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Radiating Kerr Metric

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A nonstatic solution of Einstein's equations corresponding to a field of flowing radiation is presented. The solution is algebraically special and contains two shear-free geodetic null congruences. It has the symmetry of the Kerr solution and when a certain parameter is put equal to zero the solution becomes static and reduces to the Kerr solution.

I. INTRODUCTION

The Kerr metric[1] describes a possible exterior field outside a rotating body. It is the only known example of a stationary vacuum metric with gravitating mass and rotation that is asymptotically flat. Like Schwarzschild's solution it is algebraically special. A nonstatic generalization of Schwarzschild's solution describing the gravitational field of a radiating star is well known.[2] The problem of finding a similar nonstatic generalization of the Kerr solution has recently attracted the attention of several workers. Misra[3] posed the following question. Can we obtain a nonstatic generalization of the Kerr solution which would correspond to the gravitational field of a rotating radiating body? As a first step towards finding such a nonstatic generalization he presented the Kerr metric and the radiating star metric in a unified treatment based on the Kerr-Schild form[4] of metric. Murenbeeld and Trollope[5] have used an approximation procedure and worked out a metric corresponding to slowly rotating radiating spheres.

Hughston[6] has obtained criteria under which one can construct a metric compatible with the energy tensor of a null radiation field from an algebraically special vacuum metric. The new metric bears the same relation to the original metric as does the radiating star metric to the Schwarzschild metric. As an example he has obtained a class of null radiation fields from a class of vacuum metrics without symmetry discovered by Robinson and Robinson.[7] However, the Kerr metric does not satisfy his criteria and so it is not possible to use his scheme for constructing a null radiation field corresponding to the Kerr metric.

In this paper we present a nonstatic solution of Einstein's equations corresponding to the field of flowing radiation. This solution is algebraically special and contains two shear-free geodetic null congruences. It has the symmetry of the Kerr solution and when a certain parameter is put equal to zero, the solution becomes stationary and reduces to the Kerr solution.

II. GEOMETRICAL PRELIMINARIES

We shall use a geometrical scheme developed by one of us[8] earlier to obtain a general shear-free null congruence in Minkowskian space-time and then use that congruence to construct a Kerr-Schild-type metric. This geometrical scheme is briefly described below. In a Minkowskian space-time we can find four uniform vector fields such that (i) any two of them are mutually orthogonal and (ii) one of them is timelike and the other three spacelike. Let λ_i be the unit tangent to the timelike vector passing through a point P (coordinates x^i) and A^i, B^i, and C^i be the unit tangents to the spacelike vectors. We use the signature $(-,-,-,+)$ and raise and lower indices with the help of the metric tensor η_{ik} or η^{ik} of Minkowskian space-time. These four uniform vector fields give rise to a Euclidean reference frame with coordinates x, y, z, t for P where, $x = x^i A_i$, $y = x^i B_i$, $z = x^i C_i$, and $t = x^i \lambda_i$ so that $x_{,i} = A_i$, $y_{,i} = B_i$, $z_{,i} = C_i$, and $t_{,i} = \lambda_i$, a comma indicating an ordinary derivative.

If l^i is any spacelike unit vector in the flat 3-space Π at right angles to λ^i at the point P, then

$$\xi^i = \lambda^i - l^i \tag{2.1}$$

is a null vector defining a null congruence in Minkowskian space-time.

In the flat 3-space Π at the point P, let α and β be the spherical angles of the direction l^i with reference to the triad A^i, B^i, C^i at P. Then we can define three new spacelike unit vectors $\bar{l}_i$, $\overline{m}_i$, and m_i by the following relations (see Fig. 1):

$$m^i = \cos\beta A^i + \sin\beta B^i ,$$

$$l^i = \sin\alpha\, m^i + \cos\alpha\, C^i ,$$

$$\bar{l}^i = \cos\alpha\, m^i - \sin\alpha\, C^i ,$$

$$\overline{m}^i = -\sin\beta A^i + \cos\beta B^i .$$

We take α and β as functions of the coordinates x^i of the point P. It is easy to find the derivatives of these vectors:

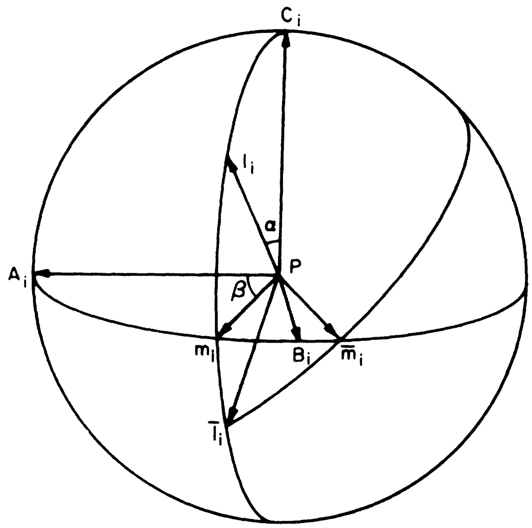

FIG. 1. The directions l^i, $\bar{l}^i$, m^i and $\bar{m}^i$ in the flat 3-space Π.

$$l^i{}_{,k} = \bar{l}^i \alpha_{,k} + \bar{m}^i \sin\alpha\, \beta_{,k} ,$$

$$m^i{}_{,k} = \bar{m}^i \beta_{,k} ,$$

$$\bar{l}^i{}_{,k} = -l^i \alpha_{,k} + \bar{m}^i \cot\alpha \sin\alpha\, \beta_{,k} ,$$

$$\bar{m}^i{}_{,k} = -m^i \beta_{,k} .$$

Now the conditions for the geodetic and shear-free null congruence ξ^i are

$$\xi^i{}_{,k}\, \xi^k = 0$$

and

$$(\xi_{i,k} + \xi_{k,i})\xi^i{}_{,l}\eta^{kl} - (\xi^i{}_{,i})^2 = 0 .$$

With ξ^i given by Eq. (2.1) these conditions after some simplification will lead to

$$\alpha_{,k}\, \xi^k = 0 , \quad \beta_{,k}\, \xi^k = 0 \tag{2.2}$$

and

$$\bar{m}^i \alpha_{,i} + \bar{l}^i \sin\alpha\, \beta_{,i} = 0 , \tag{2.3}$$

$$\bar{l}^i \alpha_{,i} - \bar{m}^i \sin\alpha\, \beta_{,i} = 0 . \tag{2.4}$$

It can be verified that if

$$u = t - x \sin\alpha \cos\beta - y \sin\alpha \sin\beta - z \cos\alpha ,$$

$$v = x \cos\alpha \cos\beta + y \cos\alpha \sin\beta - z \sin\alpha ,$$

$$w = x \sin\beta - y \cos\beta ,$$

then $u_{,i}\xi^i = 0$, $v_{,i}\xi^i = 0$, and $w_{,i}\xi^i = 0$. Therefore, the conditions (2.2) for a geodetic null congruence can be integrated and exhibited in the form

$$w = w(u, \alpha, \beta) , \quad v = v(u, \alpha, \beta) ,$$

w and v being undetermined functions of the arguments.

The conditions (2.3) and (2.4) for the vanishing shear will now lead to two partial differential equations for the functions v and w, which are being systematically studied in a series of papers.[8] For our purposes we note that if

$$v = \tfrac{1}{2} b \sin\alpha \cos\alpha , \quad v^2 + w^2 = (bu + a^2)\sin^2\alpha , \tag{2.5}$$

the conditions (2.3) and (2.4) will be satisfied, a and b being constants of integration. The relations (2.5) are equivalent to

$$x \cos\beta + y \sin\beta = z \tan\alpha + \tfrac{1}{2} b \sin\alpha \tag{2.6}$$

and

$$x \sin\beta - y \cos\beta = aU \sin\alpha , \tag{2.7}$$

where

$$a^2 U^2 = bu + a^2 - \tfrac{1}{4} b^2 \cos^2\alpha . \tag{2.8}$$

The conclusion that we reach now is that

$$\xi^i = \lambda^i - \sin\alpha \cos\beta A^i - \sin\alpha \sin\beta B^i - \cos\alpha\, C^i \tag{2.9}$$

is a tangent vector at the point $P(x, y, z, t)$ of a shear-free geodetic null congruence if α and β are given as functions of x, y, z, t by Eqs. (2.6) and (2.7). The expansion $\theta = \xi^i{}_{,i}$ of the null congruence is given by

$$\theta = 2(z \sec\alpha + \tfrac{1}{2} b \cos^2\alpha)\rho^{-2} ,$$

where

$$\rho^2 = (z \sec\alpha + \tfrac{1}{2} b \cos^2\alpha)^2 + a^2 U^2 \cos^2\alpha .$$

It may be noted that when b is put equal to zero Eq. (2.5) will give $v = 0$ and $w = a \sin\alpha$ which define the shear-free null congruence of the Kerr solution.

III. THE METRIC

Consider a Riemannian space-time described by the metric

$$g_{ik} = \eta_{ik} + H \xi_i \xi_k . \tag{3.1}$$

H is a function of the coordinates and ξ_i is a geodetic null congruence. The form of the Christoffel symbols and R_{ik} for this general metric are given in Appendix A.

We now use ξ^i given by Eq. (2.9) above and following Mas.[9] We take

$$H = M \xi^i{}_{,i} = M\theta = 2M(z \sec\alpha + \tfrac{1}{2} b^2 \cos^2\alpha)\rho^{-2} \tag{3.2}$$

with $M = M(x^i)$ such that $M_{,i}\xi^i = 0$. We shall then find that

$$R_{ik}\, \xi^i = 0 , \quad R_{ik}\, \bar{l}^i = \mu \xi_k , \quad R_{ik}\, \bar{m}^i = \nu \xi_k .$$

The forms of the scalars μ and ν are recorded in Appendix B. In order to get $R_{ik} = \sigma \xi_i \xi_k$, we must take $\mu=\nu=0$ which leads to the following two differential equations of M:

$$\frac{\partial M}{\partial \alpha}=\tfrac{1}{2}b \sin\alpha \cos\alpha \frac{\partial M}{\partial u}\ , \tag{3.3}$$

$$(\tfrac{3}{2})bM+(bu+a^2-\tfrac{1}{4}b^2\cos^2\alpha)\frac{\partial M}{\partial u}=0\ . \tag{3.4}$$

From Eqs. (3.3) and (3.4) it follows that if $b=0$ then either $a=0$ or $\partial M/\partial u=0$. If one takes the first alternative, viz., $b=0$, $a=0$, one finds that Eqs. (3.3) and (3.4) leave M as an undetermined function of u and we get the metric (3.2) as representing the spherically symmetric field of a radiating star.[2] In the second alternative, viz., $b=0$, $\partial M/\partial u=0$, and $a\neq 0$, we recover the Kerr metric.[1] Thus when $a\neq 0$, $b\neq 0$, and $\partial M/\partial u\neq 0$, we get the case corresponding to a radiating Kerr metric with

$$M=-mU^{-3}$$
$$=-m[1+u(b/a^2)-(b^2/4a^2)\cos^2\alpha]^{-3/2}\ ,$$

where m is a constant. With this choice for M, we ultimately find that

$$R_{ik}=-\frac{3mb}{a^2U^5}\xi_i\xi_k\ . \tag{3.5}$$

IV. EXPLICIT FORM OF THE METRIC

The explicit form of function H for the radiating Kerr metric can be written in the form

$$H=\frac{-2mU^{-3}(r+\tfrac{1}{2}b\cos^2\alpha)r^2}{[r^2(r+\tfrac{1}{2}b\cos^2\alpha)^2+a^2U^2z^2]^2}\ , \tag{4.1}$$

where $r=z\sec\alpha$.

The explicit form of the radiating Kerr metric in Euclidean coordinates (x, y, z, t) will be

$$ds^2=dt^2-dx^2-dy^2-dz^2-\frac{2mU^{-3}r^2(r+\tfrac{1}{2}b\cos^2\alpha)^2}{r^2(r+\tfrac{1}{2}b\cos^2\alpha)^2+a^2U^2z^2}\left(dt-\frac{z}{r}dz-\frac{(r+\tfrac{1}{2}b)(xdx+ydy)}{a^2U^2+(r+\tfrac{1}{2}b)^2}-\frac{aU(xdy-ydx)}{a^2U^2+(r+\tfrac{1}{2}b)^2}\right)^2 \tag{4.2}$$

and

$$R_{ik}=-\frac{3mb}{a^2U^5}\xi_i\xi_k\ . \tag{4.3}$$

When b is put equal to zero, R_{ik} becomes zero and the metric (4.2) becomes the familiar Kerr metric.

APPENDIX A

We give here the form of the Christoffel symbols and R_{ik} for the metric given by Eq. (3.1):

$$\begin{Bmatrix} n \\ i\ \ k \end{Bmatrix}=\xi^n(H_{,(i}\xi_{k)}+H\xi_{(i,k)}+\tfrac{1}{2}hH\xi_i\xi_k+H^2\xi_{(i}\xi_{k),a}\xi^a)$$
$$+H\xi^n{}_{,(i}\xi_{k)}-\eta^{nl}(H\xi_{(i}\xi_{k),l}+\tfrac{1}{2}H_{,l}\xi_i\xi_k)\ , \tag{A1}$$

$$-2R_{ik}=2(H\theta+h)_{,(i}\xi_{k)}+2(H\theta+h)\xi_{(i,k)}-2\eta^{ab}H_{,b}(\xi_{i,a}\xi_k+\xi_{k,a}\xi_i)-2H\eta^{ab}\xi_{i,a}\xi_{k,b}$$
$$-H\eta^{ab}(\xi_{i,ab}\xi_k+\xi_{k,ab}\xi_i)-(\eta^{ab}H_{,ab}+H^2\xi^a{}_{,b}\xi^b{}_{,a}-H^2\eta^{ab}\xi_{l,a}\xi^l{}_{,b}-Hh_{,a}\xi^a-Hh\theta)\xi_i\xi_k\ . \tag{A2}$$

In (A1) and (A2) $h=H_{,i}\xi^i$ and $\theta=\xi^i{}_{,i}$.

APPENDIX B

The forms of the scalars μ and ν of Sec. III are given by

$$\mu=-\frac{A}{\rho^4}(M_\alpha-\tfrac{1}{2}bM_u\sin\alpha\cos\alpha)-\frac{\sin\alpha\cos\alpha}{\rho^4}(M_ua^2U^2+\tfrac{3}{2}bM)\ ,$$

$$\nu=-\frac{C}{\rho^4}(M_\alpha-\tfrac{1}{2}bMu\sin\alpha\cos\alpha)-\frac{AC\tan\alpha}{\rho^4}\left(M_u+\frac{3Mb}{2a^2U^2}\right)\ ,$$

where $A=z\sec\alpha-\tfrac{1}{2}b\cos^2\alpha$, $C=w\cot\alpha$, $\rho^2=A^2+C^2$, and $bu+a^2-\tfrac{1}{4}b^2\cos^2\alpha=a^2U^2$.

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Vector-Metric Theory of Gravity*

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A theory in which gravity is produced by a massless vector field in addition to the usual metric field is presented and found to be compatible with present solar-system experiments and cosmological expansion. A special case predicts the same first post-Newtonian gravitational experimental results as general relativity.

I. INTRODUCTION

In 1961, Brans and Dicke[1] proposed a theory in which gravitation was produced by two fields—a tensor metric field and an auxiliary scalar field. The field equations were derived from the Lagrangian action[2]

$$A=\int(-g)^{1/2}\left(16\pi L_m+\phi R+\frac{\omega}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\right)d^4x\,,$$

where $L_m=L_m(g_{\mu\nu}$, matter variables) is the matter Lagrangian, ϕ is the scalar field, and ω is a dimensionless parameter. Variation of the matter variables (position, velocity, etc.) will produce equations of motion involving only $g_{\mu\nu}$, not ϕ. Matter "sees" only the metric field and free test bodies follow geodesics. Theories where matter exhibits this behavior are termed "metric theories."

If L_m contains additional gravitational scalar, vector, or tensor fields, matter will not in general follow geodesics. An analysis by Dicke[3] has shown that the high-precision null experiments—Eötvös experiments, Hughs-Drevor experiment, etc.—rule out the existence of vector or additional second-rank-tensor fields coupling directly to matter. As pointed out by Will and Nordtvedt,[4] however, such vector and tensor fields may exist along with the metric field as long as the additional fields do not couple directly to matter (i.e., do not enter L_m). While the additional fields in these metric theories do not affect the null experiments they will, in general, produce observable effects in light deflection and retardation experiments, planetary perihelion advance, orbiting gyroscope precession, nonsecular terms in planetary and satellite orbits, and geophysical phenomena. Will and Nordtvedt[4] have summarized these observational effects and their relationship to metric theories of gravity, showing that such effects are simply calculable from a parametrized post-Newtonian (PPN) metric, which exists for all metric theories.

In particular, it has been found that some metric theories predict observable effects due to the motion of the solar system relative to a preferred frame (such as the mean rest frame of the universe). These "preferred frame" or "ether" effects can occur in Lagrangian-based theories containing vector or higher-rank-tensor fields (the Brans-Dicke scalar theory exhibits no such effects). In this paper we present a metric theory of gravity containing a massless vector field in addition to the metric field. Committed to the spirit as well as to the law of general covariance in physics, we introduce no *a priori* fields or frames into the theory. We require a Lagrangian subject to the following conditions:

(1) The Lagrangian density is a four-scalar density.

(2) It generates positive-definite free-field energies for both the metric and the vector fields.

(3) It produces a "metric theory."

(4) It generates field equations containing no higher than second derivatives of the fields.

Such a Lagrangian is

$$A=\int(-g)^{1/2}(16\pi G_0L_m+R-F_{\mu\nu}F^{\mu\nu}+\omega K_\mu K^\mu R+\eta K^\mu K^\nu R_{\mu\nu})d^4x\,,\quad(1)$$

where $L_m=L_m(g_{\mu\nu}$, matter variables), $F_{\mu\nu}=K_{\mu|\nu}-K_{\nu|\mu}$ in analogy with electrodynamics, ω and η