becomes

$$\frac{d\phi}{dS}\frac{1-\kappa x^p}{Sx^p}+(\tfrac{1}{2}p+q)\phi\frac{1-\kappa x^p}{S^2x^p}+\tfrac{1}{2}p\phi\frac{\kappa}{S^2}=\frac{1}{\omega+\frac{1}{2}\eta}\left(\tfrac{8}{3}\pi G_0T_{00}-\frac{1}{S^2x^p}\right),$$

where we have defined $x\equiv S/S_0$. Multiplying by $S_0^{\,2}$ and defining

$$Y=\phi\frac{1-\kappa x^p}{x^{p+1}},$$

$$\mu=\frac{2G_0M}{S_0}=\frac{8\pi G_0T_{00}S^3}{3S_0},$$

produces the first-order equation

$$\frac{dY}{dx}+\frac{Y}{x}\left(1+q+\frac{\frac{3}{2}p}{1-\kappa x^p}\right)=\frac{1}{\omega+\frac{1}{2}\eta}\left(\frac{\mu}{x^3}-\frac{1}{x^{p+2}}\right),$$

with solution

$$\phi=\frac{(1-\kappa x^p)^{1/2}}{(\omega+\frac{1}{2}\eta)x^{q/2+1}}\int\frac{\mu x^{p/2}-x^{q/2}}{(1-\kappa x^p)^{3/2}}dx+\lambda\frac{(1-\kappa x^p)^{1/2}}{x^{q/2+1}}, \qquad (19)$$

where λ is a constant of integration.

The special case $\eta=0$, $\omega=1$ is of particular interest. First, it is the case which exactly reproduces the PPN parameters of general relativity, and it will be shown to have a cosmological solution different from the cosmology of general relativity. Second, this case represents an approximate solution for one of the possibilities allowed by Eq. (13), namely, $\eta<0.1$, $\omega\approx1$. Choosing the $\kappa=1$ of a closed cosmology, and noting $p=2$, $q=0$, we integrate Eq. (18) and obtain

$$S=[t(2S_0-t)]^{1/2}, \qquad (20)$$

and Eq. (19) yields

$$1+\phi=\frac{\mu}{x}-\lambda\frac{(1-x^2)^{1/2}}{x}\,.$$

If we demand $\phi\to0$ as $t\to0$ (the initial cosmological singularity; see Dicke[1]), then for $\phi\gg1$ (equivalent to $G_0\gg G$),

$$\phi=\frac{2G_0M}{S}[1-(1-x^2)^{1/2}]. \qquad (21)$$

Assuming $S_0\gg S$, these become

$$S\approx(2S_0t)^{1/2}, \qquad (22)$$

$$\phi\approx\frac{G_0MS}{S_0^{\,2}}\,. \qquad (23)$$

The relationship between Hubble time (T_H) and the present age of the universe t_u is

$$\frac{1}{H}\equiv T_H\equiv\frac{S}{\dot S}\approx2t_u\,. \qquad (24)$$

Writing

$$M\equiv\tfrac{4}{3}\pi\rho S^3=\tfrac{4}{3}\pi\rho(S_0T_H)^{3/2}$$

and substituting gives ϕ/G_0 in terms of observables:

$$\frac{\phi}{G_0}=\tfrac{4}{3}\pi\rho T_H^{\,2}\,.$$

Using this in Eq. (8) gives

$$\frac{1}{G}=\tfrac{2}{3}\pi\rho\eta T_H^{\,2}\sim10^4\text{ (cgs units)}, \qquad (25)$$

based on $\rho\approx10^{-31}$ g/cm^3 and $T_H\approx2\times10^{10}$ years. The actual value of $1/G$ is about 10^7, but there is substantial uncertainty in the total energy density

(ρ) of the universe. From Eq. (22) the early-time behavior of the universe is

$$S \sim t^{1/2},$$

as compared with the relationship from general relativity,

$$S \sim t^{2/3}.$$

A second limiting cosmology, compatible with the second case allowed by Eq. (13), is $\omega \approx \frac{1}{2}\eta \gg 1$. Then $p = q = 1$, and for $\kappa = 1$ we find

$$\frac{t}{S_0} = \sin^{-1}(x)^{1/2} - (x - x^2)^{1/2}, \quad (26)$$

$$\phi = 2\frac{(\mu - 1)(1-x)^{1/2}}{\eta x}\left[\frac{1}{(1-x)^{1/2}} - \frac{\sin^{-1}(x)^{1/2}}{(x)^{1/2}}\right], \quad (27)$$

where we have again assumed $s \to 0$, $\phi \to 0$ as $t \to 0$. The early-time behavior is

$$t = \tfrac{2}{3} S_0 x^{3/2} \quad (28)$$

or $S \sim t^{2/3}$, and

$$\frac{\phi}{G_0} = \frac{8MS}{15\eta S_0^2}.$$

From Eq. (28), the age of the universe is related to H by

$$H \equiv \frac{\dot{S}}{S} = \frac{2}{3t_u};$$

also,

$$M = \tfrac{4}{3}\pi\rho S_0 T_H^2,$$

therefore

$$\frac{\phi}{G_0} = \frac{32\pi}{45\eta}\rho T_H^2\left(\frac{T_H}{S_0}\right)^{2/3}.$$

Finally, using this in Eq. (8) gives

$$\frac{1}{G} = \tfrac{32}{90}\pi\rho T_H^2\left(\frac{T_H}{S_0}\right)^{2/3} \sim 10^4,$$

if $T_H/S_0 \sim 0.1$.

A final choice of parameters, compatible with Case II, is $\omega = 0$. In this case, Eq. (15) reduces to

$$\eta\frac{\ddot{S}}{S} = 0,$$

having the solution

$$S = vt,$$

where v is a constant. Equation (16) becomes

$$\frac{\dot{S}^2 + \kappa}{S^2} + \eta\phi\frac{\dot{S}^2}{S^2} + \tfrac{1}{2}\eta\dot{\phi}\frac{\dot{S}}{S} = \tfrac{8}{3}\pi G_0 T_{00},$$

which has the solution

$$\phi = \frac{2\lambda}{\eta v^3}t^{-2} + \frac{4G_0 M}{\eta v^3}t^{-1} + \frac{v^2 + \kappa}{\eta v^2},$$

and is singular at the origin.

IV. CONCLUSION

We have presented a theory of gravity in which gravity is produced by a vector field in addition to the usual metric field. It has been found that such a theory is compatible with present solar-system experiments and contains a reasonable cosmology. Future papers will examine gravitational radiation in this theory, and examine the possibility of identifying the vector field (K_μ) with the Maxwell field of electrodynamics (A_μ).

One of the authors (RH) would like to thank Cliff Will, Robert Dicke, and Bill Kinnersley for helpful discussions.

APPENDIX

The calculation of the PPN metric parameters consists of three parts. First, the solution of the dynamic (moving sources) linearized equations is made to get all potentials first order in M. Second, the static field equations are solved to second order in M, giving β and δ. Third, a method is devised to get the two-source metric potential with *PPN* coefficient involving ζ_2.

Using the PPN metric for a single source [Eq. (2)], and the expansion of K_μ [Eq. (3)], one can calculate the following derivatives to linear order in M:

$$K_0 K_{0|00} = \phi_0 \frac{GM}{r^3}(1 + a_1)\left[3\frac{(\vec{r}\cdot\vec{v})^2}{r^2} - v^2 + \vec{r}\cdot\vec{a}\right],$$

$$K_0 K_{0|0k} = K_0 K_{0|k0} = -\phi_0 \frac{GM}{r^3}(1 + a_1)\left(3\frac{\vec{r}\cdot\vec{v}}{r^2}x^k - v^k\right),$$

$$K_0 K_{0|kk} = \phi_0 \frac{GM}{r^3}(\zeta_1 - 2a_3)\left[3\frac{(\vec{r}\cdot\vec{v})^2}{r^2} - v^2\right] - 2\phi_0\frac{GM}{r^3}f\vec{r}\cdot\vec{a} - 2\pi\phi_0 GM\delta(\vec{r})[2a_1 + 2a_2 v^2 + 2 + v^2(2\gamma + 1 + \alpha_3 + \zeta_1)],$$

$$K_0K_{k|00} = -\phi_0 \frac{GM}{r^3}\left[3\frac{\vec{r}\cdot\vec{v}}{r^2}x^k - v^k\right],$$

$$K_0K_{k|0k} = \phi_0 \frac{GM}{r^3}(\zeta_1 + d' - d)\left[3\frac{(\vec{r}\cdot\vec{v})^2}{r^2} - v^2\right] + \phi_0\frac{GM}{r^3}(d'-d)\vec{r}\cdot\vec{a} - 2\pi\phi_0 GM\delta(\vec{r})[2 + v^2(2\gamma + 1 + \alpha_3 + \zeta_1)],$$

$$K_0K_{k|k0} = \phi_0\frac{GM}{r^3}(d' - d + \tfrac{1}{2}\alpha_1 - \alpha_2 + 1 + \zeta_1 - \gamma)\left[3\frac{(\vec{r}\cdot\vec{v})^2}{r^2} - v^2 + \vec{r}\cdot\vec{a}\right],$$

$$K_0K_{k|ll} - K_0K_{l|kl} = -\phi_0\frac{GM}{r^3}(d' + d)\left(3\frac{\vec{r}\cdot\vec{v}}{r^2}x^k - v^k\right) - 4\pi GM\phi_0\delta(\vec{r})dv^k.$$

Components of $R_{\mu\nu}$ are

$$R_{00} = \frac{GM}{r^3}(\gamma - 1 + \alpha_2 - \tfrac{1}{2}\alpha_1)\left[3\frac{(\vec{r}\cdot\vec{v})^2}{r^2} - v^2\right] + \frac{GM}{r^3}(\gamma - 1 + \alpha_2 - \tfrac{1}{2}\alpha_1 - \zeta_1)\vec{r}\cdot\vec{a}$$
$$- 2\pi GM\delta(\vec{r})[2 + v^2(1 + 2\gamma + \alpha_3 + \zeta_1)],$$

$$R_{0k} = \frac{GM}{r^3}(1 - \gamma + \tfrac{1}{4}\alpha_1)\left(3\frac{\vec{r}\cdot\vec{v}}{r^2}x^k - v^k\right) + \pi GM\delta(\vec{r})(4\gamma + 3 + \alpha_1 - \alpha_2 + \zeta_1)v^k.$$

We will need the δ-function part of R to static order only,

$$R = 2R_{00} + 16\pi GM\gamma\delta(r).$$

Next we write the first-order approximations to the field equations. First the g_{00} equation, (4):

$$(R_{00} - \tfrac{1}{2}R)(1 - \omega\phi_0) + (2\omega + \eta)(\phi_0 R_{00} + K_0K^0{}_{|ll}) = -8\pi G_0 T_{00}; \quad \text{(A1)}$$

next the g equation, (5):

$$R + (6\omega + 3\eta)K_0K^0{}_{|00} - (6\omega + \eta)K_0K^0{}_{|ll} - \eta K_0(K_{l|l0} + K_{l|0l}) = 8\pi G_0 T; \quad \text{(A2)}$$

and the K_0 equation, (6):

$$\omega K_0 R + \eta K_0 R_{00} = 2(K_{0|ll} - K_{l|0l}). \quad \text{(A3)}$$

It is profitable to eliminate R in the above three equations. Adding $\frac{1}{2}$ of (A2) to (A1) and subtracting $\frac{1}{2}K_0$ times (A3) gives

$$R_{00}(1 + \omega\phi_0 + \tfrac{1}{2}\eta\phi_0) + K_0K_{0|ll}(1 - \omega + \tfrac{1}{2}\eta) + K_0K_{0|00}(3\omega + \tfrac{3}{2}\eta) - \tfrac{1}{2}\eta K_0K_{l|l0} - (1 + \tfrac{1}{2}\eta)K_0K_{l|0l} = -8\pi G_0 T_{00} + 4\pi G_0 T. \quad \text{(A4)}$$

Keeping only the δ-function parts of each equation and setting v to zero, the first-order static equations can be written. Thus (A1), (A3), and (A4) become, respectively,

$$(1 - \omega\phi_0)2GM\gamma\delta(r) + (2\omega + \eta)(a_1 + 2)\phi_0 GM\delta(r) = 2G_0T_{00}, \quad \text{(A5)}$$

$$(2a_1 + 4\omega\gamma - 2\omega - \eta)\phi GM\delta(r) = 0, \quad \text{(A6)}$$

$$(1 + \omega\phi_0 + \tfrac{1}{2}\eta\phi_0)GM\delta(r) + (1 - \omega + \tfrac{1}{2}\eta)(a_1 + 1)\phi_0 GM\delta(r) - (\tfrac{1}{2}\eta + 1)\phi_0 GM\delta(r) = G_0(2T_{00} - T). \quad \text{(A7)}$$

Using some relations valid in Minkowski space,

$$\int T_{00}d^3x = \int\rho\left(\frac{dt}{ds}\right)^2 d^3x = \int\rho\frac{dt}{ds}\frac{d^3x}{(1 - v^2)^{1/2}} = \frac{M}{(1 - v^2)^{1/2}}, \quad \text{(A8)}$$

$$\int T d^3x = \int\rho\frac{dt}{ds}\frac{ds}{dt}d^3x = \int\rho\frac{dt}{ds}(1 - v^2)^{1/2}d^3x = M(1 - v^2)^{1/2}, \quad \text{(A9)}$$

Eqs. (A5)–(A7) integrate (with $v = 0$) to:

$$2\gamma(1 - \omega\phi_0) + (2\omega + \eta)\phi_0 a_1 - 2G_0/G = -2(2\omega + \eta)\phi_0, \quad \text{(A10)}$$

$$2\gamma(4\omega) + 4a_1 = 2(2\omega + \eta), \quad \text{(A11)}$$

$$(2 - 2\omega + \eta)\phi_0 a_1 - 2G_0/G = -2(1 + \tfrac{1}{2}\eta\phi_0). \quad \text{(A12)}$$

These can be solved simultaneously to give

$$\gamma = \frac{1 - \omega\phi_0(2\omega + \eta + 1)}{1 + \omega\phi_0(1 - 4\omega)} .$$

Experimental evidence indicates $\gamma \approx 1$, so we proceed with the special case

$$\omega = \tfrac{1}{2}\eta + 1 . \quad \text{(A13)}$$

(The procedure for the alternative possibility, $\omega = 0$, is similar and will not be given here.) Using (A13) in (A11) gives

$$a_1 = -1 .$$

Keeping the v^2 part of the δ-function terms and using (A13), (A4) integrates to

$$(1 + \tfrac{1}{2}\eta\phi_0)[2 + v^2(1 + 2\gamma + \alpha_3 + \zeta_1)] = 2\frac{G_0}{G}\left[\frac{2}{(1 - v^2)^{1/2}} - (1 - v^2)^{1/2}\right],$$

which gives

$$G = \frac{G_0}{1 + \frac{1}{2}\eta\phi_0} \quad \text{(A14)}$$

as a renormalization of G_0, and the relationship

$$1 + 2\gamma + \alpha_3 + \zeta_1 = 3 . \quad \text{(A15)}$$

Having handled the localized source terms, we next find solutions to the total dynamic field equations outside the source. First the g_{00} equation, (A1):

$$(2\omega + \eta)\phi_0 \frac{GM}{r^3}\left[\left(3\frac{(\vec{r}\cdot\vec{v})^2}{r^2} - v^2\right)(2\zeta_1 + 3\gamma + \alpha_2 - \tfrac{1}{2}\alpha_1 - 1 - 2\gamma - \zeta_1 - 2a_3) + \vec{r}\cdot\vec{a}(3\gamma + \alpha_2 - \tfrac{1}{2}\alpha_1 - 1 - 2\gamma - \zeta_1 - 2f)\right] = 0 . \quad \text{(A16)}$$

This gives two requirements,

$$\begin{aligned} &\zeta_1 + \alpha_2 - \tfrac{1}{2}\alpha_1 - 2a_3 = 0 , \\ &-\zeta_1 + \alpha_2 - \tfrac{1}{2}\alpha_1 - 2f = 0 , \end{aligned} \quad \text{(A17)}$$

where we have used $\gamma = 1$. Subtracting gives

$$\zeta_1 = a_3 - f . \quad \text{(A18)}$$

Equation (A3) becomes

$$(2\omega + \eta)\phi_0 \frac{GM}{r^3}\left[\left(3\frac{(\vec{r}\cdot\vec{v})^2}{r^2} - v^2\right)(\alpha_2 - \tfrac{1}{2}\alpha_1) + \vec{r}\cdot\vec{a}(\alpha_2 - \tfrac{1}{2}\alpha_1 - \zeta_1)\right] + 2\phi_0 \frac{GM}{r^3}\left[\left(3\frac{(\vec{r}\cdot\vec{v})^2}{r^2} - v^2\right)(2a_3 - d + d') + \vec{r}\cdot\vec{a}(2f - d + d')\right] = 0 , \quad \text{(A19)}$$

giving

$$\begin{aligned} &(2\omega + \eta)(\alpha_2 - \tfrac{1}{2}\alpha_1) = 2(d - d' - 2a_3) , \\ &(2\omega + \eta)(\alpha_2 - \tfrac{1}{2}\alpha_1 - \zeta_1) = 2(d - d' - 2f) . \end{aligned} \quad \text{(A20)}$$

Subtracting and using (A18) yields

$$(2\omega + \eta + 4)\zeta_1 = 0 .$$

So

$$\zeta_1 = 0 , \quad \text{(A21)}$$

unless $2\omega + \eta + 4 = 0$. This, however, combined with $\omega = \frac{1}{2}\eta + 1$, gives unique values for ω and η which, when used in Eq. (A4), give $\zeta_1 = 0$, anyway. Equation (A18) yields the additional result

$$a_3 = f . \quad \text{(A22)}$$

In Eq. (A16) it was assumed that $2\omega + \eta \neq 0$. In the special case $2\omega + \eta = 0$, Eq. (A19) gives $a_3 = f$. Using this in (A4) results again in $\zeta_1 = 0$. Therefore the same PPN parameters are obtained, regardless of special relationships between ω and η. Substituting $\zeta_1 = 0$ and $\gamma = 1$ in Eq. (A15) gives the additional result

$$\alpha_3 = 0 .$$

Using $\zeta_1 = 0$ and $a_3 = f$, (A17), (A20), and (A4) become

$$\begin{aligned} &\alpha_2 - \tfrac{1}{2}\alpha_1 = 2f , \\ &(\eta + 1)(\alpha_2 - \tfrac{1}{2}\alpha_1) = d' - d - 2f , \\ &(1 + \phi_0 + \tfrac{3}{2}\eta\phi_0)(\alpha_2 - \tfrac{1}{2}\alpha_1) = (1 + \eta)\phi_0(d' - d) . \end{aligned}$$

Solving these simultaneously gives

$$2f=\alpha_2-\tfrac{1}{2}\alpha_1=d'-d=0. \qquad \text{(A23)}$$

To get the individual values of α_1 and α_2, the K_l and g_{0l} equations are needed. Equation (6) gives

$$\eta K_0 R_{0l}=2(K_{l|mm}-K_{m|lm}-K_{l|00}),$$

or

$$\frac{GM}{r^3}\left[3\frac{\vec{r}\cdot\vec{v}}{r^2}x^l-v^l\right]\left[\tfrac{1}{4}\eta\alpha_1+2(d+d'-1)\right]=0.$$

Equation (4) becomes

$$(1+\omega\phi_0+\eta\phi_0)R_{0l}+(2\omega+\eta)K_0K^0{}_{|0l}+\tfrac{1}{2}\eta K_0(K_{l|mm}-K_{m|lm})=0,$$

yielding

$$\frac{GM}{r^3}\left[3\frac{\vec{r}\cdot\vec{v}}{r^2}x^l-v^l\right]\times\left[(1+\omega\phi_0+\eta\phi_0)(\tfrac{1}{4}\alpha_1)-\tfrac{1}{2}\eta\phi_0(d+d')\right]=0.$$

Solving for α_1 and using $\omega=\frac{1}{2}\eta+1$ yields

$$\alpha_2=\tfrac{1}{2}\alpha_1=\frac{4\eta}{4\phi_0{}^{-1}+4+6\eta+\eta^2}; \qquad \text{(A24)}$$

also,

$$d=d'=\tfrac{1}{2}\,\frac{4\phi_0{}^{-1}+4+6\eta}{4\phi_0{}^{-1}+4+6\eta+\eta^2}.$$

The only *PPN* parameters remaining involve second-order static terms in the metric. The static equations to second order are:

g_{00},

$$R_{00}(1+\omega\phi_0+\eta\phi_0)-\tfrac{1}{2}R(1-\omega\phi_0)-(2\omega+\eta)\tfrac{1}{2}(K_0K_0)_{|lm}g^{lm}-F_{0l}F_{0m}g^{lm}=0; \qquad \text{(A25)}$$

g,

$$R+(6\omega+3\eta)K^0K^0{}_{|00}+(6\omega+\eta)\tfrac{1}{2}(K_0K^0)_{|lm}g^{lm}+(6\omega+\eta)K_{l|0}K_{m|0}g^{lm}+2\eta K_{0|l}K_{m|0}g^{lm}+2\eta K^0K_{l|0m}g^{lm}+\eta R_{00}K^0K^0=0; \qquad \text{(A26)}$$

and K_0 (multiplied by K_0),

$$\omega\phi_0R+\eta\phi_0R_{00}+(K_0K_0)_{|lm}g^{lm}-2K_{0|l}K_{0|m}g^{lm}-2K_0K_{l|0m}g^{lm}=0. \qquad \text{(A27)}$$

Adding $\frac{1}{2}$ of (A26) to (A25), and subtracting $\frac{1}{2}$ of (A27) and using $\omega=\frac{1}{2}\eta+1$, gives

$$R_{00}(1+\tfrac{3}{2}\eta\phi_0+\phi_0)+(3\eta+3)K_0K^0{}_{|00}-(2\eta+2)K_{l|0}K_{l|0}+(\eta+1)K_0K^l{}_{|0l}=0,$$

where we have also used the fact that to first order the only nonzero derivative of K_μ is $K_{l|0}$. To second order,

$$R_{00}=(2-2\beta)\frac{G^2M^2}{r^4},$$

$$K_0K^0{}_{|00}=K_{l|0}K_{l|0}=\frac{G^2M^2}{r^4},$$

$$K_0K^l{}_{|0l}=(2\beta-3)\frac{G^2M^2}{r^4}.$$

Using these gives the relation

$$(2-2\beta)(1+\tfrac{1}{2}\eta\phi_0)=0,$$

or $\beta=1$, as in general relativity. To get δ, which only appears in R, we solve Eqs. (A25) and (A27) for R. Multiplying (A27) by $\omega+\frac{1}{2}\eta$ and adding to (A25) gives

$$\tfrac{1}{2}R[-1+\omega\phi_0(2\omega+\eta+1)]-F_{0l}F_{0m}g^{lm}-(2\omega+\eta)K_0K_{l|0m}g^{lm}=0,$$

where we have used $R_{00}=0$ when $\beta=1$. Solving and using $\eta=2\omega-2$,

$$\tfrac{1}{2}R=\frac{(4\omega-1)\phi_0G^2M^2/r^4}{1-\omega\phi_0(4\omega-1)}.$$

Now

$$R=(-4\beta-4\delta+8)G^2M^2/r^4$$
$$=(4-4\delta)G^2M^2/r^4,$$

so

$$\delta=1+\frac{1-4\omega}{2\phi_0{}^{-1}+2\omega(1-4\omega)}.$$

The ζ_2 parameter appearing in the two-mass interaction term can be found by means of a convenient trick. The configuration of a point mass m inside a spherical shell of mass M and radius $R \gg r$ is solved by two approaches, and the results are compared. First, M is included as a source of the metric, and we write

$$g_{00} = 1 - 2G\frac{M}{R} - 2G\frac{m}{r} + 4\beta G^2\frac{mM}{rR} - (2 - 4\beta + 2\zeta_2)G^2\frac{mM}{rR} + O(m^2, M^2), \quad \text{(A28)}$$

$$g_{ss} = -1 - 2\gamma G\frac{M}{R} - 2\gamma G\frac{m}{r}.$$

This has the asymptotic limit (as $m/r \to 0$ but, always, $R \gg r$):

$$g_{00} = 1 - 2G\frac{M}{R},$$

$$g_{ss} = -1 - 2\gamma G\frac{M}{R}. \quad \text{(A29)}$$

Second, we note that the post-Newtonian limit is valid inside the shell, so one must be able to write to linear order

$$g'_{00} = 1 - 2G^* m/r',$$

$$g'_{ss} = -1 - 2\gamma G^* m/r', \quad \text{(A30)}$$

where it is recognized that the presence of the mass shell may affect G, and that the coordinates will be different to allow a Minkowskian asymptotic limit. Comparison of (A29) and (A30) shows that the coordinate transformation must be

$$\frac{\partial x^0}{\partial x^{0\prime}} = 1 + \frac{GM}{R},$$

$$\frac{\partial r}{\partial r'} = 1 - \gamma\frac{GM}{R}.$$

Applying this to (A28) gives

$$g_{00}{}' = 1 - 2G\frac{m}{r} + (8\beta - 6 - 2\zeta_2)G^2\frac{mM}{rR};$$

but $r = (1 - \gamma GM/R)r'$, so

$$g_{00}{}' = 1 - 2G\frac{m}{r'} + (8\beta - 6 - 2\gamma - 2\zeta_2)G^2\frac{mM}{r'R}.$$

Comparison with (A30) shows the effect of M on G:

$$G^* = G\left[1 - (4\beta - 3 - \gamma - \zeta_2)\frac{GM}{R}\right].$$

However, the effect of M on G is well known from previous analysis. Equation (A14) gave

$$G^* = \frac{G_0}{1 + \frac{1}{2}\eta g^{00}K_0K_0}$$
$$= G_0[1 + \tfrac{1}{2}\eta(1 + 2M/R)\phi_0(1 - M/R)(1 - M/R)]^{-1},$$
$$= \frac{G_0}{1 + \frac{1}{2}\eta\phi_0}$$
$$= G.$$

Therefore $4\beta - 3 - \gamma - \zeta_2 = 0$ and, since $\beta = \gamma = 1$, this implies

$$\zeta_2 = 0.$$

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*Work supported in part by the National Aeronautics and Space Administration under Grant No. NGR27-001-035.

†NDEA Title IV Graduate Fellow.

‡Alfred P. Sloan Foundation Research Fellow.

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[2]In all that follows, we will use a subscript comma for ordinary partial differentiation and a subscript vertical bar for covariant differentiation. Greek indices will run from 0 to 3 and Latin indices will run from 1 to 3. All sign conventions in $g_{\mu\nu}$, R, etc., follow R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity* (McGraw-Hill, New York, 1965).

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[8]K. Nordtvedt, Jr. and C. M. Will, Astrophys J. 177, 775 (1972).

[9]The original value of $\alpha_2 < 0.03$ given by Nordtvedt and Will (Ref. 8) probably represents an overly optimistic evaluation of gravimeter results. See K. Nordtvedt, Jr., Science 178, 1157 (1972).