Kinematics of K_{l3} Decay Amplitudes and Relativistic Equations for Spin-0 Particles*

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The standard S-matrix kinematic analysis is applied to the K_{l3} decays to show that the form factors $f_\pm(t)$ are free of kinematic singularities or constraints. The general form of the matrix elements of the vector current and its divergence in the Duffin-Kemmer formalism are determined and shown to be consistent with the general kinematic analysis, in contrast to a form proposed in some recent letters.

The invariant matrix element for K_{l3} decays, e.g., $K^- \to \pi^0 + l^- + \bar{\nu}$, is commonly written

$$M = \frac{G\sin\theta_C}{\sqrt{2}}\,\bar{u}(k_1,\lambda_1)(1+\gamma_5)\gamma_\alpha v(k_2,\lambda_2) \times[(p_K+p_\pi)^\alpha f_+(t) + (p_K-p_\pi)^\alpha f_-(t)]\,, \qquad (1)$$

$$t=(p_\pi - p_K)^2\,.$$

In some recent publications,[1-3] the K_{l3} decays have been analyzed using the five-component Duffin-Kemmer wave functions for the spin-0 kaon and pion. Among other results, these authors find that the effective form factors $f_\pm(t)$ should satisfy a kinematic constraint at the scattering threshold $t=(m_K+m_\pi)^2$, which results in the linear combination $f_0(t)=(m_K^2-m_\pi^2)f_+(t)+tf_-(t)$ having a kinematic zero at that point. In this note we point out that the standard kinematic analysis of the analytic S matrix implies that the $f_\pm(t)$ are free of kinematic singularities and constraints. We also determine the general form of the matrix element of the vector current (and its divergence) in the Duffin-Kemmer formalism and find that it differs from that of Refs. 1 and 2 and, in particular, does not imply any kinematic zero in $f_0(t)$.[4]

According to the standard[5] theory of the analytic S matrix, all kinematic singularities and constraints of helicity amplitudes follow explicitly from Lorentz invariance, analyticity, and crossing. We show that all of the kinematic singularities and constraints implied by this analysis are in fact present in (1) for arbitrary $f_\pm(t)$, i.e., these functions are free of any kinematic singularity or constraint. Since the standard theory is developed for no zero-mass particles, we give the neutrino an arbitrary nonzero mass. According to the current-current description of the weak interactions implicit in (1), the properties of the matrix elements of the hadronic current [the $f_\pm(t)$] are independent of the details of the leptonic current (e.g., the neutrino mass) to which it is coupled.[6] Evaluating the Dirac spinor matrix elements of (1) in the lepton center-of-mass system, we find

$$M_{++} = \frac{G\sin\theta_C}{\sqrt{2}}[(E_l+m_l)(E_\nu+m_\nu)]^{1/2}\left(1-\frac{p_l}{E_l+m_l}\right)\left(1+\frac{p_\nu}{E_\nu+m_\nu}\right) \times[(E_K+E_\pi+2p_\pi\cos\theta)f_+(t)+(E_K-E_\pi)f_-(t)],$$
$$M_{-+} = \frac{G\sin\theta_C}{\sqrt{2}}[(E_l+m_l)(E_\nu+m_\nu)]^{1/2}\left(1+\frac{p_l}{E_l+m_l}\right)\left(1+\frac{p_\nu}{E_\nu+m_\nu}\right)[2p_\pi\sin\theta f_+(t)], \qquad (2)$$

$\theta = \sphericalangle(\vec{p}_\pi, \vec{p}_l)$ and $p_l = p_\nu$, $p_\pi = p_K$ in the lepton c.m. system.

We do not write the matrix elements M_{+-}, M_{--} which are proportional to the square of the neutrino mass. We now enumerate the kinematic singularities and constraints in the variable t determined by the standard analysis: (i) no singularity at the threshold or pseudothreshold $t=(m_K\pm m_\pi)^2$, i.e., $p_\pi=0$, because the kaon and pion have no spin; (ii) singularities $[t-(m_l+m_\nu)^2]^{-1/2}$ and $[t-(m_l-m_\nu)^2]^{-1/2}$ at $p_l=0$ threshold and pseudothreshold [these singularities are correctly given by the $\cos\theta$ and $\sin\theta$ factors in (2)]; (iii) zeros on the boundary of the physical region proportional to $(\cos\frac{1}{2}\theta)^{|\lambda_1-\lambda_2|}(\sin\frac{1}{2}\theta)^{|\lambda_1-\lambda_2|}$, i.e., M_{-+} should vanish as $\sin\theta$ in the forward and backward directions, which it does according to (2). Thus the invariant matrix elements (2) contain explicitly all of the kinematic singularities and constraints implied by the general S-matrix analysis, for arbitrary $f_\pm(t)$.[7]

What then is the situation in the Duffin-Kemmer

(D-K) formalism?[8] We work with the real timelike metric $g_{\mu\nu}=(+1,-1,-1,-1)$; then the D-K momentum-space wave function is $u(p)=(1, p^0/m,\ p^1/m,\ p^2/m,\ p^3/m)$, and the relativistic adjoint is $\bar{u}(p)=(1,\ p^0/m,\ -p^1/m,\ -p^2/m,\ -p^3/m)$. These wave functions satisfy the D-K equation

$$(\beta\cdot p-m)u(p)=0\ , \tag{3}$$

where the β^μ are five-by-five matrices (for spin 0) which satisfy the D-K algebra

$$\beta^\mu\beta^\nu\beta^\lambda+\beta^\lambda\beta^\nu\beta^\mu=\beta^\mu g^{\nu\lambda}+\beta^\lambda g^{\nu\mu}\ . \tag{4}$$

A complete set of 25 independent five-by-five matrices is

$$\begin{aligned}&1,\ \zeta,\ \beta^\mu,\ \zeta\beta^\mu,\ \Sigma^{\mu\nu}=\tfrac{1}{2}(\beta^\mu\beta^\nu-\beta^\nu\beta^\mu)\ ,\\ &S^{\mu\nu}=\tfrac{1}{2}(\beta^\mu\beta^\nu+\beta^\nu\beta^\mu)-\tfrac{1}{4}g^{\mu\nu}(\beta^\lambda\beta_\lambda)\ .\end{aligned} \tag{5}$$

Note that there are only nine independent $S^{\mu\nu}$ because of the condition $S^\mu{}_\mu=0$. The matrix ζ is chosen to be the diagonal matrix $(-1,+1,+1,+1,+1)$. It is a linear combination of the two Lorentz scalar matrices 1 and $\beta\cdot\beta$;

$$\zeta=\tfrac{5}{3}1-\tfrac{2}{3}\beta\cdot\beta\ , \tag{6}$$

$$\zeta\beta^\mu+\beta^\mu\zeta=0\ . \tag{7}$$

Also useful is the equation, which can be derived[9] from (3) and (4),

$$\beta\cdot p\beta^\mu u(p)=p^\mu u(p)\ . \tag{8}$$

Now the situation is seen most simply by looking directly at the hadronic matrix element (ME) of the divergence of the strangeness-changing current. It is well known that the ME of a scalar operator between single-scalar-particle states is determined by a single function of t (scalar form factor),

$$\begin{aligned}i\langle\pi^0(p_\pi)|\partial_\mu V^\mu(0)|K^-(p_K)\rangle&=(m_K{}^2-m_\pi{}^2)f_+(t)\\&\quad+tf_-(t)\\&\equiv f_0(t)\ .\end{aligned} \tag{9}$$

If one insists on expressing this ME in terms of five-component D-K momentum-space wave functions, then one must consider all scalars which can be formed from the two vectors p_π^μ, p_K^μ and the elements of the D-K algebra (5):

$$i\langle\pi^0(p_\pi)|\partial_\mu V^\mu(0)|K^-(p_K)\rangle=\bar{u}(p_\pi)\Gamma(p_\pi,p_K,1,\zeta,\beta,\Sigma,S)u(p_K)\ , \tag{10a}$$

i.e., Γ is some function of various products of powers of the D-K invariants $p_\pi\cdot p_K$, 1, ζ, $\beta\cdot p_\pi$, and $\beta\cdot p_K$. By use of (3)–(8) all such terms can be reduced to a linear combination of functions of t times the two D-K scalar matrices 1 and ζ;

$$i\langle\pi^0(p_\pi)|\partial_\mu V^\mu(0)|K^-(p_K)\rangle=\bar{u}(p_\pi)[1G_1(t)+\zeta G_2(t)]u(p_K)\ . \tag{10b}$$

Of course, the ME is determined by only one function of t, i.e., it must depend only one one particular linear combination of the two functions G_1 and G_2. By explicitly carrying out the five-by-five matrix multiplication on the right-hand side of (10b), we find

$$i\langle\pi^0(p_\pi)|\partial_\mu V^\mu(0)|K^-(p_K)\rangle=(2m_\pi m_K)^{-1}\{[(m_K+m_\pi)^2-t]G_1(t)+[(m_K-m_\pi)^2-t]G_2(t)\} \tag{10c}$$

$$\equiv f_0(t)\ . \tag{10d}$$

Since the D-K formalism does not place any restrictions on the functions $G_1(t)$ and $G_2(t)$, we see from (10c) and (10d) that the formalism does not imply any kinematic zero for $f_0(t)$,[10] in agreement with the general helicity analysis outlined above.

There is a similar proliferation of D-K terms in the general Lorentz-covariant form of the ME of the current; by application of (3)–(8) to all possible D-K vectors, one can reduce the ME to the form

$$\langle\pi^0(p_\pi)|V^\mu(0)|K^-(p_K)\rangle=\bar{u}(p_\pi)\{p_K^\mu 1g_1(t)+p_\pi^\mu 1g_2(t)+\beta^\mu g_3(t)+p_K^\mu\zeta g_4(t)+p_\pi^\mu\zeta g_5(t)+\beta^\mu\zeta g_6(t)\}u(p_K)\ . \tag{11a}$$

This ME, and hence all of the physics of the K_{l3} decays (in the current-current description), is well known to be determined by two functions of t, e.g., the $f_\pm(t)$ of (1); hence, it must depend on just two linear combinations of the functions $g_1(t),\ldots,g_6(t)$ on the right-hand side of (11a). Again, explicitly carrying out the five-by-five matrix multiplication on the right-hand side of (11a), we verify this in the following equation:

$$\langle \pi^0(p_\pi)|V^\mu(0)|K^-(p_K)\rangle = \tfrac{1}{2}(p_K+p_\pi)^\mu \left\{ \frac{(m_K+m_\pi)^2-t}{2m_K m_\pi}[g_1(t)+g_2(t)] + \frac{(m_K-m_\pi)^2-t}{2m_K m_\pi}[g_4(t)+g_5(t)] + \frac{m_K+m_\pi}{m_K m_\pi} g_3(t) + \frac{m_K-m_\pi}{m_K m_\pi} g_6(t) \right\}$$

$$+ \tfrac{1}{2}(p_K-p_\pi)^\mu \left\{ \frac{(m_K+m_\pi)^2-t}{2m_K m_\pi}[g_1(t)-g_2(t)] + \frac{(m_K-m_\pi)^2-t}{2m_K m_\pi}[g_4(t)-g_5(t)] - \frac{m_K-m_\pi}{m_K m_\pi} g_3(t) - \frac{m_K+m_\pi}{m_K m_\pi} g_6(t) \right\} \quad (11b)$$

$$\equiv (p_K+p_\pi)^\mu f_+(t) + (p_K-p_\pi)^\mu f_-(t) . \quad (11c)$$

Since the ME does depend on just two linear combinations of the six $g_i(t)$, four of them can be eliminated from (11a); however, as explained in Ref. 10, for the simpler case of the scalar ME (10), this elimination necessarily introduces kinematic poles into the remaining functions – which are canceled by kinematic zeros of the matrix products.

The results of the K_{l3} analysis of Refs. 1–3 depend on several things. First is the parametrization of the K, π ME by two effective Klein-Gordon form factors $f_\pm(t)$ which are written as the product of definite kinematic factors times unknown D-K form factors [Eq. (7) of Ref. 1; Eq. (8) of Ref. 3]. Their parametrization can be obtained from our (11a) by eliminating g_4, g_5, g_6, and the linear combination g_1+g_2 in terms of g_3 and g_1-g_2. Then the kinematic factors can be determined by comparison of (11b) and (11c). Second is the assumption that the resulting D-K functions $g_V(t)$ and $g_S(t)$ are smooth functions. This leads immediately to the prediction of the zero in the effective Klein-Gordon scalar form factor $f_0(t)$. Third are further dynamical assumptions, e.g., K^* pole dominance for $g_V(0)$ and $g_S(0)$ or application of SU_3 to obtain $g_V(0)=2^{-1/2}$ rather than $f_+(0)=2^{-1/2}$. These approximations lead to different values for $\xi \equiv f_-(0)/f_+(0)$ and the vector Cabibbo angle θ_V than do the same approximations applied directly to the Klein-Gordon form factors $f_\pm(t)$, because of the presence of the kinematic factors described above. However, our general kinematic analysis, as well as explicit algebraic calculation,[10] indicates that the $g_V(t)$ and $g_S(t)$ obtained this way are not smooth functions; rather they contain kinematic poles which cancel the kinematic zeros of the D-K wave functions.

In Ref. 3 the authors point out that zeros and poles may be dynamic as well as (or rather than) kinematic. They suggest that when they choose a specific limited set of D-K covariants they are specifying the form of some underlying Lagrangian theory of K_{l3} decays, i.e., specifying the dynamics. However, this does not justify their assumptions about the form of the ME; the absence of certain couplings from a Lagrangian does not imply the absence of the corresponding forms from the phenomenological ME. Once one goes beyond the lowest order of perturbation theory, all forms not forbidden by some symmetry principle will appear in the ME's whether or not they appear in some Lagrangian. This is well known, for example, in the case of the Schwinger $g-2$ term in quantum electrodynamics. One has minimal electromagnetic coupling, i.e., $\bar{\psi}\gamma^\mu\psi A_\mu$, *not* $\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}$, in the Lagrangian; yet $g-2$ is the coefficient of the $\sigma_{\mu\nu}q^\nu$ term in the ME of the current and is computed to be nonzero in all orders beyond the lowest order [similarly, the experimental existence of the anomalous magnetic moments (Pauli form factor, F_2) of the nucleon is not taken as evidence against minimal electromagnetic coupling]. In Ref. 3, the authors compute the ME for K_{l3} decays only in lowest order with their model Lagrangian, so of course the result has only the form of the input Lagrangian. If they had computed radiative corrections they would have found the more general form of the current ME with no zero at $t=(m_K+m_\pi)^2$.

Note added in proof. Fischbach, Nieto, and Scott [following paper, Phys. Rev. D 7, 3544 (1973)] agree that starting from a Lagrangian including only $\bar{\psi}\beta_\lambda\psi$, one will induce additional effective couplings. The only one they list is $q_\lambda\bar{\psi}\psi$, which also leads to a zero in $f_0(t)$ at $t=(m_K+m_\pi)^2$. Our point is that other couplings, which lead to $f_0((m_K+m_\pi)^2)\neq 0$, will also be induced. A final general comment is that we have not made any objection to the parametrization of K_{l3} decays by Fischbach *et al.* regarded as a purely empirical model. Our contention is only that the Duffin-Kemmer formalism does not provide any *a priori* theoretical basis for their parametrization. We leave it to others to argue whether or not that parametrization provides a better representation of the experimental data than various theoretical models of the $f_\pm(t)$.

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[1]E. Fischbach, F. Iachello, A. Lande, M. M. Nieto, and C. K. Scott, Phys. Rev. Lett. 26, 1200 (1971).

[2]E. Fischbach, M. M. Nieto, H. Primakoff, C. K. Scott, and J. Smith, Phys. Rev. Lett. 27, 1403 (1971).

[3]E. Fischbach, M. M. Nieto, and C. K. Scott, Phys. Rev. D 6, 726 (1972).

[4]B. Nagel and H. Snellman [Phys. Rev. Lett. 27, 761 (1971)] have already pointed out that not all Duffin-Kemmer vector forms lead to a kinematic zero at $t=(m_K+m_\pi)^2$. In a subsequent comment [Phys. Rev. D 6, 731 (1972)], they have again clearly emphasized that the presence of a zero in the scalar form factor at $t=(m_K+m_\pi)^2$ is an *ad hoc* assumption, not a consequence of describing the spin-0 mesons by D-K wave functions. The present comment goes beyond this observation by applying the S-matrix kinematic analysis to show the freedom of the Klein-Gordon form factors from kinematic constraints, whether or not they are taken as fundamental or derived from some D-K form factors, and in listing in a more systematic way all possible D-K covariants and invariants and carrying out their reduction to Klein-Gordon forms. We also comment on Ref. 3, in which some of the authors of Refs. 1 and 2 raise the issue of an underlying Lagrangian as justification for their particular choice of D-K forms.

[5]The most detailed treatment of the kinematic properties of the analytic S-matrix may be found in A. D. Martin and T. D. Spearman, *Elementary Particle Theory* (Wiley, New York, 1970). References to the original literature may be found in this book.

[6]The properties of the hadronic current are also independent of whether it couples directly to the leptonic current or to an intermediate W boson. In the latter case, the invariant matrix element acquires a dynamic pole at $t=m_W^2$ from the W propagator.

[7]Once the $f_\pm(t)$ have been determined to be free of kinematic singularities or constraints from the comparison with the standard analysis for $m_\nu \neq 0$, it is very interesting to look at the invariant matrix elements in the $m_\nu=0$ limit. Then a naive extrapolation of the general formulas (see Ref. 5) derived for $m_\nu\neq 0$ would give a $(t-m_l^2)^{-1}$ singularity from the coalescence of the threshold and pseudothreshold ($p_l=p_\nu=0$) singularities. However, for $m_\nu=0$, the $(E_\nu+m_\nu)^{1/2}$ factor in (2) becomes simply $\sqrt{p_\nu}$ and the matrix elements have only a $(t-m_l^2)^{-1/2}$ singularity.

[8]R. J. Duffin, Phys. Rev. 54, 1114 (1938); N. Kemmer, Proc. R. Soc. A173, 91 (1939). For a textbook treatment see, e.g., P. Roman, *Theory of Elementary Particles* (North-Holland, New York, 1960).

[9]We are indebted to Fischbach, Nieto, and Scott for calling our attention to this equation in the paper of Kemmer (see Ref. 8).

[10]The equations which lead from (10b) to (10c) are

$$\bar u(p_\pi)1u(p_K)=1+\frac{p_\pi\cdot p_K}{m_\pi m_K}=\frac{(m_K+m_\pi)^2-t}{2m_K m_\pi}\,,$$

$$\bar u(p_\pi)\zeta u(p_K)=-1+\frac{p_\pi\cdot p_K}{m_\pi m_K}=\frac{(m_K-m_\pi)^2-t}{2m_K m_\pi}\,,$$

from which follows

$$\bar u(p_\pi)\zeta u(p_K)=\frac{(m_K-m_\pi)^2-t}{(m_K+m_\pi)^2-t}\,u(p_\pi)1u(p_K)\,,$$

i.e., one can eliminate the G_2 term from (10b) and write

$$i\langle\pi^0(p_\pi)|\,\partial_\mu V^\mu(0)|K^-(p_K)\rangle=\bar u(p_\pi)1\,G(t)u(p_K)\,.$$

However, this elimination of the matrix ζ necessarily introduces a *kinematic pole* at $t=(m_K+m_\pi)^2$ into the function $G(t)$,

$$G(t)=G_1(t)+\frac{(m_K-m_\pi)^2-t}{(m_K+m_\pi)^2-t}\,G_2(t)\,,$$

which just cancels the kinematic zero from $\bar u(p_\pi)1\,u(p_K)$, again in agreement with the general helicity analysis that $f_0(t)$ does not have any kinematic constraint.