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Lepton Pair Production from Two-Photon Processes*

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An exact expression for the lepton-pair mass spectrum for an α^4 two-photon process in lepton-lepton, lepton-hadron, and hadron-hadron scattering processes is derived. This result is applied to muon pair production in proton-proton scattering to show that such a process is an important background to the α^2 one-photon process in certain energy ranges and can become physically significant by itself at very high energies. The general physical significance of such a two-photon process in hadron-hadron scattering is discussed, and comparison of our exact expression with some approximation schemes is made. The main differences between this work and earlier papers on the subject are that (1) exact calculations are given and (2) the inelastic contributions are included.

I. INTRODUCTION

Recently, lepton pair production in high-energy collisions has been the subject of various studies. The reactions under consideration are of the type

$$a_1(p_1)+a_2(p_2)\to l(l_1)+\bar{l}(l_2)+X, \qquad (1.1)$$

where a_i are the incident particles with momenta p_i, l and $\bar{l}$ are the produced lepton pair with momenta l_i and total invariant mass squared $Q^2=(l_1+l_2)^2$, and X may be either a definite exclusive state or anything inclusive. This type of reaction is important in studying the electromagnetic structure of hadrons and the purely electromagnetic interaction at high energies.

There are two important mechanisms contributing to the reactions (1.1), namely, the α^2 one-photon process and the α^4 two-photon process.

For the α^2 one-photon process, we have

$$a_1 + a_2 \to \gamma^* + X \to l + \bar{l} + X, \tag{1.2}$$

where the incident particles emit a timelike virtual photon γ^* which turns into an odd-C lepton pair [Fig. 1(a)]. In the case when the incident particles are a lepton-antilepton pair, we have the familiar one-photon annihilation mechanism $l + \bar{l} \to \gamma^* \to l + \bar{l}$, with X = vacuum, which is important at low energies. When a_1 and a_2 are hadrons, the detailed mechanism is much less understood. One version of the parton model[1] predicts a scaling behavior for the differential cross section,

$$\frac{d\sigma}{d(Q^2)^{1/2}} = \frac{\alpha^2}{(Q^2)^{3/2}} F(s/Q^2), \tag{1.3}$$

at large s and Q^2, where $s = (p_1 + p_2)^2$ is the total incident invariant mass squared. However, the magnitude and detailed behavior of the scaling function $F(s/Q^2)$ cannot be unambiguously calculated at the present stage. A different approach to the same process is an analysis by Altarelli, Brandt, and Preparata of the light-cone singularity of products of currents.[2] This analysis predicts a nonscaling behavior with exponential decrease of the differential cross section at large Q^2.

In addition to the α^2 one-photon process, it is important to study the α^4 two-photon process

$$a_1 + a_2 \to X_1(P_1) + \gamma_1^*(q_1) + \gamma_2^*(q_2) + X_2(P_2)$$
$$\to X_1 + l + \bar{l} + X_2, \tag{1.4}$$

where the incident particles a_i emit virtual photon γ_i^* with momentum q_i and turns into final state X_i with momentum P_i and invariant mass squared $s_i = P_i^2$, the virtual photons in turn annihilated into an even-C lepton pair [Fig. 1(b)].

The importance of the two-photon process in electron-positron (electron) colliding beams, where a_i are leptons and X_i are single lepton states, has been noted by three independent groups.[3-5] At high energies, this process becomes the dominant one, and various differential cross sections are calculated in Refs. 3-5 by use of an equivalent-photon approximation.[6]

It is also important to know the full contribution to the differential cross sections by the two-photon process, as shown in Fig. 1(b), when a_i are hadrons and X_i includes both elastic states ($X_i = a_i'$) and inelastic states. Knowledge of this contribution is necessary to assess the background for the extraction of the α^2 one-photon process. Since the amplitudes for the one- and two-photon processes do not interfere in the differential cross section, the two-photon process will also provide a lower bound to the expected cross section. Furthermore, in regions of s and Q^2 for certain reactions, the two-photon process itself may be important and useful in testing the scaling of the hadron inelastic structure functions[7] and measuring the pion and kaon electromagnetic form factors in the spacelike q_i^2 region.[8]

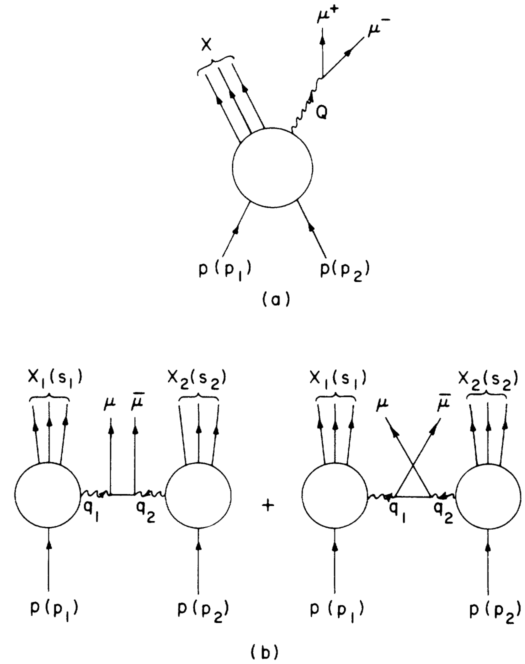

FIG. 1. (a) Feynman graph for α^2 one-photon annihilation into a μ pair. (b) Feynman graph for α^4 two-photon annihilation into a μ pair via a two-photon process.

The relevance of the two-photon process in $p + p \to \mu + \bar{\mu} + X$, where p is the proton, has been emphasized earlier by Budnev, Ginzburg, Meledin, and Serbo,[9] who estimated the elastic contribution ($X_i = p$) by an equivalent-photon approximation which neglects the effects of the proton form factors at the vertex $p \to p + \gamma^*$. As subsequently pointed out in Ref. 5, such effects are important at high Q^2, where the minimum momentum transfers for q_i^2 to the protons are not negligible. This point was confirmed by a calculation of the elastic contribution to $d\sigma/d(Q^2)^{1/2}$ by Fujikawa,[10] who still neglected some dependence on q_i^2 for the purely electromagnetic process $\gamma_1^* + \gamma_2^* \to \mu + \bar{\mu}$. The results of an exact calculation, using the formalism of this work, of both the elastic and the full contribution, where X also includes inelastic states, have been presented in an earlier publication.[7]

In this work, we present a general formalism for the two-photon process (1.4), which includes the electron-electron (positron) collision process with a_1, a_2, X_1, and X_2 all being single-electron (positron) states, the trident production with a_1 and X_1 being single-lepton states and a_2 being a hadron, and the lepton pair production in hadron-

hadron collisions. This formalism can also be easily generalized and applied to the photoproduction of lepton pairs, where we have $a_1 = \gamma_1^*$ being a real photon and X_1 being the vacuum.

In Sec. II, we formulate the general framework using the helicity methods previously developed[11-13] and derive the exact differential cross section in terms of the usual invariant variables s, Q^2, q_1^2, q_2^2, s_1, s_2, $s_{12} = (p_1 + q_2)^2$, and $s_{21} = (p_2 + q_1)^2$. In Sec. III, we apply the results to the muon pair production in proton-proton scattering and also discuss the possible modification to the contribution given by Fig. 2(b) due to the initial- and final-state interactions. Comparison with equivalent photon approximation and other possible physical applications are also discussed.

II. DETAILS OF THE CALCULATION

The calculation of the Feynman graphs of the two-photon process is straightforward but tedious. We present here a complete summary of the kinematics, the relevant matrix elements, and the phase-space variables used in this calculation. The amplitude is expressed in both O(2, 1) and Lorentz-invariant variables.

The matrix element for the process shown in Fig. 1(b) is

$$T_{fi} = -i(2\pi)^4 \delta^4(P_1 + P_2 + l_1 + l_2 - p_1 - p_2) \times \frac{1}{(2\pi)^3} \frac{\mu}{l_{10}} \frac{\mu}{l_{20}} M_{fi}\,, \tag{2.1}$$

where M_{fi} is the usual Møller amplitude

$$M_{fi} = e^2 \langle X(P_1) | J_\mu(0) | p_1 \rangle \langle X(P_2) | J_\alpha(0) | p_2 \rangle \times D^{\mu\nu}(q_1) D^{\alpha\beta}(q_2) T_{\nu\beta}(l_1, l_2, q_1, q_2)\,, \tag{2.2}$$

$D^{\mu\nu}(q_i) = -ig^{\mu\nu}/q_i^2$ is the photon propagator $(i = 1, 2)$, $T_{\nu\beta}(l_1, l_2; q_1, q_2)$ is the quantum-electrodynamic amplitude for the process shown in Fig. 1(b) and is given in second-order perturbation theory by

$$T_{\nu\beta}(l_1, l_2; q_1, q_2) = \bar{u}(l_1)\left[(-ie\gamma_\nu)\frac{1}{\not{l}_1 - \not{q}_1 - \mu}(-ie\gamma_\beta) + (-ie\gamma_\beta)\frac{1}{\not{l}_1 - \not{q}_2 - \mu}(-ie\gamma_\nu)\right]v(l_2)\,, \tag{2.3}$$

J_μ is the electromagnetic current, $q_i = p_i - P_i$ $(i = 1, 2)$ is the momentum of the spacelike virtual photon with invariant mass squared $q_i^2 < 0$, and μ is the lepton rest mass.

Upon squaring the amplitude and introducing the flux factor and phase-space volume element in the usual way, the differential cross section is given by

$$d\sigma = \frac{M_1 M_2}{(sk^2)^{1/2}} \frac{1}{4} \sum_{\text{spins}} |M_{fi}|^2 (2\pi)^4 \delta^4(P_1 + P_2 + l_1 + l_2 - p_1 - p_2) \frac{\mu}{l_1^0} \frac{\mu}{l_2^0} \frac{d^3 l_1}{(2\pi)^3} \frac{d^3 l_2}{(2\pi)^3} d\phi_1 d\phi_2\,, \tag{2.4}$$

where M_1 and M_2 are the masses of the incident particles, $d\phi_1$ and $d\phi_2$ are the phase-space volume elements for the multiparticle systems X_1 and X_2, and k is the absolute value of the incident momentum in the center-of-mass (c.m.) frame and is given by

$$k = \frac{1}{2(s)^{1/2}} (s^2 + M_1^4 + M_2^4 - 2sM_1^2 - 2sM_2^2 - 2M_1^2 M_2^2)^{1/2}\,, \tag{2.5}$$

which becomes $k = \frac{1}{2}(s - 4M^2)^{1/2}$ when $M_1 = M_2 = M$.

Using the identity

$$\delta^4(P_1 + P_2 + l_1 + l_2 - p_1 - p_2) = \int d^4q\, d^4q_2\, \delta^4(P_1 - p_1 + q_1)\delta^4(P_2 - p_2 + q_2)\delta^4(l_1 + l_2 - q_1 - q_2)$$

and the expression for the Møller amplitude, the differential cross section can be converted into a Lorentz-invariant integral of the virtual Compton amplitude and the absorptive part of the electromagnetic amplitude for $\gamma^*(q_1) + \gamma^*(q_2) \to l(l_1) + \bar{l}(l_2) \to \gamma(q_1) + \gamma(q_2)$:

$$\sigma = \frac{M_1 M_2}{(s)^{1/2} k} \left(\frac{\alpha}{\pi}\right)^2 \frac{1}{(2\pi)^2} \int \frac{d^4 q_1}{q_1^4} \int \frac{d^4 q_2}{q_2^4} W^{(1)}_{\mu\lambda}(-q_1, p_1) a^{\lambda\beta\mu\alpha}(q_1, q_2) W^{(2)}_{\alpha\beta}(-q_2, p_2)\,, \tag{2.6a}$$

where $a^{\lambda\beta\mu\alpha}(q_1, q_2)$ is the absorptive part of the electromagnetic amplitude $\gamma^*\gamma^* \to l\bar{l} \to \gamma^*\gamma^*$ and is given by

$$a^{\lambda\beta\mu\alpha}(q_1, q_2) = \sum_{\substack{\text{lepton}\\ \text{spins}}} \int \frac{d^3 l_1}{(2\pi)^3} \frac{d^3 l_2}{(2\pi)^3} \frac{\mu^2}{l_1^0 l_2^0} (2\pi)^4 \delta^4(l_1 + l_2 - q_1 - q_2) T^{\mu\alpha} T^{\lambda\beta *}\,, \tag{2.6b}$$

and $W^{(i)}_{\mu\lambda}$ is the forward Compton amplitude

$$W^{(i)}_{\mu\lambda}(-q_i, p_i) = \int d\phi_i \, \tfrac{1}{2} \sum_{\substack{\text{nucleon}\\ \text{spins}}} (2\pi)^3 \langle X_i(P_i) | J_\mu(0) | p_i \rangle \langle X_i(P_i) | J_\lambda(0) | p_i \rangle^* \delta^4(P_i - (p_i - q_i)) \qquad (i=1,2). \tag{2.7}$$

This has the usual expression in terms of the invariant structure functions W_1 and W_2:

$$W^{(i)}_{\mu\lambda}(-q_i, p_i) = -W_1^{(i)}(q_i^2, -q_i\cdot p_i)\left(g_{\mu\lambda} - \frac{q_{i\mu}q_{i\lambda}}{q_i^2}\right) + \frac{W_2^{(i)}(q_i^2, -q_i\cdot p_i)}{M_i^2}\left(p_{i\mu} - \frac{q_{i\mu}q_i\cdot p_i}{q_i^2}\right)\left(p_{i\lambda} - \frac{q_{i\lambda}q_i\cdot p_i}{q_i^2}\right). \tag{2.8}$$

The relevant differential cross section for the lepton-pair mass spectrum is

$$\frac{d\sigma}{d\sqrt{Q^2}} = \frac{M_1 M_2}{(s)^{1/2}k}\left(\frac{\alpha}{\pi}\right)^2 \frac{1}{(2\pi)^2} \int \frac{d^4q_1}{q_1^4}\frac{d^4q_2}{q_2^4} W^{(1)}_{\mu\lambda}(-q_1, p_1) a^{\lambda\beta\mu\alpha}(q_1, q_2) W^{(2)}_{\alpha\beta}(-q_2, p_2) 2\sqrt{Q^2}\, \delta(Q^2 - (q_1+q_2)^2). \tag{2.9}$$

Since $a^{\lambda\beta\mu\alpha}$ can be straightforwardly calculated according to Eqs. (2.3) and (2.6b), Eq. (2.9) can be readily applied to calculate the differential cross sections for muon pair production in $e^\pm e^- \to e^\pm e^- \mu^+\mu^-$, muon trident production, and electron or muon pair production from the two-photon processes in hadron-hadron collisions by suitable choice of the structure functions W_1 and W_2. To make the practical calculation extremely simple, we can take advantage of the multiperipheral character of the process and the factorization property of the matrix element and phase-space volume element. To facilitate this simplification, we use a variant of the O(2, 1) formalism developed earlier for the strong interactions and expand the integrand of Eq. (2.9) in terms of the photon-photon helicity amplitudes evaluated in their own c.m. frame. In Sec. II A, we present this O(2, 1) formalism and in Sec. II D we also express all our results in terms of the usual Lorentz-invariant variables. While the group theory is not necessary for the following, it simplifies greatly the subsequent analytic and numerical integrations.

A. O(2,1) Formalism

In this framework, all the momenta are parametrized in terms of the O(3, 1) group elements. We define a special Breit frame B, in which

$$q_1 = \hat{q}_1 \equiv (0, 0, 0, (-q_1^2)^{1/2}), \tag{2.10a}$$

$$q_2 = \exp(-i\pi J_y)\exp(-i\alpha' K_z)\hat{q}_2 = (\sinh\alpha'(-q_2^2)^{1/2}, 0, 0, -\cosh\alpha'(-q_2^2)^{1/2}). \tag{2.10b}$$

In Eq. (2.10) and the following discussions, J_i and K_i represent generators for rotations and boosts along axes labeled by i ($i=x, y, z$). We can easily relate the boost parameter α' to the invariants by calculating Q^2:

$$Q^2 = (q_1+q_2)^2 = q_1^2 + q_2^2 + 2(-q_1^2)^{1/2}(-q_2^2)^{1/2}\cosh\alpha'. \tag{2.11}$$

Similarly, we can express the external momenta in terms of group elements in such a frame by the relations:

$$\begin{aligned} p_1 &= \exp(-i\zeta_1 K_x)\exp(-i\alpha_1 K_z)\hat{p}_1 \\ &= (M_1\cosh\zeta_1\cosh\alpha_1, M_1\sinh\zeta_1\cosh\alpha_1, 0, M_1\sinh\alpha_1), \\ p_2 &= \exp(+i\alpha' K_z)\exp(-i\psi J_z)\exp(-i\zeta_2 K_x) \\ &\quad\times\exp(-i\pi J_y)\exp(-i\alpha_2 K_z)\hat{p}_2 \\ &= (M_2(\cosh\alpha'\cosh\zeta_2\cosh\alpha_2 + \sinh\alpha'\sinh\alpha_2), \\ &\quad M_2\sinh\zeta_2\cosh\alpha_2\cos\psi, \\ &\quad M_2\cosh\alpha_2\sinh\zeta_2\sin\psi, \\ &\quad -M_2(\cosh\alpha'\sinh\alpha_2 + \sinh\alpha'\cosh\zeta_2\cosh\alpha_2)), \end{aligned} \tag{2.12}$$

where $\hat{p}_i = (M_i, 0, 0, 0)$, $i=1,2$.

The last two relations follow from the fact that the external timelike momenta p_i can always be brought to the zt plane by an O(2, 1) transformation with the general parametrization

$$\begin{aligned} &g \in O(2,1), \\ &g = \exp(-i\beta J_z)\exp(-i\tau K_x)\exp(-i\gamma J_z) \\ &0 \le \beta, \quad \gamma \le 2\pi, \quad \tau \text{ real}. \end{aligned} \tag{2.13}$$

The O(2, 1) parameters, ζ_1 and ζ_2, for the external momenta p_i assume special significance in terms of invariants when evaluated in the frame where the adjacent q_i momenta are in their respective Breit frames $q_i = \hat{q}_i$.

It then follows by evaluation of invariants that:

$$s_1 = (-q_1 + p_1)^2 = M_1^2 + q_1^2 + 2M_1(-q_1^2)^{1/2}\sinh\alpha_1 ,$$
$$s_{12} = (p_1 + q_2)^2 = M_1^2 + q_2^2 + 2M_1(-q_2^2)^{1/2}\sinh\alpha_{12} , \quad (2.14a)$$
$$\sinh\alpha_{12} = \cosh\alpha_1 \sinh\alpha' \cosh\zeta_1 + \sinh\alpha_1 \cosh\alpha' ,$$

and

$$s_2 = (-q_2 + p_2)^2 = M_2^2 + q_2^2 + 2M_2(-q_2^2)^{1/2}\sinh\alpha_2 ,$$
$$s_{21} = (p_2 + q_1)^2 = M_2^2 + q_1^2 + 2M_2(-q_1^2)^{1/2}\sinh\alpha_{21} , \quad (2.14b)$$
$$\sinh\alpha_{21} = \cosh\alpha_2 \sinh\alpha' \cosh\zeta_2 + \sinh\alpha_2 \cosh\alpha' .$$

Positivity of energy and the mass spectrum and conservation of energy require the inequalities

$$\alpha_1 \geq 0, \quad \alpha_2 \geq 0, \quad \alpha_{12} \geq 0, \quad \alpha_{21} \geq 0, \quad \alpha' \geq 0, \quad \zeta_1 \geq 0, \text{ and } \zeta_2 \geq 0 . \quad (2.14c)$$

Equations (2.11), (2.14a), and (2.14b) provide the necessary relations of the O(2, 1) variables to the natural invariants of the graph in Fig. 1. The azimuthal angle ψ describes the orientation of the normals of the planes defined by $\vec{p}_1, \vec{q}_1$ and $\vec{p}_2, \vec{q}_2$:

$$\cos\psi = (\vec{p}_1 \times \vec{q}_1) \cdot (\vec{p}_2 \times \vec{q}_2) / |\vec{p}_1 \times \vec{q}_1||\vec{p}_2 \times \vec{q}_2| . \quad (2.15)$$

Since the frame B defined by Eqs. (2.10) and (2.12) is related to the photon-photon c.m. frame by a z boost, the c.m. helicity amplitudes for $\gamma^*(q_1) + \gamma^*(q_1) \to \gamma^*(q_1) + \gamma^*(q_1)$ are numerically equal to the helicity amplitudes defined in the frame B. This follows from the fact that the helicity which is the eigenvalue of J_z in this case commutes with z boosts. The expansion of the integrand in Eq. (2.7) in terms of c.m. helicity amplitudes and the O(2, 1) parameters given in Eq. (2.12) is

$$W^{(1)}_{\mu\lambda}(-q_1, P_1) a^{\lambda\beta\mu\alpha}(q_1, q_2) W^{(2)}_{\alpha\beta}(-q_2, P_2) = \sum_{(m,n)} W_{m_1}(\sinh\alpha_1, q_1^2) W_{m_2}(\sinh\alpha_2, q_2^2) d^1_{m_1 n_1}(\zeta_1) d^1_{m_1 n_1'}(\zeta_1)$$
$$\times a_{n_1' n_2' n_1 n_2}(\cosh\alpha', q_1^2, q_2^2) D^1_{n_2 m_2}(\psi, \zeta_2, 0) D^1_{n_2' m_2}(\psi, \zeta_2, 0) . \quad (2.16)$$

The quantities $a_{n_1' n_2' n_1 n_2}(\cosh\alpha', q_1^2, q_2^2)$ are the c.m. helicity amplitudes with the standard definition in terms of polarization vectors

$$a_{n_1' n_2' n_1 n_2}(\cosh\alpha', q_1^2, q_2^2) = \epsilon^{\lambda *}_{n_1'}(q_1) \epsilon^{\beta *}_{n_2'}(q_2) \epsilon^{\mu}_{n_1}(q_1) \epsilon^{\alpha}_{n_2}(q_2) a_{\lambda\beta\mu\alpha}(q_1, q_2) , \quad (2.17)$$

and $W_{m_1}(\sinh\alpha_1, q_1^2)$ and $W_{m_2}(\sinh\alpha_2, q_2^2)$ are the helicity projections (transverse and longitudinal) of the tensor in Eq. (2.8) and are given by an expression analogous to Eq. (2.17). In terms of the invariant amplitudes, the nonvanishing helicity components in the frame where p, and the adjacent q, are collinear are given by

$$W^{(i)}_{\text{trans}} = W^{(i)}_{m=1} = -W^{(i)}_1 = W^{(i)}_{m=-1} ,$$
$$W^{(i)}_{\text{long}} = W^{(i)}_{m=0} = -W^{(i)}_1 + (\cosh\alpha_i)^2 W^{(i)}_2 . \quad (2.18)$$

The polarization vectors $\epsilon^{\mu}_m(q)$ have the standard orthogonality and completeness relations

$$\epsilon^{\mu}_m(q) \epsilon^{\lambda}_{m'}(q) g_{\mu\lambda} = (-1)^m \delta_{mm'} ,$$
$$\sum_m (-1)^m \epsilon^{\mu}_m(q) \epsilon^{\lambda}_m(q) = g^{\mu\lambda} - q^{\mu} q^{\lambda}/q^2 , \quad (2.19)$$
$$q^{\lambda} \epsilon^{\mu}_m(q) g_{\mu\lambda} = 0 .$$

The polarization representation most useful for Eq. (2.16) is

$$\epsilon_0(q) = (-q^2)^{-1/2}(|\vec{q}|, q_0 \hat{z}) ,$$
$$\epsilon_{\pm 1}(q) = \mp \frac{1}{\sqrt{2}} (\hat{x} \pm i\hat{y}) , \quad (2.20)$$

where q is along the z axis and $\hat{x}, \hat{y}, \hat{z}$ are unit vectors along the x, y, z axes.

The representations of the O(2, 1) group are given by the functions $D^1_{mn}(\beta, \tau, \gamma)$:

$$D^1_{mn}(\beta, \tau, \gamma) = e^{-im\beta} d^1_{mn}(\tau) e^{-in\gamma} , \quad (2.21a)$$
$$d^1_{00}(\tau) = \cosh\tau ,$$
$$d^1_{01}(\tau) = -d^1_{10}(\tau) = -d^1_{0\,-1}(\tau) = d^1_{-10}(\tau) = 2^{-1/2}\sinh\tau , \quad (2.21b)$$
$$d^1_{11}(\tau) - d^1_{-1\,-1}(\tau) = \tfrac{1}{2}(1 + \cosh\tau) ,$$
$$d^1_{1\,-1}(\tau) = d^1_{-11}(\tau) = \tfrac{1}{2}(1 - \cosh\tau) .$$

We next proceed with the expression for the volume element $d^4q_1 d^4q_2$ in terms of group elements. We elect to perform the integrations in a particular order; namely, we perform the integration over q_1 first at fixed values of q_2 [see Fig.

2(a)], next the integration over q_2 at fixed values of the external momenta [see Fig. 2(b)]. For this purpose, it is useful to evaluate the absorptive part of the graph in Fig. 2(a) in a frame where p_1 and q_2 are collinear. The explicit form of the momenta is given by

$$p_1 = \exp(-i\alpha_{12}K_z)\hat{p}_1 = (M_1\cosh\alpha_{12}, 0, 0, M_1\sinh\alpha_{12})$$

and (2.22)

$$q_2 = \hat{q}_2 = (0, 0, 0, (-q_2^2)^{1/2}) .$$

This frame is reached from B by application of a z boost of $-\alpha'$ to all vectors. The momenta p_1 and q_1 then have the parametrization

$$p_1 = \exp(i\alpha' K_z)\exp(-i\zeta_1 K_x)\exp(-i\alpha_1 K_z)\hat{p}_1 ,$$

$$q_1 = \exp(i\alpha' K_z)\hat{q}_1 . \tag{2.23}$$

Next, we use the identity

$$\exp(i\alpha' K_z)\exp(-i\zeta_1 K_x)\exp(-i\alpha_1 K_z) = \exp(-i\zeta_1' K_x)\exp(-i\alpha_{12}K_z)\exp(-i\theta_1 J_y) , \tag{2.24a}$$

which implies the relation

$$\sinh\alpha_1 = \sinh\alpha_{12}\cosh\alpha' - \sinh\alpha' \cosh\alpha_{12}\cosh\zeta_1' . \tag{2.24b}$$

Equation (2.24) follows from the parametrization of p_1 in Eqs. (2.23) and the fact that a timelike vector in the xzt hyperplane can always be brought to the zt plane by application of an O(2, 1) x boost. The rotation by θ_1 around the y axis leaves $\hat{p}_1$ invariant. The z boost α_{12} on the right-hand side is related to s_{12} by Eq. (2.14a), which can be seen by evaluation of invariants in this frame. Next, we apply the x boost $\exp(i\zeta_1' K_x)$ to all vectors and the final frame is achieved with p_1 and q_2 given by Eqs. (2.22) and with q_1 and p_2 given by

$$q_1 = \exp(i\zeta_1' K_x)\exp(i\alpha' K_z)\hat{q}_1 ,$$

$$p_2 = \exp(+i\zeta_1' K_x)\exp(-i\psi J_z)\exp(-i\zeta_2 K_x) \times\exp(-i\pi J_y)\exp(-i\alpha_2 K_z)\hat{p}_2 . \tag{2.25}$$

The expression for p_2 can be simplified if we realize that the canonical form for the parametrization of the O(2, 1) group, Eq. (2.21a), allows us to use the identity

$$\exp(i\zeta_1' K_x)\exp(-i\psi J_z)\exp(-i\zeta_2 K_x) = \exp(i\varphi J_z)\exp(-iuK_x)\exp(i\psi' J_z) ; \tag{2.26}$$

then p_2 becomes

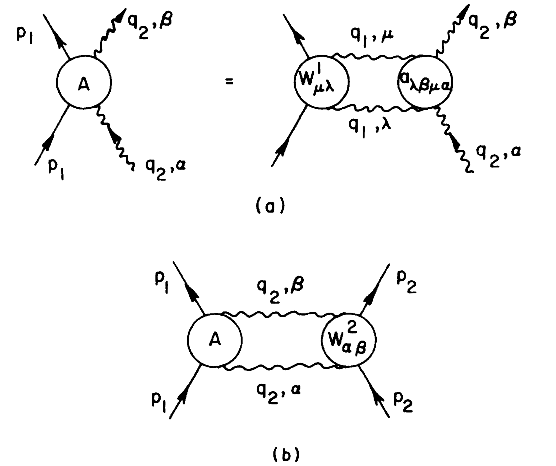

FIG. 2. (a) Graphical representation of absorptive part in Eq. (2.6a) for integration over q_1 at fixed values of q_2. (b) Graphical representation of absorptive part for pp amplitude in terms of absorptive part in Fig. 2(a).

$$p_2 = \exp(-i\varphi J_z)\exp(-iuK_x)\exp(-i\pi J_y) \times\exp(-i\alpha_2 K_z)\hat{p}_2 . \tag{2.27}$$

The azimuthal angle ψ' does not contribute because it leaves invariant the vector $\exp(-i\pi J_y) \times\exp(-i\alpha_2 K_z)\hat{p}_2$.

Next, applying the φ rotation we obtain:

$$q_2 = \hat{q}_2 ,$$

$$p_1 = \exp(-i\alpha_{12}K_z)\hat{p}_1 = (M_1\cosh\alpha_{12}, 0, 0, M_1\sinh\alpha_{12}) ,$$

$$q_1 = \exp(i\varphi J_z)\exp(i\zeta_1' K_x)\exp(i\alpha' K_z)\hat{q}_1 = ((-q_1^2)^{1/2}\sinh\alpha'\cosh\zeta_1', (-q_1^2)^{1/2}\sinh\alpha'\sinh\zeta_1'\cos\varphi , (-q_1^2)^{1/2}\sinh\alpha'\sinh\zeta_1'\sin\varphi, (-q_1^2)^{1/2}\cosh\alpha') , \tag{2.28}$$

$$p_2 = \exp(-iuK_x)\exp(-i\pi J_y)\exp(-i\alpha_2 K_z)\hat{p}_2 ,$$

$$p_2 = (M_2\cosh u\cosh\alpha_2, M_2\sinh u\cosh\alpha_2, 0, -M_2\sinh\alpha_2) .$$

From Eqs. (2.28), an elementary calculation gives

$$d^4q_1 = \tfrac{1}{2}d\varphi(-q_1^2)dq_1^2 d\cosh\zeta_1', \sinh\alpha' d\cosh\alpha' = \tfrac{1}{2}d\varphi(-q_1^2)dq_1^2 \frac{d\sinh\alpha_1\, d\cosh\alpha'}{\cosh\alpha_{12}} . \tag{2.29}$$

The last statement follows from Eq. (2.24b).

Finally, we combine the absorptive parts in Fig. 2(b), referring the external momenta to a collinear frame. In particular, we choose

$$p_1 = \hat{p}_1 ,$$

$$p_2 = \exp(-i\alpha K_z)\hat{p}_2 = (M_2\cosh\alpha, 0, 0, M_2\sinh\alpha) , \tag{2.30}$$

with

$$s = M_1^2 + M_2^2 + 2M_1M_2\cosh\alpha \, .$$

This frame is reached by application of a z boost $\exp(i\alpha_{12}K_z)$ from that of Eq. (2.28),

$$\begin{aligned} p_2 &= \exp(i\alpha_{12}K_z)\exp(-iuK_x)\exp(-i\alpha_2K_z)\hat{p}_2 \, , \\ q_2 &= \exp(i\alpha_{12}K_z)\hat{q}_2 \, . \end{aligned} \tag{2.31}$$

Next, we use the identity

$$\exp(i\alpha_{12}K_z)\exp(-iuK_x)\exp(-i\alpha_2K_z)\hat{p}_2 = \exp(-i\beta_2J_y)\exp(-i\alpha K_z)\hat{p}_i \, ,$$

which merely is a polar coordinate parametrization of p_2; application of the rotation through $-\beta_2$ produces the desired frame where

$$\begin{aligned} p_1 &= \hat{p}_1 \, , \\ p_2 &= (M_2\cosh\alpha, 0, 0, M_2\sinh\alpha) \, , \\ q_2 &= ((-q_2^2)^{1/2}\sinh\alpha_{12}, (-q_2^2)^{1/2}\cosh\alpha_{12}\sin\beta, \\ &\qquad 0, (-q_2^2)^{1/2}\cosh\alpha_{12}\cos\beta) \, . \end{aligned} \tag{2.32}$$

From Eqs. (2.31), then it follows that

$$\cosh\alpha = \cosh\alpha_{12}\cosh\alpha_2\cosh u + \sinh\alpha_{12}\sinh\alpha_2 \, . \tag{2.33}$$

In the frame defined by Eq. (2.32),

$$\begin{aligned} d^4q_2 &= \pi(-q_2^2)d(-q_2^2)d\cos\beta\, d\sinh\alpha_{12}\cosh\alpha_{12} \\ &\equiv \pi(-q_2^2)d(-q_2^2)\frac{d\sinh\alpha_{12}\, d\sinh\alpha_2}{\sinh\alpha} \, . \end{aligned} \tag{2.34}$$

Now we have the necessary Jacobian relations and variable assignments to perform the integration in Eq. (2.9). A straightforward expansion of the helicity sum in Eq. (2.16) and use of Eqs. (2.29), (2.34), and (2.9) with subsequent integration over φ yields the result:

$$\begin{aligned} \frac{d\sigma}{d(Q^2)^{1/2}} = \frac{2\alpha^4}{\pi}\frac{1}{\sinh\alpha}\frac{M_1M_2}{(s)^{1/2}k}(Q^2)^{1/2} & \\ \times \int \frac{dq_2^2}{(-q_2^2)^{3/2}} \int \frac{dq_1^2}{(-q_1^2)^{3/2}} \int d\alpha_{12} \int d\sinh\alpha_1 \int d\sinh\alpha_2 & \\ \times \{ (W_1^{(2)} + \tfrac{1}{2}\cosh^2\alpha_2\sinh^2u\, W_2^{(2)}) [(W_1^{(1)} + \tfrac{1}{2}\cosh^2\alpha_1\sinh^2\zeta_1\, W_2^{(1)})\mathrm{tr}A & \\ - (\tfrac{1}{2}\cosh^2\alpha_1(3\sinh^2\zeta_1 + 2)W_2^{(1)})A_{0\alpha}{}^{\alpha}] & \\ + [\tfrac{1}{2}(3\sinh^2u + 2)\cosh^2\alpha_2\, W_2^{(2)}] [\tfrac{1}{2}\sinh^2\zeta_1'\, W_1^{(1)}\mathrm{tr}A - \tfrac{1}{2}(3\cosh^2\zeta_1' - 1)W_1^{(1)}A^{\lambda}{}_{\lambda 0} & \\ + \cosh^2\alpha_1\sinh^2\zeta_1\sinh^2\zeta_1'\, W_2^{(1)}A_{yyyy} & \\ + \cosh^2\alpha_1\sinh^2\zeta_1\cosh^2\zeta_1'\, W_2^{(1)}A_{1010} & \\ + \cosh^2\alpha_1\cosh^2\zeta_1\sinh^2\zeta_1'\, W_2^{(1)}A_{0101} & \\ + \cosh^2\alpha_1\cosh^2\zeta_1\cosh^2\zeta_1'\, W_2^{(1)}A_{0000} & \\ + 2\cosh^2\alpha_1\cosh\zeta_1\sinh\zeta_1\cosh\zeta_1'\sinh\zeta_1'(A_{1100} - A_{01\,-10})] \} \, , & \end{aligned} \tag{2.35}$$

where μ is the lepton mass, the quantities A are various helicity projections of the central photon-photon amplitudes redefined by

$$A_{m_1'm_2'm_1m_2} = 4\pi\alpha^2 a_{m_1'm_2'm_1m_2}$$

and will be given explicitly in Sec. II C, and ζ_1, ζ_1', and u are given in terms of the integration variables by Eqs. (2.14a), (2.24b), and (2.33), respectively. The φ integration is elementary and enters Eq. (2.16) only through ζ_2 and ψ. This follows since the spin-1 nature of the electromagnetic current allows the integrand in Eq. (2.9) to depend only upon the φ angle from polynomials in $\cos\varphi$ which enter through the relations

$$\cosh\zeta_2 = \cosh\zeta_1'\cosh u - \cos\varphi\sinh\zeta_1'\sinh u \, , \tag{2.36}$$

$$\cos\psi\sinh\zeta_2 = \sinh\zeta_1'\cosh u - \cos\varphi\cosh\zeta_1'\sinh u \, , \tag{2.37}$$

which follow from Eq. (2.26). The expression for the boost parameters ζ_1, ζ_1', and u are given in Eqs. (2.24a), (2.24b), and (2.33). The region of integration in Eq. (2.35) is defined through the Kibble conditions

$$\alpha_{12} \geq \alpha_1 + \alpha' ,\qquad \alpha \geq \alpha_{12} + \alpha_2 , \tag{2.38}$$

which follow from Eqs. (2.14a), (2.33), and the inequalities in Eq. (2.14c), and the corresponding maximum and minimum values allowed by threshold conditions. The Kibble conditions, Eq. (2.38), the physical thresholds, and the range of the integration define the complete restrictions upon the variables of integration given by phase space. Also, notice that in the physical region, we always have $q_1^2 \leq 0$ and $q_2^2 \leq 0$.

At this stage, we can see the advantage of the O(2, 1) formalism. From Eqs. (2.14a), (2.24b), and (2.33), it can be shown that the kinematical factors multiplying the structure functions $W_1^{(i)}$ and $W_2^{(i)}$ are rational functions of $\sinh\alpha_1$ and $\sinh\alpha_2$, while the amplitudes a are independent of these two variables. Furthermore, if the integration over these two variables are performed first, their integration limits are independent of each other. Therefore, for a simple enough parametrization of $W_1^{(i)}$ and $W_2^{(i)}$, the integration over these two variables can be performed analytically separately. More detailed discussions will be given in Sec. II B and the Appendix.

B. Structure Functions

To apply Eq. (2.35) to the process (1.4) mentioned in the Introduction, we need to know the structure functions W_1 and W_2. Obviously for the elastic contribution, where $X_i = a_i$, the structure functions are given for a spin-$\frac{1}{2}$ particle by

$$\begin{aligned} W_1^{(i)} &= (-q_i^2)^{-1/2}\delta\left(\sinh\alpha_i - \frac{(-q_i^2)^{1/2}}{2M_i}\right) \times\left(\frac{-q_i^2}{4M_i^2}\right)G_M^2(q_i^2) , \\ W_2^{(i)} &= (-q_i^2)^{-1/2}\delta\left(\sinh\alpha_i - \frac{(-q_i^2)^{1/2}}{2M_i}\right) \times \frac{G_E^2(q_i^2) - (q_i^2/4M_i^2)G_M^2(q_i^2)}{1 - q_i^2/4M_i^2} , \end{aligned} \tag{2.39}$$

where G_E and G_M are the usual electric and magnetic form factors and have the normalization

$$G_E(0) = 1 , \quad G_M(0) = \mu_i ,$$

where μ_i is the total magnetic moment of the particle a_i. ($\mu_p = 2.79$ for the proton.) When a_i is an electron, as in the case of electron-positron (electron) scattering and trident production, we have $G_E(q^2) = G_M(q^2)/\mu = 1$ ($\mu = 1$), and if a_i is a nucleon, G_E and G_M can be given by a well-known empirical parametrization. In particular, we have chosen the dipole form factor

$$G_M(q_i^2)/\mu = G_E(q_i^2) = \left(1 - \frac{q_i^2}{0.71}\right)^{-2} \tag{2.40}$$

for the application in Sec. III.

As seen from Eq. (2.39), the δ function in $W_1^{(i)}$ and $W_2^{(i)}$ makes the integration over $\sinh\alpha_i$ trivial. When the integration over both $\sinh\alpha_1$ and $\sinh\alpha_2$ involve such δ functions, the integrand in Eq. (2.35) furthermore becomes a simple function of α_{12}, which can be integrated analytically.

For the inelastic contribution, if a_i is a nucleon and X_i represents inclusive multiparticle states, the observed scaling property of the structure function makes the integration over $\sinh\alpha_i$ very simple, also. For the application in Sec. III, we choose the parametrization of Bloom and Gilman,[14]

$$\nu W_2^{(i)} = \left[0.557\left(1 - \frac{1}{\omega'}\right)^3 + 2.1978\left(1 - \frac{1}{\omega'}\right)^4 - 2.5954\left(1 - \frac{1}{\omega'}\right)^5\right], \tag{2.41}$$

with $R = (W_1^{(i)} + \cosh^2\alpha_i\, W_2^{(i)})/W_1^{(i)} \approx 0.18$, for deep-inelastic scattering on a proton target. Also, ν is the usual $p\cdot q/M$ and ω' is the modified scaling variable,

$$\omega' = \frac{2M\nu}{(-q^2)} + \frac{M^2}{(-q^2)} .$$

We modify W_2 in Eq. (2.41) by the factor $-q^2/(-q^2 + a^2)$ with $a^2 = 0.15$ GeV^2 to satisfy the gauge-invariance condition $W_2(0, \nu) = 0$, and to match on to the observed real photoabsorption cross section at large s, $\sigma_{\gamma p}(s) = 8\pi^2\alpha\, W_1(0, \nu)/(s - M^2) \approx 150\ \mu$b, at $q^2 = 0$. Since $\nu = (-q^2)^{1/2}\sinh\alpha$ and the parametrization is simple powers of ω'^{-1}, the integrations over the subenergies $\sinh\alpha_1$ and $\sinh\alpha_2$ can be performed analytically in Eq. (2.35). The details of the subenergy integration are included in an appendix.

C. Photon-Photon Helicity Amplitudes

We give here the c.m. helicity amplitudes for the process $\gamma^*(q_1) + \gamma^*(q_2) \to \mu(l_1) + \bar{\mu}(l_2) \to \gamma^*(q_1) + \gamma^*(q_2)$. These amplitudes are found by a straightforward, but tedious, calculation with the specialization of the polarization basis Eq. (2.20) to the center-of-mass system. The results are:

$$A_{0000}=\frac{(-q_1^2)(-q_2^2)}{q^4}\rho(Q^2)\{q^2Q^2[C_0(-2,0)-C_0(-1,-1)]-2K^2q^2[C_2(-2,0)-C_2(-1,-1)]\},$$

$$\begin{aligned}\operatorname{tr}A&=g^{\lambda\mu}g^{\alpha\beta}q_{\mu\alpha\lambda\beta}\\&=\rho(Q^2)\{-8+4C_0(-1,0)(-Q^2+q_1^2+q_2^2+2\mu^2)-4C_0(-2,0)(2\mu^2+q_1^2)(2\mu^2+q_2^2)\\&\quad+4C_0(-1,-1)[Q^2(q_1^2+q_2^2+\mu^2)-\mu^2(q_1^2+q_2^2)-4\mu^2]\}\\&=2A_{1111}+2A_{1-11-1}-2A_{0101}-2A_{1010}+A_{0000},\end{aligned}$$

$$\begin{aligned}A_{0\alpha}{}^{\alpha}&=-2A_{0101}+A_{0000}\\&=-q_1^2\frac{1}{q^2Q^2}\rho(Q^2)\{4Q^2+4\mu^2C_0(-1,0)(Q^2+q_2^2-q_1^2)+2q_2^2Q^2C_0(-2,0)(2m^2+q_2^2)\\&\quad-2q_1^2C_0(-1,-1)[Q^2(q_1^2+q_2^2)+2\mu^2(q_2^2-q_1^2)]\},\end{aligned} \tag{2.42}$$

$$A^{\lambda}{}_{\lambda_0}=-2A_{1010}+A_{0000}-2A_{0101}+A_{0000},\quad \text{with } q_1^2\to q_2^2;$$

$$\begin{aligned}A_{yyyy}&=\tfrac{1}{2}(A_{1111}+A_{1-11-1}+A_{11-1-1})\\&=\rho(Q^2)[q^2Q^2C_0(-2,0)-4K^2q^2C_2(-2,0)+2(q_1^2+q_2^2)K^2S_2(-2,0)-4K^2S_2(-1,0)-6K^4S_4(-2,0)\\&\quad+2K^2Q^2S_2(-2,0)-q^2Q^2C_0(-1,-1)+4K^2q^2C_2(-1,-1)-6K^4S_4(-1,-1)],\end{aligned}$$

$$A_{01-10}-A_{1100}=[(-q_1^2)^{1/2}(-q_2^2)^{1/2}/q^2]\tfrac{1}{2}\left(A_{llll}\frac{q^4}{(-q_1^2)(-q_2^2)}A_{0000}-A_{1010}\frac{q^2}{-q_1^2}-A_{0101}\frac{q^2}{q_2^2}-A_{yyyy}\right),$$

where A_{yyyy} is the projection of Eq. (2.6b) with the vector (0, 0, 1, 0) and where A_{llll} is the projection of the $\lambda\beta\mu\alpha$ of Eq. (2.6b) with a lightlike vector $l=(1,0,1,0)$ and is given by

$$\begin{aligned}A_{llll}=\rho(Q^2)\Big\{&(q_2^2-q_1^2)^2[C_0(-2,0)-C_0(-1,-1)]-(q_2^2-q_1^2)^2\frac{2K^2}{Q^2}[S_2(-2,0)-S_2(-1,-1)]\\&+2Q^2K^2[S_2(-2,0)+S_2(-1,-1)]-6K^4[S_4(-2,0)+S_4(-1,-1)]\Big\}.\end{aligned} \tag{2.43}$$

In the above equations, we have used the usual phase-space and c.m. quantities

$$\begin{aligned}\rho(Q^2)&=\left(\frac{Q^2-4\mu^2}{Q^2}\right)^{1/2},\\K^2&=\frac{Q^2-4\mu^2}{4},\\q^2&=\frac{1}{4Q^2}[(Q^2)^2-2Q^2(q_1^2+q_2^2)-(q_1^2-q_2^2)^2].\end{aligned} \tag{2.44}$$

The functions $S_i(m,n)$ and $C_i(m,n)$, $i=0,1,2$, $m,n=0,-1,-2$, arise from angular integrations over products of the lepton propagators in Eq. (2.3):

$$\begin{aligned}S_i(m,n)&=\int^1 d\cos\theta[(l_1-q_1)^2-\mu^2]^m[(l_1-q_2)^2-\mu^2]^n(\sin\theta)^i,\\C_i(m,n)&=\int^1 d\cos\theta[(l_1-q_1)^2-\mu^2]^m[(l_1-q_2)^2-\mu^2]^n(\cos\theta)^i,\end{aligned} \tag{2.45}$$

where θ is the polar angle of the muon pair in the two-photon center-of-mass system. The results are:

$$
\begin{aligned}
S_0 &= C_0\,,\\
S_2 &= C_0 - C_2\,,\\
S_4 &= C_0 - 2C_2 + C_4\,,\\
C_2 &= \lambda^2 C_0 + \frac{\lambda}{Kq} C_0 + \frac{1}{2K^2q^2}\,,\\
C_4 &= \lambda^4 C_0 + \frac{1}{4K^2q^2}(6\lambda + \tfrac{2}{3}) + \frac{\lambda^3}{Kq} C_0\,,\\
C_0(-1,0) &= \frac{1}{2Kq} \ln\frac{t_1^+ - \mu^2}{t_1^- - \mu^2}\,,\\
C_0(-2,0) &= \frac{2}{(t_1^+ - \mu^2)(t_1^- - \mu^2)}\,,\\
C_0(-1,-1) &= \frac{1}{\frac{1}{2}(Q^2 - q_1^2 - q_2^2)} C_0(-1,0)\,,\\
t_1^{\pm} - \mu^2 &= \tfrac{1}{2}(Q^2 - q_1^2 - q_2^2) \pm 2Kq\,,
\end{aligned}
\tag{2.46}
$$

and

$$\lambda = \tfrac{1}{2}(Q^2 - q_1^2 - q_2^2)\,\frac{1}{2Kq}\;.$$

This completes the preliminaries as far as kinematics and contains all of the necessary helicity projections for Eq. (2.35) and Eq. (2.50) in Sec. II D.

D. Formulation in Terms of Invariants

For completeness and some particular applications, it is desirable to have an expression for the integrand directly in terms of the invariant variables of integration as defined in Eq. (2.14). First, the volume element can easily be translated in terms of invariants by use of Eqs. (2.35), (2.24), (2.14a), and (2.14b) in Eqs. (2.29) and (2.34), which gives the result

$$d^4q_1 d^4q_2 = \frac{1}{128(s)^{1/2}k}\,\frac{1}{\tau}\, dq_1^2 dq_2^2 ds_1 ds_2 ds_{12} ds_{21} ds' d\psi'\,, \tag{2.47}$$

where

$$
\begin{aligned}
\tau = \{[&-q_1^2q_2^2 + q_1^2\nu_{12}^2 + \nu_{21}^2q_2^2 + q_2^2\nu_1^2 + q_2^2\nu_2^2 + (q_1\cdot q_2)^2]M_1^2M_2^2 + (q_1\cdot q_2)^2q_1^2q_2^2 - \nu_{12}^2\nu_{21}^2M_1^2M_2^2\\
&- (q_1\cdot q_2)^2(p_1\cdot p_2)^2 - \nu_1^2\nu_2^2M_1^2M_2^2 - 2p_1\cdot p_2M_1^2q_2^2\nu_{12}\nu_2 - 2p_1\cdot p_2M_2^2q_2^2\nu_{21}\nu_1 + 2q_1\cdot q_2M_1^2M_2^2\nu_{12}\nu_1\\
&+ 2q_1\cdot q_2M_1^2M_2^2\nu_{21}\nu_2 + 2M_1M_2\nu_{12}\nu_{21}q_1\cdot q_2p_1\cdot p_2 + 2\nu_{12}\nu_{21}\nu_2M_1^2M_2^2 + 2\nu_1\nu_2q_1\cdot q_2p_1\cdot p_2M_1M_2\}^{1/2}\,,
\end{aligned}
\tag{2.48}
$$

with the obvious identifications

$$
\begin{aligned}
\nu_i &= p_i\cdot q_i/M_i\\
&= (s_i - q_i^2 - M_i^2)/2M_i\,, \quad i = 1, 2\\
\nu_{ij} &= p_i\cdot q_j/M\\
&= (s_{ij} - q_j^2 - M_i^2)/2M_i\,, \quad i \neq j = 1, 2\,.
\end{aligned}
\tag{2.49}
$$

The formula for the differential cross section then takes the symmetric form

$$
\begin{aligned}
\frac{d\sigma}{d(Q^2)^{1/2}} = \frac{M_1M_2(Q^2)^{1/2}}{64(sk^2)^{1/2}}\, 2\left(\frac{\alpha^4}{\pi^2}\right)
&\times \int \frac{1}{\tau}\, dq_1^2 dq_2^2 ds_1 ds_2 ds_{12} ds_{21} d\psi'\\
&\times \Big\{ W_1^{(1)} W_2^{(2)} \mathrm{tr}A - W_1^{(1)} W_2^{(2)} \cosh^2\alpha_2 \sum_{n_1} (A_{n_1 1 n_1 1}\sinh^2\zeta_2 + A_{n_1 0 n_1 0}\cosh^2\zeta_2)(-1)^n\\
&\quad - W_2^{(1)} W_1^{(2)} \cosh^2\alpha_1 \sum_{n_2} (A_{1 n_2 1 n_2}\sinh^2\zeta_1 + A_{0 n_2 0 n_2}\cosh^2\zeta_1)(-1)^{n_2}\\
&\quad + W_2^{(1)}\cosh^2\alpha_1 W_2^{(2)}\cosh^2\alpha_2 [\tfrac{1}{2}(A_{1111} + A_{1\,-1 1\,-1} + A_{11\,-1\,-1}\cos 2\psi)\sinh^2\zeta_1\sinh^2\zeta_2\\
&\qquad + A_{1010}\sinh^2\zeta_1\cosh^2\zeta_2 + A_{1010}\cosh^2\zeta_1\sinh^2\zeta_2\\
&\qquad + A_{0000}\cosh^2\zeta_1\cosh^2\zeta_2\\
&\qquad + 2(A_{1100} - A_{100\,-1})\sinh\zeta_1\cosh\zeta_1\sinh\zeta_2\cosh\zeta_2\cos\psi] \Big\}\;.
\end{aligned}
\tag{2.50}
$$

The kinematical factors are written in the most economical form for the differential cross sections when expressed in terms of the hyperbolic angles. However, we can easily express the kinematic coefficients in terms of invariants by use of Eqs. (2.12), (2.14a), (2.14b). The azimuthal angle can be eliminated by calculating $p_1\cdot p_2$ in the Breit frame B which gives the result

$$p_1 \cdot p_2 = M_1 M_2 [\cosh\zeta_1 \cosh\alpha_1 (\cosh\alpha' \cosh\zeta_2 \cosh\alpha_2 + \sinh\alpha' \sinh\alpha_2) + \sinh\alpha_1 (\cosh\alpha' \sinh\alpha_2 + \sinh\alpha' \cosh\zeta_2 \cosh\alpha_2) - \sinh\zeta_1 \cosh\alpha_1 \sinh\zeta_2 \cosh\alpha_2 \cos\psi]. \quad (2.51)$$

The other variables are expressed in terms of invariants trivially; we list here a complete set of relations:

$$\begin{aligned}
s_1 &= M_1^2 + q_1^2 + 2M_1(-q_1^2)^{1/2}\sinh\alpha_1 ,\\
s_{12} &= M_1^2 + q_2^2 + 2M_1(-q_2^2)^{1/2}\sinh\alpha_{12} ,\\
\cosh\alpha_1 \cosh\zeta_1 &= \frac{\sinh\alpha_{12} - \sinh\alpha_1 \cosh\alpha'}{\sinh\alpha'} ,\\
s_2 &= M_2^2 + q_2^2 + 2M_2(-q_2^2)^{1/2}\sinh\alpha_2 ,\\
s_{21} &= M_2^2 + q_1^2 + 2M_2(-q_1^2)^{1/2}\sinh\alpha_{21} , \qquad (2.52)\\
\cosh\alpha_2 \cosh\zeta_2 &= \frac{\sinh\alpha_{21} - \sinh\alpha_2 \cosh\alpha'}{\sinh\alpha'} ,\\
\cosh\alpha' &= (Q^2 - q_1^2 - q_2^2)/2(-q_1^2)^{1/2}(-q_2^2)^{1/2} ,
\end{aligned}$$

and

$$\sinh\alpha' \geq 0,$$

which follows from positivity of the energy of $Q_0 \geq 0$ and

$$\sinh\zeta_1 \text{ and } \sinh\zeta_2 \geq 0 ,$$

which are a restatement of the Kibble conditions [Eqs. (2.38)].

The region of integration for the invariants is easily computed by mapping Eqs. (2.38) onto the invariants and requiring that

$$\tau^2 \geq 0 .$$

We have demonstrated that this latter requirement is equivalent to the condition that the azimuthal angles $\cos\psi$ and $\cos\varphi$ of Eqs. (2.15) and (2.26) be physical, viz $|\cos\varphi|$ and $|\cos\psi| \leq 1$.

III. APPLICATION TO PROTON-PROTON SCATTERING

In this section, we apply the results of Sec. II to compute the contribution to the muon-pair mass spectrum from the two-photon process in Fig. 1(b) for proton-proton scattering, discuss the physical implications, and compare our exact calculation with some approximation schemes.

For the proton-proton scattering, we include the elastic contribution with the structure functions given by Eqs. (2.39) and (2.41), and the deep-inelastic contribution with the structure functions given by Eq. (2.41), while the resonance contributions are included in Eq. (2.41) in the average sense.[14] The total incident energies $(s)^{1/2}$ are chosen to agree with the Brookhaven-Columbia experiments at $s = 56.3$ GeV^2, and the future experiments at the National Accelerator Laboratory (NAL), CERN Intersecting Storage Rings (ISR), and Isabelle with $s = 1000$, 2500, and 10^5 GeV^2, respectively. For the inelastic contribution at the high values of s, namely, $s = 2500$ and 10^5 GeV^2, we use a modified asymptotic form for the structure functions with

$$\nu W_2 = 0.2\left(\frac{-q^2}{-q^2 + 0.15}\right) , \qquad (3.1)$$

which well approximates the Bloom and Gilman form given by Eq. (2.41) over most region of integration and improves the stability of the numerical integration to be described below. We checked Eq. (3.1) against Eq. (2.41) at $s = 1000$ GeV^2 and also partially at $s = 2500$ GeV^2 with an agreement good to a few percent.

The numerical evaluation of Eq. (2.35) is performed in the following way: The integrations over $\sinh\alpha_1$ and $\sinh\alpha_2$ are performed analytically for both the elastic and inelastic contributions as described in Sec. II and in the appendix. Then the integrations over α_{12}, q_1^2, and q_2^2 are performed numerically with a Monte Carlo integration routine to an estimated accuracy of less than 1%. Tables I through IV and Fig. 3 include a summary of the results.

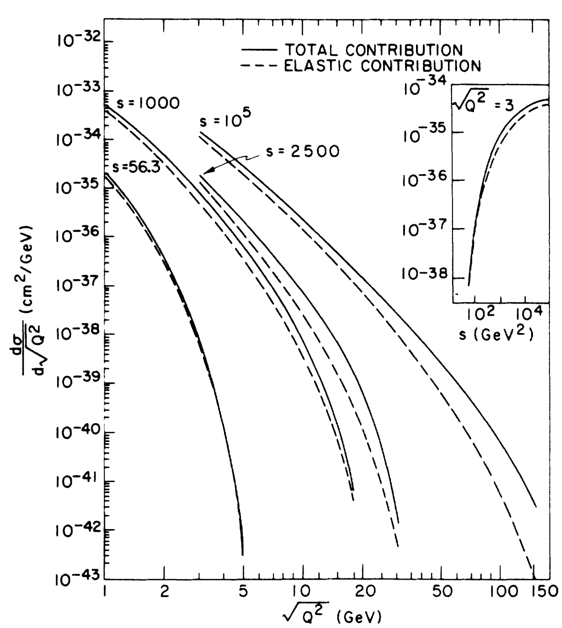

FIG. 3. Differential cross section $d\sigma/d(Q^2)^{1/2}$ for various s values as a function of Q^2; the dashed line represents elastic-elastic contribution [Eq. (2.39)]; solid line represents the total contribution.

TABLE I. Numerical results at $s=56.3$ GeV^2. The error represents the estimated accuracy in the Monte Carlo integration.

		el.-el.		tot	
$(Q^2)^{1/2}$ (GeV)	$\rho=Q^2/s$	$d\sigma/d(Q^2)^{1/2}$ (cm^2/GeV)	$f(s,Q^2)$	$d\sigma/d(Q^2)^{1/2}$ (cm^2/GeV)	$f(s,Q^2)$
0.3	0.001 60	$(1.016\pm0.003)\times10^{-33}$	2.743×10^{-35}	$(1.047\pm0.003)\times10^{-33}$	2.826×10^{-35}
1	0.0178	$(1.736\pm0.005)\times10^{-35}$	1.736×10^{-35}	$(2.049\pm0.006)\times10^{-35}$	2.049×10^{-35}
2	0.0775	$(2.943\pm0.009)\times10^{-37}$	2.345×10^{-36}	$(3.654\pm0.011)\times10^{-37}$	2.923×10^{-36}
3	0.1599	$(7.561\pm0.025)\times10^{-39}$	2.041×10^{-37}	$(8.868\pm0.027)\times10^{-39}$	2.394×10^{-37}
4	0.2842	$(1.125\pm0.004)\times10^{-40}$	7.200×10^{-39}	$(1.204\pm0.004)\times10^{-40}$	7.707×10^{-39}
5	0.4441	$(2.252\pm0.007)\times10^{-43}$	2.814×10^{-41}	$(2.262\pm0.007)\times10^{-43}$	2.827×10^{-41}

The results are shown for both the elastic contribution, namely, $p+p\to p+p+\mu^++\mu^-$, and the total contribution as functions of $(Q^2)^{1/2}$ at different values of s. We would like to make the following general comments concerning the physical implication of the results.

(1) At $s=56.3$ GeV^2, the two-photon contribution is negligible compared to the experimental data of Ref. 15, which indicates that at this energy there is very little background due to the two-photon process.

(2) The cross section increases very rapidly with s at fixed values of $(Q^2)^{1/2}$ between $s=56.3$ and 1000 GeV^2 and gives observable values at s greater than 1000 GeV^2 even at $(Q^2)^{1/2}=10$ GeV. The cross section asymptotically increases as $(\ln s)^3$. If one assumes that the α^2 one-photon process dominates the observed data at $s=56.3$ GeV^2 and in addition, that this process has already reached the scaling limit in s/Q^2, then the two-photon process becomes comparable with the one-photon process extrapolated to $s=10^5$ GeV^2. Therefore, the two-photon process has a good chance to be one of the main contributions at high s and very important.

(3) At high s values, the inelastic and elastic contributions become comparable and independently observable. The nonelastic part receives a substantial contribution from the scaling region with large s_i and $-q_i^2$. Since in this region the muon pair will have a sizable total transverse momentum, this part can be separated out to provide a test of the Bjorken scaling in W_1 and W_2 in the average sense at very high energies.

(4) At $s=1000$ GeV^2, the elastic contribution is large enough so that one may have a good chance to scatter pions and kaons coherently on heavy nucleon targets to measure their electromagnetic structure as suggested by Geshkenbein and Terentyev.[8]

(5) One might worry that the angular distribution of the lepton pair from the two-photon process might be forward peaked in the beam direction to the extent that the pair may not be well enough separated from the beam direction to be observable. Although the angular distribution can be straightforwardly calculated, it is sufficient to have a qualitative estimation at the present stage. Since each of the muons are preferentially emitted in the direction of the muon-pair total momentum $\vec{Q}$, it is sufficient to estimate the portion of the cross section in which $\vec{Q}$ has a substantial opening angle with respect to the beam. From the previous studies of the two-photon process for π^0 and η^0 production where the π^0 (η^0) angle relative to the beam direction corresponds to the opening angle

TABLE II: Numerical results at $s=1000$ GeV^2. The error represents the error in the Monte Carlo integration.

		el.-el.		tot	
$(Q^2)^{1/2}$ (GeV)	ρ	$d\sigma/d(Q^2)^{1/2}$ (cm^2/GeV)	$f(s,Q^2)$	$d\sigma/d(Q^2)^{1/2}$ (cm^2/GeV)	$f(s,Q^2)$
1	0.001	$(4.209\pm0.016)\times10^{-34}$	4.209×10^{-34}	$(5.326\pm0.020)\times10^{-34}$	5.326×10^{-34}
2	0.004	$(2.959\pm0.011)\times10^{-35}$	2.367×10^{-34}	$(4.537\pm0.018)\times10^{-35}$	3.630×10^{-34}
3	0.009	$(4.887\pm0.020)\times10^{-36}$	1.319×10^{-34}	$(8.473\pm0.033)\times10^{-36}$	2.288×10^{-34}
6	0.036	$(1.235\pm0.006)\times10^{-37}$	2.668×10^{-35}	$(2.529\pm0.011)\times10^{-37}$	5.463×10^{-35}
9	0.081	$(8.101\pm0.042)\times10^{-39}$	5.906×10^{-36}	$(1.700\pm0.079)\times10^{-38}$	1.239×10^{-35}
12	0.144	$(7.096\pm0.044)\times10^{-40}$	1.266×10^{-36}	$(1.406\pm0.007)\times10^{-39}$	2.429×10^{-36}
15	0.255	$(6.245\pm0.044)\times10^{-41}$	2.108×10^{-37}	$(1.106\pm0.006)\times10^{-40}$	3.732×10^{-37}
18	0.324	$(4.206\pm0.034)\times10^{-42}$	2.453×10^{-38}	$(6.732\pm0.044)\times10^{-42}$	3.926×10^{-38}

TABLE III. Numerical results at $s=2500\ \mathrm{GeV}^2$. The error represents the error in the Monte Carlo integration.

		el.-el.		tot	
$(Q^2)^{1/2}$ (GeV)	ρ	$d\sigma/d(Q^2)^{1/2}$ (cm^2/GeV)	$f(s,Q^2)$	$d\sigma/d(Q^2)^{1/2}$ (cm^2/GeV)	$f(s,Q^2)$
3	0.0036	$(1.207\pm0.005)\times10^{-35}$	3.260×10^{-34}	$(1.870\pm0.007)\times10^{-35}$	5.051×10^{-34}
6	0.0144	$(4.689\pm0.022)\times10^{-37}$	1.013×10^{-34}	$(9.774\pm0.035)\times10^{-37}$	2.111×10^{-34}
9	0.0326	$(4.963\pm0.026)\times10^{-38}$	3.618×10^{-35}	$(1.316\pm0.005)\times10^{-37}$	9.592×10^{-35}
12	0.0576	$(7.958\pm0.047)\times10^{-39}$	1.375×10^{-35}	$(2.642\pm0.010)\times10^{-38}$	4.565×10^{-35}
15	0.09	$(1.552\pm0.010)\times10^{-39}$	5.237×10^{-36}	$(6.545\pm0.023)\times10^{-39}$	2.209×10^{-35}
20	0.16	$(1.242\pm0.010)\times10^{-40}$	9.933×10^{-37}	$(8.133\pm0.026)\times10^{-40}$	6.506×10^{-36}
30	0.36	$(4.877\pm0.009)\times10^{-43}$	1.317×10^{-38}	$(1.628\pm0.004)\times10^{-41}$	4.396×10^{-37}

for $\vec{Q}$, only about half of the cross section is lost by making an angular cut of $(m/E)^{1/4}$, where $E=\frac{1}{2}s^{1/2}$. This cutoff is of the order of 15° even for $E=200$ GeV. In addition, as noted in (3), the deep-inelastic region will always give $\vec{Q}$ a larger angle.

Next, we make the following remarks concerning the comparison of our exact result with some approximation schemes and some general features of the results.

(i) For the elastic contribution, the results of this calculation at $s=56.3\ \mathrm{GeV}^2$ and $q^2=4\ \mathrm{GeV}^2$ agrees with that of Fujikawa within a factor of 2. At $s=1000\ \mathrm{GeV}^2$ and $(Q^2)^{1/2}\leq 3$ GeV, the agreement is within 25%, while for $(Q^2)^{1/2}\geq 3$ GeV, the agreement is almost exact.

(ii) We set the proton form factor equal to unity and the total moment equal to the Dirac moment in the elastic contribution Eq. (2.39) and compared the result of this calculation with the equivalent-photon result:

$$\frac{d\sigma}{d(Q^2)^{1/2}}\simeq 4\left(\frac{\alpha}{\pi}\right)^2\left(\ln\frac{(s)^{1/2}}{2M}-\frac{1}{2}\right)^2\frac{\sigma_T(Q^2)}{(Q^2)^{1/2}}\times\left[\frac{1}{2}\left(2+\frac{Q^2}{s}\right)^2\ln\frac{s}{Q^2}-\left(1-\frac{Q^2}{s}\right)\left(3+\frac{Q^2}{s}\right)\right],$$

where

$$\sigma_T(Q^2)\simeq\frac{4\pi\alpha^2}{Q^2}\left[\ln\frac{Q^2}{\mu^2}-1\right],\quad Q^2\gg\mu^2. \tag{3.2}$$

The result of this comparison is shown in Fig. 4 at $s=1000\ \mathrm{GeV}^2$ and $s=10^5\ \mathrm{GeV}^2$. It is easily seen that the equivalent-photon spectrum agrees with the result of the exact calculation to within 20%. This does not imply that the equivalent-photon spectrum Eq. (3.2) agrees with the exact calculation with proton structure, Fig. 3, or that this agreement in Fig. 4 is license to assume that the equivalent-photon spectrum is always accurate for other particles, e.g., electrons, instead of protons at high values of s. We hope to report on this question for electron-positron colliding beams in a future effort.

(iii) As shown in Fig. 3, for a fixed value of Q^2, the cross section increases rapidly with s at intermediate energies and only increases logarithmically at very high energies. A leading-log estimation of the cross section in Eq. (2.35) gives a $(\ln s)^3$ increase of cross section similar to that shown in Eq. (3.2). A theoretical estimation shows that the effective rapid increasing of the cross

TABLE IV. Numerical results at $s=10^5\ \mathrm{GeV}^2$. The error represents the error in the Monte Carlo integration.

		el.-el.		tot	
$(Q^2)^{1/2}$ (GeV)	ρ	$d\sigma/d(Q^2)^{1/2}$ (cm^2/GeV)	$f(s,Q^2)$	$d\sigma/d(Q^2)^{1/2}$ (cm^2/GeV)	$f(s,Q^2)$
3	9×10^{-5}	$(1.166\pm0.005)\times10^{-34}$	3.147×10^{-33}	$(1.481\pm0.008)\times10^{-34}$	3.999×10^{-33}
6	3.6×10^{-4}	$(9.516\pm0.044)\times10^{-36}$	2.055×10^{-33}	$(1.424\pm0.006)\times10^{-35}$	3.075×10^{-33}
9	8.1×10^{-4}	$(1.967\pm0.010)\times10^{-36}$	1.434×10^{-33}	$(3.368\pm0.019)\times10^{-36}$	2.455×10^{-33}
12	0.001 44	$(6.088\pm0.033)\times10^{-37}$	1.052×10^{-33}	$(1.167\pm0.005)\times10^{-36}$	2.016×10^{-33}
15	0.002 25	$(2.368\pm0.013)\times10^{-37}$	7.990×10^{-34}	$(4.976\pm0.029)\times10^{-37}$	1.680×10^{-33}
20	0.004	$(6.617\pm0.040)\times10^{-38}$	5.294×10^{-34}	$(1.599\pm0.007)\times10^{-37}$	1.279×10^{-33}
30	0.009	$(9.743\pm0.067)\times10^{-39}$	2.631×10^{-34}	$(2.941\pm0.013)\times10^{-38}$	7.941×10^{-34}
40	0.016	$(2.205\pm0.016)\times10^{-39}$	1.411×10^{-34}	$(8.165\pm0.042)\times10^{-39}$	5.226×10^{-34}
50	0.025	$(6.330\pm0.053)\times10^{-40}$	7.912×10^{-35}	$(2.839\pm0.013)\times10^{-39}$	3.549×10^{-34}
60	0.036	$(2.143\pm0.019)\times10^{-40}$	4.629×10^{-35}	$(1.150\pm0.005)\times10^{-39}$	2.484×10^{-34}

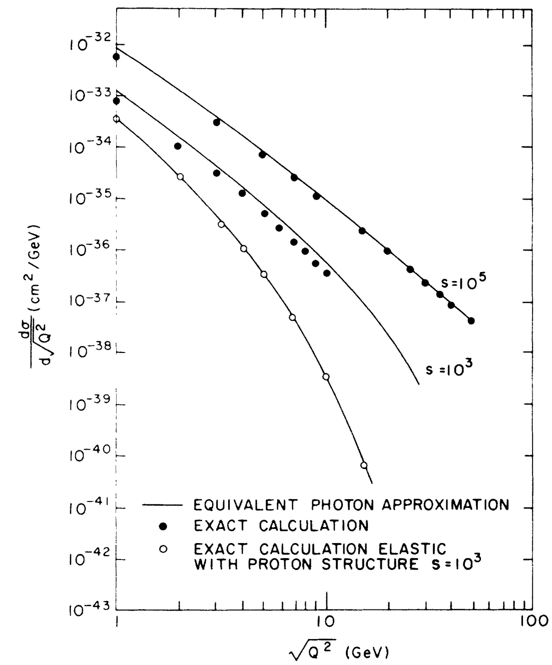

FIG. 4. Comparison of equivalent-photon approximation to exact calculation for elastic with no proton structure.

section is a result of the cancellation of the leading-log contribution by the next leading contribution at low energies. Thus, we have to be very careful in using leading-log approximations and, of course, the region of validity for such approximations depends on the particular processes under consideration. Of course, we have left out many other contributions to the same order in α, some of which are sketched in Fig. 5.

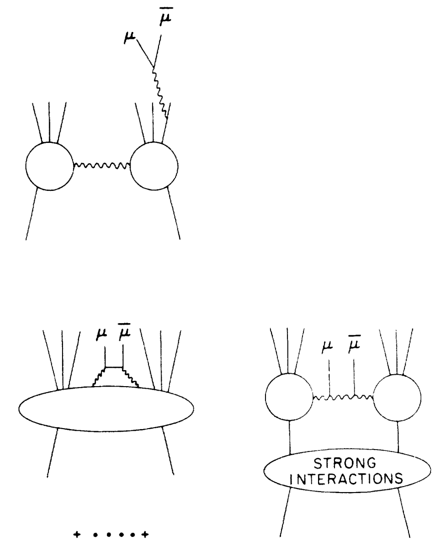

FIG. 5. Other α^4 contributions.

At low energies, it is difficult to assess the magnitude of the other α^4 graphs. However, it is safe to say that their effects will generally be additive since one does not expect the interference terms between different graphs (non-positive-definite terms) to be important except in very small regions of phase space. At high energies, when the leading-log behavior takes hand, we expect the other graphs, with the exception of those of the initial- and final-state interactions, to be less important. A rough guide to the magnitude of the initial-state interaction, the last graph in Fig. 5, can be obtained by the following argument.

The effects of initial- and final-state interactions can be estimated by a distorted-wave approximation where the corrected electromagnetic amplitude is given by a formula of the form

$$B(s,\vec{\Delta}) = B^0(s,\vec{\Delta}) + \frac{i}{4\pi k}\int d^2q\, B^0(s,\vec{q})f^{st}(s,\vec{\Delta}-\vec{q}), \quad (3.3)$$

where $B^0(s,\vec{\Delta})$ is the α^2 contribution of Fig. 1(a) or the α^4 amplitude of Fig. 1(b). In Eq. (3.3), f^{st} is the strong-interaction amplitudes given approximately by

$$f^{st}(s,q) = i\frac{k}{4\pi}\sigma_T e^{(b/2)q^2}, \quad (3.4)$$

where σ_T is the total cross section for the hadron-hadron scattering, b is the width of the diffraction peak, and k the c.m. momentum.

We assume that $B^0(s,\vec{q})$ is more slowly varying in momentum transfer than the strong amplitude Eq. (3.4); an application of Eq. (3.3) gives us the following approximate formula at small momentum transfer:

$$B(s,0) \approx B^{(0)}(s,0)\left(1-\frac{\sigma_T}{8\pi b}\right), \quad (3.5)$$

which gives a reduction of the magnitude, but not the form, of the electromagnetic amplitude. This argument which has been used to a great extent in the literature and is only to serve as a rough guide of the size of the absorptive effects. (In the context of the one photon exchange, see, for example, Low and Treiman in Ref. 16.) Typically, the size of the effect in Eq. (3.5) is 20% to 30%, and perhaps as large as 60%, of both initial- and final-state interactions of the hadrons coming from the clusters in Fig. 1(b) are taken into account by a formula of the type in Eq. (3.4).

(iv) We have also noticed an approximate scaling in the function $f(s, Q^2)$ defined by

$$f(s, Q^2) = (Q^2)^{3/2} d\sigma / d(Q^2)^{1/2}$$

in the invariable $Q^2/s = \rho$. This is similar to the parton-model prediction for the α^2 one-photon process that $f(s, Q^2)$ becomes a function of only one variable Q^2/s, except that, for the two-photon process, such a scaling is only approximate and has a logarithmic violation in the variable s. In particular, for the elastic contribution at very small values of Q^2/s, where the effects of the proton form factor can be neglected, $f(s, Q^2)$ can be estimated from Eqs. (3.2) and (3.3) and clearly show an approximation scaling with logarithmic violation. For other regions of Q^2/s, and also for the total contribution, similar properties can also be shown from analytical but tedious estimations of Eq. (2.35), and the logarithmic factors violating the scaling are different in different regions of Q^2/s. We feel that it is more transparent to exhibit the numerical results in Fig. 6, where $f(s, Q^2)$ is plotted as a function of Q^2/s at several different values of s for both the elastic and total contributions.

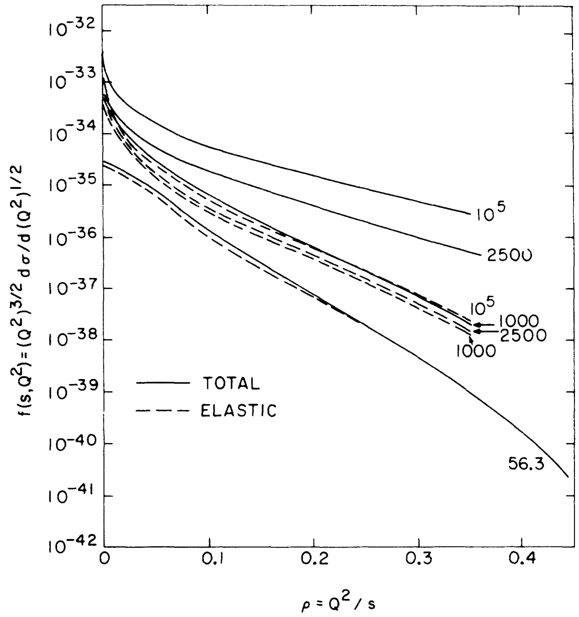

FIG. 6. Proposed scaling function $f(s, Q^2) = (Q^2)^{3/2} d\sigma / d(Q^2)^{1/2}$.

IV. CONCLUSION

To conclude, we have formulated an exact expression, given by Eqs. (2.35) and (2.50), for the lepton pair mass spectrum from the two-photon processes shown in Fig. 1(b). This expression can be easily applied to lepton-lepton, lepton-hadron, and hadron-hadron scattering processes. Depending on each particular reaction and the energy, such α^4 processes can be either the dominating process or an important and calculable background.

In particular, we have applied our results to the muon pair production in proton-proton scattering. We would like to stress the importance of the two-photon process for the muon pair production in hadron-hadron collisions. Namely, it will be necessary to have knowledge of the two-photon contribution in order to extract the α^2 one-photon process. In addition, the two-photon process will reveal its own physical significance if the one-photon contribution happens to be small at high s and moderately high Q^2 such as predicted by Altarelli, Brandt, and Preparata,[2] for example.

Furthermore, the graphs in Fig. 1(b) for the two-photon contribution may allow for the determination of the pion and kaon electromagnetic structure if the angular distribution can be sorted out. Finally, we would like to emphasize that the forthcoming experiments at the ISR and NAL accelerators will provide for much-needed information on the physics of lepton pair production.

Finally, we would like to comment that two other papers have come to our attention since preparing this work. A paper by Budnev, Ginzburg, Meledin, and Serbo[17] suggests using the process $pp \to pp + e^+e^-$ as a monitor for the luminosity of colliding-beam accelerators. Their emphasis and kinematic region in s and Q^2 for these studies are quite different from this work. They give some approximate analytic formulas for the region of small dilepton mass.

Another work by Soni[18] gives some approximate analytic formula for the leading-logarithmic contribution of the graphs in Fig. 1(b).

APPENDIX

In this appendix, we discuss the integration over $\sinh\alpha_1$ and $\sinh\alpha_2$ for the proton-proton collision process. This result can also be trivially modified to apply to the trident production process. To separately perform the integration in Eq. (2.35) over $\sinh\alpha_1$ and $\sinh\alpha_2$ analytically, we choose the particular order $\sinh\alpha_1$, $\sinh\alpha_2$, α_{12}, $q_1{}^2$, and $q_2{}^2$. The integration limits for this order can be solved from Eqs. (2.38) as

$$\begin{aligned}
\alpha_{1\,\mathrm{min}} &= \alpha_{10} = \sinh^{-1}\left[\frac{p_{10}-q_1^2-m_p^2}{2m_p(-q_1^2)^{1/2}}\right],\\
\alpha_{1\,\mathrm{max}} &= \alpha_{12}-\alpha',\\
\alpha_{2\,\mathrm{min}} &= \alpha_{20} = \sinh^{-1}\left[\frac{p_{20}-q_2^2-m_p^2}{2m_p(-q_2^2)^{1/2}}\right],\\
\alpha_{2\,\mathrm{max}} &= \alpha-\alpha_{12},\\
\alpha_{12\,\mathrm{min}} &= \alpha'+\alpha_{10},\\
\alpha_{12\,\mathrm{max}} &= \alpha-\alpha_{20},\\
(q_1^2)_{\mathrm{max,min}} &= m_p^2+s_{10}-\frac{1}{2s_{12\,\mathrm{max}}}\{(s_{12\,\mathrm{max}}+m_p^2-q_2^2)(s_{12\,\mathrm{max}}+s_{10}-Q^2)\\
&\qquad \pm[(s_{12\,\mathrm{max}}+m_p^2-q_2^2)^2-4s_{12\,\mathrm{max}}m_p^2]^{1/2}[(s_{12\,\mathrm{max}}+s_{10}-Q^2)^2-4s_{12\,\mathrm{max}}s_{10}]^{1/2}\},\\
(q_2^2)_{\mathrm{max,min}} &= m_p^2+s_{20}-\tfrac{1}{2}(s+s_{20}-s_{12\,\mathrm{min}})\pm\frac{1}{2s}(s^2-4ms)^{1/2}[(s+s_{20}+s_{12\,\mathrm{min}})^2-4ss_{20}]^{1/2},
\end{aligned} \tag{A1}$$

where

$$s_{12\,\mathrm{max}} = m_p^2+q_2^2+2m_p(-q_2^2)^{1/2}\sinh\alpha_{12\,\mathrm{max}}\,,$$

$$s_{12\,\mathrm{min}} = [(s_{10})^{1/2}+(Q^2)^{1/2}]^2,$$

$$\alpha' = \cosh\left[\frac{Q^2-q_1^2-q_2^2}{2(-q_1^2)^{1/2}(-q_2^2)^{1/2}}\right].$$

s_{i0} is the lowest value of the invariant mass squared for the system X_i, that is, $s_{i0}=m_p^2$ for the elastic contribution and $s_{i0}=(m_p+m_\pi)^2$ for the inelastic contribution, $i=1,2$, and m_p and m_π are the proton mass and the pion mass, respectively. We can also always fix $s_{i0}=m_p^2$ and put appropriate δ functions in the integrand for inelastic contributions.

Since the elastic structure functions contain θ functions of $\sinh\alpha_1$ and $\sinh\alpha_2$, the integration over these two variables is trivial. For the inelastic contribution, since the photon-photon amplitude is independent of $\sinh\alpha_1$ and $\sinh\alpha_2$, and the limits of integration for these two variables are independent of each other, we can perform the integrations separately. With a simple parametrization of the inelastic structure functions given by Eq. (2.41), the result can be written in terms of integrals of some rational functions of $\sinh\alpha_1$ and $\sinh\alpha_2$. We present the result

$$\frac{d\sigma}{d(Q^2)^{1/2}} = \frac{4\alpha^4}{\pi}\frac{1}{\sinh\alpha}\frac{1}{(s)^{1/2}k}(Q^2)^{1/2}m_p^2\int\frac{dq_2^2}{(-q_2^2)^2}\frac{dq_1^2}{(-q_1^2)^2}d\alpha_{12}\,F(\alpha_{12},q_1^2,q_2^2), \tag{A2}$$

where

$$\begin{aligned}
F(\alpha_{12},q_1^2,q_2^2) = {}&\{(I_{12}+I_{22})[(I_{11}+I_{21})\mathrm{tr}A-(3I_{21}+I_{31})A_{0\alpha}{}^{\alpha}]+(3I_{22}+2I_{32})[\tfrac{1}{2}(I_{41}-I_{11})\mathrm{tr}A\\
&-\tfrac{1}{2}(3I_{41}-I_{11})A^{\lambda}{}_{\lambda 0}+(I_{51}-2I_{21})A_{yyyy}+I_{51}A_{1010}+(I_{51}-2I_{21}+I_{61}-2I_{31})A_{0101}\\
&+(I_{51}+I_{61})A_{0000}+I_{71}(A_{1100}-A_{01-10})\}
\end{aligned} \tag{A3}$$

and

$$I_{ij} = E_{ij}+\frac{-q_j^2}{-q_j^2+0.15}N_{ij} \tag{A4}$$

are sums of elastic and inelastic contributions. The elastic contributions are given by:

$$
\begin{aligned}
E_{11} &= F_1(q_1^2)\,,\\
E_{21} &= \tfrac{1}{2}(\cosh^2\alpha_1\sinh^2\zeta 1)_{\mathrm{el}}H_2(q_1^2)\,,\\
E_{31} &= (\cosh\alpha_1)_{\mathrm{el}}H_2(q_1^2)\,,\\
E_{41} &= (\cosh^2\zeta_1')_{\mathrm{el}}H_1(q_1^2)\,,\\
E_{51} &= (\cosh^2\alpha_1\sinh^2\zeta_1\cosh^2\zeta_1')_{\mathrm{el}}H_2(q_1^2)\,,\\
E_{61} &= (\cosh^2\alpha_1\cosh^2\zeta_1')_{\mathrm{el}}H_2(q_1^2)\,,\\
E_{71} &= \cosh^2\alpha_{12}(\cosh\alpha_1\cosh\zeta_1\cosh\zeta_1'\sinh^2\zeta_1')_{\mathrm{el}}H_2(q_1^2)\,,\\
E_{12} &= F_1(q_2^2)\,,\\
E_{22} &= \tfrac{1}{2}(\cosh^2\alpha_2\sinh^2 u)_{\mathrm{el}}H_2(q_2^2)\,,\\
E_{32} &= \tfrac{1}{2}(\cosh^2\alpha_2)_{\mathrm{el}}H_2(q_2^2)\,,
\end{aligned}
\tag{A5}
$$

where the subscript el means that the kinematical factors are evaluated at the elastic point with

$$
(\sinh\alpha_i)_{\mathrm{el}} = \frac{(-q_i^2)^{1/2}}{2m_p} \quad (i=1,2)\,, \tag{A6}
$$

and

$$
\begin{aligned}
H_1(q_i^2) &= \frac{-q_i^2}{4m_p^2}G_M(q_i^2)\,,\\
H_2(q_i^2) &= \frac{G_E(q_i^2)-(q_i^2/4m_p^2)G_M(q_i^2)}{1-q_i^2/4m_p^2}\,,
\end{aligned}
\tag{A7}
$$

where G_E and G_M are the usual electromagnetic form factors, Eq. (2.40), to be approximated by Eq. (2.40). The inelastic contributions are given by:

$$
\begin{aligned}
N_{11} &= \frac{1}{1+R}(J_2^{(1)}+J_0^{(1)})\,,\\
N_{21} &= \frac{1}{2\sinh^2\alpha'}[J_2^{(1)}-2\sinh\alpha_{12}\cosh\alpha' J_1^{(1)}+(\sinh^2\alpha_{12}-\sinh^2\alpha')J_0^{(1)}]\,,\\
N_{31} &= J_2^{(1)}+J_0^{(1)}\,,\\
N_{41} &= \frac{1}{(1+R)\cosh^2\alpha_{12}\sinh^2\alpha'}[J_4^{(1)}-2\sinh\alpha_{12}\cosh\alpha' J_3^{(1)}+\sinh^2\alpha_{12}\cosh^2\alpha' J_2^{(1)}\\
&\qquad -2\sinh\alpha_{12}\cosh\alpha' J_1^{(1)}+\sinh^2\alpha_{12}\cosh^2\alpha' J_0^{(1)}]\,,\\
N_{51} &= \frac{1}{\cosh^2\alpha_{12}\sinh^4\alpha'}[J_4^{(1)}-4\sinh\alpha_{12}\cosh\alpha' J_3^{(1)}+(5\sinh^2\alpha_{12}\cosh^2\alpha'+\sinh^2\alpha_{12}-\sinh^2\alpha')J_2^{(1)}\\
&\qquad -2\sinh\alpha_{12}\cosh\alpha'(\sinh^2\alpha_{12}\cosh^2\alpha'+\sinh^2\alpha_{12}-\sinh^2\alpha')J_1^{(1)}\\
&\qquad +\sinh^2\alpha_{12}\cosh^2\alpha'(\sinh^2\alpha_{12}-\sinh^2\alpha')J_0^{(1)}]\,,\\
N_{61} &= (1+R)N_{41}\,,\\
N_{71} &= \frac{1}{\cosh^2\alpha_{12}\sinh^2\alpha'}\{\cosh\alpha' J_4^{(1)}-\sinh\alpha_{12}(3\cosh\alpha'+1)J_3^{(1)}\\
&\qquad +[3\sinh^2\alpha_{12}\cosh\alpha'(\cosh^2\alpha'+1)-\cosh^2\alpha_{12}\sinh^2\alpha'\cosh\alpha']J_2^{(1)}\\
&\qquad -\sinh\alpha_{12}(\cosh^2\alpha'+1)(\sinh^2\alpha_{12}\cosh^2\alpha'-\cosh^2\alpha_{12}\sinh^2\alpha')J_1^{(1)}\\
&\qquad +\sinh^2\alpha_{12}\cosh\alpha'(\sinh^2\alpha_{12}\cosh^2\alpha'-\cosh^2\alpha_{12}\sinh^2\alpha')J_0^{(1)}\}\,,\\
N_{12} &= \frac{1}{1+R}(J_2^{(i)}+J_0^{(i)})\,,\\
N_{22} &= \frac{1}{2\cosh^2\alpha_{12}}[-J_2^{(2)}-2\cosh\alpha\sinh\alpha_{12}J_1^{(2)}+(\cosh^2\alpha\ \cosh^2\alpha_{12})J_0^{(2)}]\,,\\
N_{32} &= \tfrac{1}{2}(J_2^{(2)}+J_0^{(2)})\,.
\end{aligned}
\tag{A8}
$$

The integrals $J_n^{(i)}$ ($n=1,4$; $i=1,2$) are defined by

$$J_n^{(i)} = \frac{1}{2^n}\left[\frac{m_p}{(-q_i^2)^{1/2}}\right]^{n-1} \times \int_{\omega_{i\,\min}}^{\omega_{i\,\max}} f(\omega)\left[\frac{-q_i^2}{m_p^2}\,\omega - 1\right]^{n-1} d\omega\,, \qquad \text{(A9)}$$

where $f(\omega)$ is the scaling function

$$\begin{aligned} f(\omega) &= \frac{\nu W_2}{m_p} \\ &= a\left(1-\frac{1}{\omega}\right)^3 + b\left(1-\frac{1}{\omega}\right)^4 + c\left(1-\frac{1}{\omega}\right)^5 \\ &\equiv \sum_{i=0}^{5} c_i\omega^{-i}\,, \end{aligned} \qquad \text{(A10)}$$

$$\omega_{i\,\max,\,\min} = \frac{2m_p}{(-q_i^2)^{1/2}}\sinh\alpha_{i\,\max,\,\min} - \frac{m_p^2}{-q_i^2} \qquad (i=1,2)\,. \qquad \text{(A11)}$$

These integrals can be explicitly evaluated to be

$$\int f(\omega)\left(\frac{-q^2}{m_p^2}\,\omega - 1\right)^{n-1} d\omega = \sum_{\substack{m=-4\\ m\neq 0}}^{n} \frac{a_{n,m}\omega^m}{m} + a_{n,0}\ln\omega + \delta_{n0}a'_{n,0}\ln\left(1+\frac{q^2}{m_p^2}\,\omega\right), \qquad \text{(A12)}$$

where the coefficients $a_{n,m}$ can be trivially calculated. For the sake of completeness, we also present the results:

$$\begin{aligned} a_{0,m} &= -\sum_{i=-m+1}^{5} c_i\left(\frac{-q^2}{m_p^2}\right)^{i+m-1} \quad (m=4,0)\,,\\ a'_{0,0} &= -a_{0,0} + c_0\,,\\ a_{1,m} &= c_{-m+1} \quad (m=-4,1)\,,\\ a_{2,m} &= -\left(c_{-m+1} + \frac{q^2}{m_p^2}\,c_{-m+2}\right) \quad (m=-4,2)\,,\\ a_{3,m} &= c_{-m+1} + \frac{2q^2}{m_p^2}\,c_{-m+2} + \left(\frac{q^2}{m_p^2}\right)^2 c_{-m+3} \quad (m=-4,2)\,,\\ a_{4,m} &= -\left[c_{-m+1} + \frac{3q^2}{m_p^2}\,c_{-m+2} + 3\left(\frac{q^2}{m_p^2}\right)^2 c_{-m+3} + \left(\frac{q^2}{m_p^2}\right)^3 c_{-m+4}\right] \quad (m=-4,2)\,, \end{aligned} \qquad \text{(A13)}$$

where the constants c_i are defined by Eq. (A10) for $i=0.5$ and $c_i=0$ otherwise.

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