*Work supported by the National Science Council, Ireland.

[1]A. Pais and S. B. Treiman, Phys. Rev. Letters 25, 975 (1970).

[2]See, for example, S. J. Brodsky, T. Kinoshita, and H. Terazawa, Phys. Rev. Letters 25, 972 (1970); Phys. Rev. D 4, 1532 (1971).

[3]Pais and Treiman (Ref. 1) suggested an experimental test on this problem involving the detection of neutral soft pions.

[4]S. Fubini and G. Furlan, Ann. Phys. (N.Y.) 48, 322 (1969); M. Ademollo, G. Denardo, and G. Furlan, Nuovo Cimento 57A, 1 (1968).

[5]F. J. Gilman and H. Harari, Phys. Rev. 165, 1803 (1968); J. Ballam *et al.*, Phys. Rev. Letters 21, 934 (1968).

[6]See, for example, M. A. B. Bég *et al.*, Phys. Rev. Letters 25, 1231 (1970).

[7]S. Weinberg, Phys. Rev. Letters 27, 1688 (1971); T. D. Lee, Phys. Rev. D 3, 801 (1971); J. Schechter and Y. Ueda, *ibid.* 2, 736 (1970).

[8]C. Bacci *et al.*, Phys. Letters 38B, 551 (1972).

[9]S. Weinberg, Phys. Rev. Letters 18, 507 (1967).

[10]See, for example, R. E. Marshak, Riazuddin, and C. P. Ryan, *Theory of Weak Interactions in Particle Physics* (Wiley-Interscience, New York, 1969), p. 723.

[11]This observation has been made by several authors, most notably by L.-F. Li and E. A. Paschos [Phys. Rev. D 3, 1178 (1971)]. See also J. D. Bjorken, Phys. Rev. 148, 1467 (1966), footnote 34. However, the accuracy of colliding-beam experiments has recently been improved (Ref. 8). We have not considered the isoscalar part of the spectral function, which is expected to be about a third of the isovector piece, since errors in the experimental relation (25) and the assumption $c \simeq c'$ may be comparable with this omission. The IVB decay width into specific hadronic channels may be small (Ref. 10), but for very large m_W many such channels are open and the total hadronic decay width becomes large.

PHYSICAL REVIEW D VOLUME 7, NUMBER 11 1 JUNE 1973

Symmetry-Breaking Effects on Singularities of Multichannel Amplitudes near Thresholds*

J. Curiale and S. W. MacDowell

Physics Department, Yale University, New Haven, Connecticut 06520

(Received 29 November 1972)

The effects of the breaking of an internal symmetry on the singularities of the scattering matrix, especially near thresholds, is discussed in terms of the K-matrix formalism. It is shown that, in first order in the symmetry-breaking coupling constant, the K matrix transforms like the symmetry-breaking Hamiltonian. This result provides a justification for the Gell-Mann–Okubo mass formula for narrow resonances with nonzero orbital angular momentum. The position of poles in different sheets of the energy Riemann surface and their displacement with the symmetry-breaking parameter is investigated in detail, particularly for s waves. The specific case of baryon-baryon scattering in the triplet s-d states is discussed using a potential model with broken SU_3 symmetry. On the basis of information from low-energy np scattering and Λp final-state interaction we suggest that a resonance probably exists in the $I = 1$, ΞN system, below the $\Sigma\Lambda$ threshold. The Gell-Mann–Okubo mass formula would have placed the resonance close to the $\Sigma\Sigma$ threshold.

INTRODUCTION

An analysis of the effects of a symmetry-breaking interaction on bound-state poles and resonances in multichannel amplitudes was carried out by Yang and Oakes.[1] They investigated the splitting and displacement relative to thresholds of these singularities as the symmetry-breaking parameter λ is gradually turned on. Their analysis cast doubt on the validity or, more precisely, on the theoretical foundation of the Gell-Mann–Okubo mass formula. The basic objection is as follows. Let us assume that in the exact symmetry limit the scattering amplitude for transitions in a given partial wave and irreducible representation of the symmetry group has a bound-state pole (or an antibound state). Now, in order for such a pole to change sheet or to become a resonance as the symmetry-breaking interaction is gradually turned on, it has either to cross a threshold or meet with another pole (a shadow pole) or both. In each case the position of the pole as a function of λ has a branch point at the value of λ for which this occurs. Therefore, the linear approximation which is the basis for the Gell-Mann–Okubo mass formula is unwarranted. Moreover, they pointed out that for some range of values of the symmetry-breaking parameter, a resonance pole might move into a Riemann sheet far removed from the physical one. If this should occur for the actual strength of the interaction, the effect of the pole on the physical scattering amplitude would not be easily observable. As a result, some members of a multiplet of poles would appear to be missing. This work aroused a great deal of interest, prompting several authors to discussion of the problem.[2-5] Owing to centrifugal-barrier effects, there is an essential difference between the singularity structure associated with resonance in s waves and in higher partial waves, particularly near thresholds.

The discussion of Yang and Oakes,[1] although addressed to the meson-baryon resonances in a p wave, is actually more pertinent to s waves. Eden and Taylor[2] described in detail an s-wave model in which a number of poles appear close together, but in different sheets of the scattering amplitude. Then it might occur that, as the strength of the coupling constant is varied, one pole always would be on a sheet adjacent to the physical sheet, producing a resonance. However, for this to happen, one would need a certain condition on the coupling constants of the model. As they pointed out, if that condition is not satisfied, the preceding result does not obtain. In addition, in the symmetry limit this model would actually have two dynamical poles close to the physical region. A similar model was discussed by Nauenberg and Nearing.[3]

Ross[4] called attention to the difference in the structure of resonances in s waves and in higher partial waves, but only the latter case was discussed in detail. The main point is that for p and higher waves, resonance poles near a threshold always occur in pairs in sheets that are adjacent to the physical sheet on the left-hand and the right-hand side of that threshold. Then as one varies the symmetry-breaking parameter, there is always a pole adjacent to the physical region. The resonance pole does not change sheet as it moves past a threshold, but is replaced by its pair on another sheet. This is not generally the case for s-wave resonance poles.

In this paper we shall discuss the displacement of poles in multi-channel scattering amplitudes as a function of an internal-symmetry-breaking coupling constant λ. In Sec. I a general discussion of the problem is given based on the K-matrix formalism. A theorem is proved stating in what intervals of the real axis between two consecutive two-particle branch points a pole is allowed to cross from one sheet to another. If follows from two-particle unitarity generalized to unphysical sheets. In Sec. II the displacement of poles for partial waves with angular momentum $l \geq 1$ is reviewed. We supplement the analysis of Ross[4] by showing that at $\lambda = 0$, $\partial K/\partial\lambda$ transforms under the symmetry group as matrix elements of the symmetry-breaking Hamiltonian. In the narrow-resonance limit the real part of the position of a resonance pole is given by the zeros of det$K$. To first order in λ it will obey a Gell-Mann–Okubo mass formula. Here however, there is, in principle, no objection to the linear approximation, at least in so far as production thresholds may be neglected. In Sec. III we discuss the particular case of s waves. It is shown that the location of shadow poles in the sheets of the energy Riemann surface depend qualitatively on the Clebsch-Gordan coefficients for the decomposition of the product of the representations of the scattering particles, into irreducible representations of the symmetry group. This is essentially a kinematical threshold effect. As the symmetry is broken the poles are displaced and may move into different sheets. The displacement of the poles also depends on the amplitudes for transitions in states belonging to representations that, through symmetry breaking, can couple to that to which the pole belongs. This happens even if dynamical representation mixing, as given by the nondiagonal matrix elements of the K matrix, is neglected. It results from kinematical representation mixing. An application of this discussion to low-energy baryon-baryon scattering in the triplet s and d states is carried out in Sec. IV. There we use dynamical parameters taken from a broken-SU_3 model of baryon-baryon scattering whose details will appear in a separate publication.[6] A summary of results and conclusions is presented in Sec. V.

I. THE K-MATRIX FORMALISM

Let us consider a two-body transition in a definite state of total angular momentum J and energy square s. We denote by α a set of conserved internal quantum numbers, such as isospin and hypercharge, specifying a state of a pair b of hadrons and by l the orbital angular momentum. We shall refer to this specification of the states as the particle basis. The matrix element for a transition $(b, l) \to (b', l')$ of states with quantum numbers (α, J) will be denoted by $T(s; J, \alpha)_{b'l';bl}$. The K matrix will be defined by

$$k_{b'}{}^{l'} T^{-1}(s; J, \alpha)_{b'l';bl}\, k_b{}^l = K(s; J, \alpha)_{b'l';bl} - ik_b{}^{2l+1}\delta_{b'b}\delta_{l'l} \,, \tag{1.1}$$

where k_b is the center-of-mass momentum of the pair b, given by

$$k_b{}^2 = \frac{1}{4s}[s-(m_i+m_j)^2][s-(m_i-m_j)^2] \,, \tag{1.2}$$

and m_i, m_j are the masses of the particles in the pair b.

The elements of the K matrix have the same singularities in s as those of the elements of the T matrix except for the two-particle unitarity branch points. In addition the K matrix will have poles at the positions of zeros of $\det T$. We shall assume invariance under time reversal which implies that the K matrix is symmetric. We also assume an underlying internal symmetry which is broken by an interaction whose strength is given by a parameter λ. In the limit of exact symmetry, i.e., for $\lambda=0$, baryons and mesons belong to irreducible representations of the symmetry group (e.g., octets in SU_3). We shall also use the basis of irreducible representations B, of the symmetry group, which is related to the particle basis in the following way:

$$|s;J,\alpha;B,l\rangle=\sum_b C_{Bb}(\alpha)|s;J,\alpha;b,l\rangle\ . \tag{1.3}$$

The $C_{Bb}(\alpha)$'s are Clebsch-Gordan coefficients for the decomposition of products of representations of the symmetry group. They form an orthogonal matrix. For $\lambda=0$ the T and K matrices are diagonal in the B basis, if each irreducible representation of the symmetry group appears only once in the decomposition of the states b, or block diagonal if they appear more than once (such as, for instance, in the decomposition of two octets, where the octet representation appears twice). At $\lambda=0$ all the k's are identical. As the symmetry is broken, the masses and thresholds split and for a given s the k's will be different. The k matrix, which is diagonal in the particle basis, will take on, in the B basis, the form

$$k_{B'B}=\sum_b C_{B'b}(\alpha)C_{Bb}(\alpha)k_b\ . \tag{1.4}$$

We shall consider the Riemann sheets that are reached from the physical sheet by crossing the two-channel unitarity cuts in arbitrary order. They are characterized by the set $\{\epsilon_b\}$ of signs of $\mathrm{Im}k_b$ in the complex s plane or, in a given k_b-Riemann surface, by the set of signs $\{\eta_{b'}\}$ of $\mathrm{Im}k_{b'}$ relative to the sign of $\mathrm{Im}k_b$. If the number of two-body channels is n, the number of sheets so defined is 2^n in the s-Riemann surface and 2^{n-1} in a k_b-Riemann surface. The physical sheet is defined by the condition $\mathrm{Im}k_b\geq 0$ for all b. The poles of the T matrix are given by the roots of the equation

$$\Delta=\det(K_{b'l';bl}-ik_b^{2l+1}\delta_{b'b}\delta_{l'l})$$
$$=0\ . \tag{1.5}$$

In the physical sheet of the variable s, the requirement of causality precludes the existence of poles, except along the real axis below the lowest threshold. In addition, unitarity places restrictions on the location of poles in other sheets, on the real axis above threshold. In fact unitarity in an arbitrary sheet gives

$$T-T^\dagger=2iT^\dagger k\theta T\ , \tag{1.6}$$

where θ is a diagonal matrix given by

$$\theta_{b'b}=\delta_{b'b}\theta(s-s_b)\ , \tag{1.7}$$

where s_b is the energy at the threshold b. If there is a pole of T on the real axis at $s=s_0$, $s_b<s_0<s_{b+1}$, then (1.6) gives

$$R^\dagger k_0\theta R=0, \tag{1.8}$$

where R is the matrix of the residues of the T matrix. If in this sheet all the k_l's have the same sign for $b'\leq b$, then (1.8) can be written as

$$(k_0^{1/2}\theta R)^\dagger(k_0^{1/2}\theta R)=0\ , \tag{1.9}$$

which implies $\theta R=0$. Hence, we have the following result:

Theorem 1. A pole of the T matrix between thresholds b and $(b+1)$, in a sheet such that the signs $\epsilon_{b'}$ are the same for all $b'\leq b$, decouples from the open channels. (The thresholds are labeled in ascending order from left to right.)

Such a situation cannot generally happen and should be regarded as anomalous. Barring this, it follows from the above theorem, that there cannot be a real pole between the first and second threshold in any sheet.

Suppose now that there is a pole of the T matrix very near a threshold b. By this we mean that the distance of the pole to this threshold is much smaller than the distance to any other threshold. Then near the pole we can expand Δ in power series of k_b. The expansion will be of the form

$$\Delta=a_0+a_1k_b^2+\cdots-ik_b^{2l+1}(a_0'+a_1'k_b^2+\cdots)\ , \tag{1.10}$$

where the coefficients are in general complex, and l is the lowest value of the orbital angular momentum. If $l\geq 1$, the roots of $\Delta=0$ near the threshold b will be approximately given by $k_b=\pm(-a_0/a_1)^{1/2}$, that is, there are, in general, two nearly symmetric roots on some sheet of the k_b plane. In the s plane these roots are reached from one another by going around the branch point at s_b. If one root is on the sheet adjacent to the physical sheet below s_b, the other is on the sheet adjacent to the physical sheet above s_b. If b is the lowest threshold, the coefficients in (1.10) are real; then for a_0/a_1 negative, the poles will be on the same sheet (the second sheet) corresponding to a resonance just above threshold, whereas for a_0/a_1 positive, the poles will be on the real axis of the s plane, below threshold, one on the physical sheet being a bound state, the other on the second sheet being an antibound state.

In the case of s waves, $l=0$, this situation does not generally occur; one might have just one pole at $ik_b \simeq a_0/a_0'$.

II. DISPLACEMENT OF POLES WITH THE SYMMETRY-BREAKING PARAMETER

Let us study the poles of the T matrix as functions of the parameter λ. Let us assume that for $\lambda=0$ there is a pole just below threshold in the irreducible amplitudes B. For $l \geq 1$ the pole occurs in both the physical and second sheets. For $l=0$ there may just be *either* a bound state *or* an antibound state. As the symmetry-breaking interaction is turned on, the thresholds split and a manifold of new sheets is coupled to the physical sheet. It is clear that, for $l \geq 1$, a pair of poles similarly located appear in all the sheets of the k plane. For $l=0$, if $|a_1 a_0| \ll a_0'^2$ in (1.10), there will be only *one* pole in every sheet of the k plane whose position is given by

$$ik = \frac{a_0}{a_0'}\left(\sum_b C_{B_0 b}(\alpha)^2 \eta_b\right)^{-1} . \tag{2.1}$$

If for some $\{\eta_b\}$ we have $\sum_b C_{B_0 b}(\alpha)^2 \eta_b = 0$, then on that sheet there will be a pair of poles symmetrically disposed with respect to the origin.

As λ varies, the poles may approach the lowest threshold. The case $l=0$ will be discussed in detail in Sec. III. In the case $l \geq 1$ the poles will meet at threshold and then move along the branch line in the second sheet, becoming a resonance. As they move past a threshold there will always be a pole on the sheet adjacent to the physical sheet as described before and in Ref. 4. To first order in λ the position of the pole is given by

$$\det\left[K_{B_0 l';B_0 l} - i\,\delta_{l'l}\sum_b C_{B_0 b}(\alpha)^2 k_b^{\,2l+1}\right] = 0 . \tag{2.2}$$

To simplify the analysis let us assume now that the orbital angular momentum is also a constant of motion. Then (2.2) becomes

$$K_{B_0 l}(s,\lambda) - i\sum C_{B_0 b}(\alpha)^2 k_b^{\,2l+1} = 0 . \tag{2.3}$$

Let s_0 be the solution of this equation. Then the displacement of the pole for sufficiently small values of λ will be, to first order in λ, given by

$$\frac{\partial K_{B_0 l}}{\partial s}\delta s + \frac{\partial K_{B_0 l}}{\partial \lambda}\lambda - i\sum C_{B_0 b}(\alpha)^2\left(\frac{\partial k_b^{\,2l+1}}{\partial s}\delta s + \frac{\partial k_b^{\,2l+1}}{\partial \lambda}\lambda\right) = 0, \tag{2.4}$$

where all the derivatives are taken at the symmetry point $s=s_0$, $\lambda=0$. Now

$$\frac{\partial k_b^{\,2}}{\partial \lambda} = \frac{\partial k_b^{\,2}}{\partial m_1}\frac{\partial m_1}{\partial \lambda} + \frac{\partial k_b^{\,2}}{\partial m_2}\frac{\partial m_2}{\partial \lambda} , \tag{2.5}$$

where m_1 and m_2 are the masses of the two particles in the pair b. Then in (2.4) we have

$$\sum C_{B_0 b}(\alpha)^2 \frac{\partial k_b^{\,2l+1}}{\partial s}\delta s = \frac{\partial k^{2l+1}}{\partial s}\delta s , \tag{2.6}$$

$$\sum C_{B_0 b}(\alpha)^2 \frac{\partial k_b^{\,2l+1}}{\partial \lambda} = \sum C_{B_0 b}(\alpha)^2 (l+\tfrac{1}{2}) k^{2l-1} \times\left(\frac{\partial k^2}{\partial m_1}\frac{\partial m_1}{\partial \lambda} + \frac{\partial k^2}{\partial m_2}\frac{\partial m_2}{\partial \lambda}\right) . \tag{2.7}$$

Let us consider a basis $\{|b\rangle\}$ for a (reducible) representation of the internal-symmetry group which is the direct product of the irreducible vector spaces of particles 1 and 2, $|b\rangle = |1\rangle \otimes |2\rangle$. In this basis $\partial m_1/\partial\lambda$ is a diagonal operator, transforming under the symmetry group as the symmetry-breaking Hamiltonian. Therefore we have

$$\langle B_0|\partial m_i/\partial\lambda|B_0\rangle = \sum C_{B_0 b'}\langle b'|\partial m_i/\partial\lambda|b\rangle C_{B_0 b} = \sum C_{B_0 b}{}^2\langle b|\partial m_i/\partial\lambda|b\rangle . \tag{2.8}$$

The matrix element on the left-hand side transforms like matrix elements of the symmetry-breaking Hamiltonian. Taking this into account it follows that the last term on the left-hand side of (2.4) also transforms in the same way, and therefore so does

$$\left.\frac{\partial K_{B_0}}{\partial \lambda}\right|_{s=s_0;\lambda=0} .$$

This result can be generalized to arbitrary energies (within the two-particle approximation). In fact, if $\lambda\mathcal{H}'$ is the symmetry-breaking Hamiltonian density, including all the counterterms for mass and coupling-constant renormalization to first order in λ, then we have

$$\left.\frac{\partial T}{\partial \lambda}\right|_{\lambda=0} = \mathcal{H}' , \tag{2.9}$$

where the derivative is taken with fixed external momenta. Then the total derivative with respect to λ of the two-particle amplitude $T_{b'b}$, at fixed center-of-mass energy and scattering angle, is given by

$$\frac{dT_{b'b}}{d\lambda}=\frac{\partial T_{b'b}}{\partial\lambda}+\sum_{i=1,2}\left(\frac{\partial m_i}{\partial\lambda}\frac{\partial}{\partial m_i}+\frac{\partial m_i'}{\partial\lambda}\frac{\partial}{\partial m_i'}\right)T_{b'b}$$

$$=\frac{\partial T_{b'b}}{\partial\lambda}+\sum_{i=1,2}\left(\frac{\partial T}{\partial m_i}\frac{\partial m_i}{\partial\lambda}+\frac{\partial m_i'}{\partial\lambda}\frac{\partial T}{\partial m_i'}\right)_{b'b},$$

where m_i and m_i' refer to the masses of the particles in the initial and final states b and b', respectively. Now, at $\lambda=0$, $\partial T/\partial m_i$ and $\partial T/\partial m_i'$ are invariant under the symmetry group and, as has been observed, the diagonal matrices $(\partial m_i/\partial\lambda)_{bb}$ and $(\partial m_i'/\partial\lambda)_{b'b'}$ transform as matrix elements of $\mathcal{H}'$ in the product space of one-particle states. Therefore we can write

$$\frac{dT_{B'B}}{d\lambda}=\mathcal{K}_{B'B}+\sum_{i=1,2}\left(\frac{\partial T}{\partial m_i}\frac{\partial m_i}{\partial\lambda}+\frac{\partial m_i'}{\partial\lambda}\frac{\partial T}{\partial m_i'}\right)_{B'B}, \tag{2.10}$$

where all the derivatives are taken at $\lambda=0$, and

$$\left(\frac{\partial m_i}{\partial\lambda}\right)_{B'B}=\sum_b C_{B'b}(\alpha)\left(\frac{\partial m_i}{\partial\lambda}\right)_{bb}C_{Bb}(\alpha), \tag{2.11}$$

with an analogous expression for $(\partial m_i'/\partial\lambda)_{B'B}$. Now

$$\frac{\partial}{\partial\lambda}(k^l T^{-1}k^l)=\frac{\partial k^l}{\partial\lambda}T^{-1}k^l+k^l T^{-1}\frac{\partial T}{\partial\lambda}T^{-1}k^l+k^l T^{-1}\frac{\partial k^l}{\partial\lambda}, \tag{2.12}$$

where

$$\left(\frac{\partial k^l}{\partial\lambda}\right)_{B'B}=\sum_{i=1,2}\frac{\partial k^l}{\partial m_i}\left(\frac{\partial m_i}{\partial\lambda}\right)_{B'B}. \tag{2.13}$$

Since $(\partial m_i/\partial\lambda)_{B'B}$ as given by (2.11) transforms like the symmetry-breaking Hamiltonian, it follows that $(\partial k^l/\partial\lambda)_{B'B}$ and $dT_{B'B}/d\lambda$ also do so. Therefore, it follows from (2.2) and (1.1) that at fixed energy, $dK_{B'B}/d\lambda$ transforms in the same way. Thus, we have the following result:

Theorem 2. To first order in λ, one can write

$$K=K_0+\lambda K_1, \tag{2.14}$$

where K_0 is invariant under the symmetry group and K_1 transforms like the symmetry-breaking Hamiltonian.

Since the K matrix has no two-particle branch points, one might expect the linear approximation to be good at low energies. An application of this result to a phenomenological discussion of the decuplet of meson-baryon resonances will appear in a separate publication.

III. BEHAVIOR OF s-WAVE POLES

To fix ideas we consider a two-channel s wave. Let $b=1$ be the channel with lowest threshold when the symmetry is broken. The k_1 plane has two sheets I and II defined according to whether Imk_1 and Imk_2 have the same or opposite signs. In the symmetry limit we have $k_2=\pm k_1$ and the two sheets disconnect from each other. In the s plane there will be four sheets defined by the signs of Imk_1 and Imk_2 as follows: $S_1=(+,+)$, $S_2=(-,-)$, $S_3=(+,-)$, $S_4=(-,+)$. S_1 and S_2 in the s plane correspond to the upper and lower half of sheet I in the k_1 plane, S_3 and S_4 to the upper and lower half of sheet II. Let $B=1$ be the representation which has a pole in the symmetry limit. In the basis of irreducible representations of the symmetry group the position of the pole in sheet I is given by

$$K_{11}-ik=0. \tag{3.1}$$

In sheet II there will be a pole given by

$$(K_{11}-ik\cos2\beta)(K_{22}+ik\cos2\beta)+k^2\sin2\beta=0, \tag{3.2}$$

where the angle β is defined by the Clebsch-Gordan matrix C_{Bb} which is of the form

$$C_{Bb}=\begin{pmatrix}\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta\end{pmatrix}. \tag{3.3}$$

Without loss of generality we can take $-\frac{1}{2}\pi<\beta<\frac{1}{2}\pi$. We shall distinguish three cases:

(a) $\cos\beta>|\sin\beta|$ $(\cos2\beta>0)$. If the pole on the first sheet is sufficiently close to threshold, the pole on the second sheet is approximately given by

$$K_{11}-ik\cos2\beta=0 \tag{3.4}$$

and is located on the same side of the real axis as the pole on the first sheet. As λ varies, the poles will move and may change sheets, but they will not come together. Thus a bound state may become antibound and vice versa, but it will not turn into a resonance [see Fig. 1(a)].

(b) $\cos\beta=|\sin\beta|=(\frac{1}{2})^{1/2}$ $(\cos2\beta=0)$. Equation (3.2) reduces to

$$K_{11}K_{22}+k^2=0. \tag{3.5}$$

Let a_{11} and a_{22} be the scattering lengths for channels $B=1$ and $B=2$, respectively, and r_{11} and r_{22}, the corresponding effective ranges. We assume that $|a_{22}|\ll|a_{11}|$. Then the solutions of (3.2) are approximately given by

$$k_\pm=\pm[-a_{11}(a_{22}+\tfrac{1}{2}r_{11})]^{-1/2}. \tag{3.6}$$

Unless r_{11} is very large, these poles are not as close to threshold as that on the first sheet. If these roots are real, they will become complex as we change λ. For small λ, (3.5) will be modified in the following way:

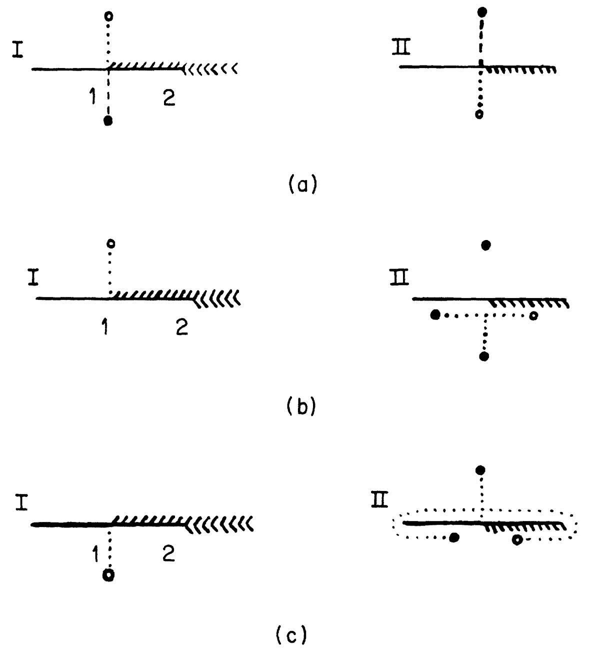

FIG. 1. The two sheets of the k_1-Riemann surface in two coupled two-particle channels. The hatches indicate the physical region and the region adjacent to it. The position of poles and shadow poles and their displacements are indicated: in (a) for the case $\cos^2\beta > \sin^2\beta$; in (b) for $\cos^2\beta = \sin^2\beta$, with the original pole being a bound state; in (c) for $\cos^2\beta < \sin^2\beta$, with the original pole being an anti-bound-state pole; for a bound-state pole, we would have a behavior similar to that shown in (b).

$$K_{11}K_{22}+\tfrac{1}{4}(k_1-k_2)^2-\tfrac{1}{2}i(k_1+k_2)(K_{11}+K_{22}) - i(k_1-k_2)K_{12}=0 \; . \quad (3.7)$$

In sheet II, $k_2 \to -k_1$ as $\lambda \to 0$. The imaginary part of $k_\pm$ will be approximately given by

$$\mathrm{Im}k_\pm \simeq -\tfrac{1}{8}a_{11}(k_1^2-k_2^2) \quad (3.8)$$

and will have the sign of $-a_{11}$. If $a_{11}<0$ corresponding to a bound state pole at $\lambda=0$, then $\mathrm{Im}k_\pm >0$ and the pole cannot become a resonance. If $a_{11}>0$ corresponding to an antibound state, then $\mathrm{Im}k_\pm <0$. These poles on the lower half of sheet II might eventually appear as a resonance in the channel $b=1$, below the threshold for $b=2$, if the position of this threshold moves past the position of the pole. On the other hand, if the roots (3.6) are imaginary, then it may happen that the pole in sheet I moves towards the real axis crossing into the second sheet and meeting with one of the poles there. Then they will move apart alongside the interval of the real axis between thresholds. If they are on the lower half plane, they will produce a resonance [Fig. 1(b)]. But if, on the other hand, they are in the upper half plane, they will not produce a physical resonance unless they turn around the second threshold and back between the thresholds. In both cases the resonance will appear in channel $b=1$ below the threshold for $b=2$. Here we have a situation as conjectured by Yang and Oakes, in which for some interval of values of λ the pole moves into a sheet not adjacent to the physical sheet, its effect on the physical amplitude being expected to be small. As λ increases, however, it may emerge as a resonance pole below the second threshold.

(c) $\cos\beta < |\sin\beta|$ $(\cos 2\beta < 0)$. The pole on the second sheet is given by Eq. (3.4) but is now located on the opposite side of the real axis with respect to the pole on the first sheet. As λ is changed the pole on the first sheet may move towards the real axis and into the second sheet and we have a situation analogous to that discussed in case (b) when the roots (3.6) are imaginary [Figs. 1(b) and 1(c)].

Let us investigate further under what conditions poles on sheet S_3 and S_4 can cross the real axis beyond the second threshold. The position of the pole on the real axis is determined by the simultaneous equations

$$K_{11}K_{22}-K_{12}^2-k_1k_2=0 \; , \quad (3.9)$$

$$K_{11}(k_1\cos^2\beta+k_2\sin^2\beta)+K_{22}(k_1\sin^2\beta+k_2\cos^2\beta) - 2K_{12}(k_1-k_2)\sin\beta\cos\beta=0 \; . \quad (3.10)$$

Since $k_1^2>k_2^2$, Eq. (3.10) has solutions only if the following condition holds:

$$(K_{22}+K_{11})[(K_{22}-K_{11})\cos 2\beta+K_{12}\sin 2\beta]<0 \; . \quad (3.11)$$

If the pole is associated with a state with little dynamical configuration mixing, then at the pole we should have $|K_{22}| \gg |K_{11}|$ and $|K_{22}| \gg |K_{12}|$. Under these circumstances (3.11) would imply $\cos 2\beta < \epsilon$, where ϵ is a small number. Therefore the crossing of the real axis is more likely to occur if $\cos^2\beta < \sin^2\beta$. In general a pole is likely to cross an interval of the real axis between two thresholds b and $b+1$, if the equation

$$\sum_{b'<b+1} C_{B_0 b'}(\alpha)^2 k_{b'}=0 \quad (3.12)$$

has solutions in that interval. The signs of the $k_{b'}$'s determine the sheets on which crossing may occur.

It should be pointed out that these conclusions are valid only if the s wave is not coupled to another partial wave such as a d wave. In the case of coupled waves the argument presented here no longer applies. [See, for instance, the discussion of coupled partial waves in Sec. IV and, in particular, the condition (4.1).]

IV. APPLICATION TO BARYON-BARYON SCATTERING

We have studied a simple meson exchange model of baryon-baryon scattering as applied to triplet s and d waves in SU_3 antisymmetric states. We used an SU_3-symmetric coupling of pseudoscalar and vector mesons to baryons and an SU_3-symmetric hard core. The symmetry is broken only through the splitting of masses of the mesons and baryons, which are taken to obey a Gell-Mann–Okubo type mass formula linear in a symmetry-breaking parameter λ. For $\lambda=0$ the masses are degenerate within a multiplet, corresponding to exact symmetry. For $\lambda=1$ one obtains the actual physical masses. We have calculated the scattering amplitude in the triplet s and d coupled channels in terms of the parameter λ. These states, being symmetric in coordinate and spin space, belong to antisymmetric representations of SU_3. (The full description of the model and results of the calculation will be given in a separate paper.[6])

It was found that for $\lambda=0$, the model gives a pole in the amplitude for scattering in states belonging to the decuplet representation $\underline{10}^*$. The pole is on the second sheet very near threshold, at $ik=0.11$ fm^{-1}. We have investigated how this pole is displaced in the multichannel amplitudes for different components of the decuplet, as λ varies from zero to one.

In Table I, we list the components in the particle basis of each member of the $\underline{10}^*$ multiplet and give the Clebsch-Gordan coefficients for the decomposition of each pair in terms of three antisymmetric irreducible representations in the product $\underline{8}\otimes\underline{8}$.

The proton-neutron channel with $I=0$, $Y=2$, a pure decuplet, and the coupled ΛN, ΣN channels with $I=\frac{1}{2}$, $Y=1$, were studied in detail. In the p-n amplitude the pole moves into the first sheet to become a bound state. In fact, one of the free parameters in the model, namely the hard core radius for the decuplet, was adjusted so as to obtain the correct binding energy of the deuteron at $\lambda=1$.

The $I=\frac{1}{2}$, $Y=1$ channels are superpositions of the decuplet $\underline{10}^*$ and the antisymmetric $\underline{8}_F$. One can see from Table I that the Clebsch-Gordan matrix for these channels corresponds to case (b) discussed above with $\cos\beta=\sin\beta=1/\sqrt{2}$. The position of the poles on the second sheet of the k_1 plane is given by (3.6) with $a_{11}=a_{\underline{10}^*}=8.1$ fm, $r_{\underline{10}^*}=2$ fm, and $a_{22}=a_8=0.58$ fm. These are the values for these parameters at $\lambda=0$. It was found (see Fig. 1) that the anti-bound pole moves into the upper-half of sheet II and meets the pole in this sheet for $\lambda=0.02$. As λ increases they move away from each other, staying close to the real axis on the upper half of sheet II. No resonance appears in the physical amplitudes. At $\lambda=0.6$, the pole on the positive side of the plane crosses the real axis around the $N\Sigma$-threshold branch point. For $\lambda>0.6$ the pole emerges as a resonance in the $N\Lambda$ amplitude, below the $N\Sigma$ threshold. It turned out in our model that the resonance (for $\lambda=1$) is almost purely in the d wave. This suggests that the centrifugal barrier might be responsible for keeping the pole close to the real axis. If one neglects representation mixing, at the position where the pole crosses the real axis we have

$$(k_1+k_2)K_{dd}+(k_1^5+k_2^5)K_{ss}=0\ . \tag{4.1}$$

Notice that K_{dd} and K_{ss} should have opposite signs for this equation to have a solution.

We have not done a numerical calculation of the T matrix in the $Y=0$, $I=1$ channels. However, from the results of the calculations for the other members of the $\underline{10}^*$ multiplet, it will be possible to make a qualitative analysis of these channels, based on the effective-range approximation.

Let us denote by 1, 2, 3 the (antisymmetric) channels ΞN, $\Sigma\Lambda$, $\Sigma\Sigma$, respectively, and by $K_{\underline{10}^*}$, K_8, K_{10}, the diagonal blocks of the $J^P=1^+$ K matrix in the basis of irreducible representations of SU_3. The nondiagonal blocks are zero in the limit $\lambda=0$. These blocks are 2×2 matrices corresponding to transitions in s and d waves.

At very small values of λ and energies near

TABLE I. Clebsch-Gordan coefficients for the reduction of two-baryon antisymmetric states into irreducible representations of SU_3.

Y	I	Particle pair	Irreducible representations $\underline{10}^*$	$\underline{8}_F$	$\underline{10}$
2	0	NN	1	0	0
1	$\frac{1}{2}$	ΛN	$1/\sqrt{2}$	$1/\sqrt{2}$	0
		ΣN	$-1/\sqrt{2}$	$1/\sqrt{2}$	0
0	1	ΞN	$1/\sqrt{3}$	$1/\sqrt{3}$	$-1/\sqrt{3}$
		$\Sigma\Lambda$	$-1/\sqrt{2}$	0	$-1/\sqrt{2}$
		$\Sigma\Sigma$	$-1/\sqrt{6}$	$2/\sqrt{6}$	$1/\sqrt{6}$
−1	$\frac{3}{2}$	$\Xi\Sigma$	1	0	0

threshold, one can argue that the off-diagonal blocks of the K matrix may be neglected. Likewise, in the matrix ik^{2l+1} in (1.1), we shall keep only the elements linear in k, corresponding to s waves, and neglect the d-wave elements which are of 5th order in k.

Then, with these approximations, the position of the pole of the T matrix will be given by

$$[A_{\underline{10}*}^{-1}-i(\tfrac{1}{3}k_1+\tfrac{1}{2}k_2+\tfrac{1}{6}k_3)][A_8^{-1}-i(\tfrac{1}{2}k_1+\tfrac{1}{2}k_3)][A_{10}^{-1}-i(\tfrac{1}{3}k_1+\tfrac{1}{2}k_2+\tfrac{1}{6}k_3)]$$
$$+\tfrac{1}{9}(k_1-k_3)^2[A_{\underline{10}}*^{-1}+A_{10}^{-1}-2i(\tfrac{1}{3}k_1+\tfrac{1}{2}k_2+\tfrac{1}{6}k_3)]+(-\tfrac{1}{3}k_1+\tfrac{1}{2}k_2-\tfrac{1}{6}k_3)^2[A_8^{-1}-i(\tfrac{1}{2}k_1+\tfrac{1}{2}k_3)]=0\ , \tag{4.2}$$

where

$$A_B^{-1}=(\det K_B)/K_{B,d}$$
$$=K_{B,s}-K_{B,sd}{}^2/K_{B,d}\ . \tag{4.3}$$

Since A_8^{-1} and A_{10}^{-1} should be large as compared to $A_{\underline{10}*}^{-1}$ and the momenta k_i, one can simplify the above equation so as to bring it to the form

$$[A_{\underline{10}*}^{-1}-i(\tfrac{1}{3}k_1+\tfrac{1}{2}k_2+\tfrac{1}{6}k_3)]A_8^{-1}A_{10}^{-1}+\tfrac{1}{9}(k_1-k_3)^2A_{10}^{-1}+(-\tfrac{1}{3}k_1+\tfrac{1}{2}k_2-\tfrac{1}{6}k_3)^2A_8^{-1}=0\ . \tag{4.4}$$

In discussing the roots of this equation near $k_1^2=0$ and for small values of λ, it is sufficient to use an effective-range formula for $A_{\underline{10}*}^{-1}$ and to take A_8^{-1} and A_{10}^{-1} as constants. So we write

$$A_{\underline{10}*}^{-1}=\frac{1}{a_{\underline{10}*}}+\tfrac{1}{2}r_{\underline{10}*}k_{\underline{10}*}^2,\quad A_8^{-1}=\frac{1}{a_8},\quad A_{10}^{-1}=\frac{1}{a_{10}}\ , \tag{4.5}$$

where the variable $k_{\underline{10}*}^2$ is the average center-of-mass momentum square in the $\underline{10}*$ representation,

$$k_{\underline{10}*}^2=\tfrac{1}{3}k_1^2+\tfrac{1}{2}k_2^2+\tfrac{1}{6}k_3^2\ , \tag{4.6}$$

and the a's are the scattering lengths of the irreducible amplitudes at $\lambda=0$. To first order in λ the parameters of the K matrix, according to Theorem 2 of Sec. II, are proportional to Y. Since $Y=0$ in the channels we are considering here, these parameters are constant to first order in λ. According to the results (2.7) and (2.8) of Sec. II, $\partial k_{\underline{10}*}^2/\partial\lambda$, for fixed energy and $\lambda=0$, transforms in the same way as the symmetry-breaking Hamiltonian. In the limit $\lambda\to 0$ the solutions of Eq. (4.2) in the different sheets of the k_1 plane are approximately given by

$$\mathrm{I}(++)\quad ik_1=\frac{2}{a_{\underline{10}*}}[1+(1+2r_{\underline{10}*}/a_{\underline{10}*})^{1/2}]^{-1}\ , \tag{4.7}$$

$$\mathrm{II}(--)\quad ik_1=-\frac{6}{a_{\underline{10}*}}\{1+[1+2(9r_{\underline{10}*}+8a_8+8a_{10})/a_{\underline{10}*}]^{1/2}\}^{-1}\ , \tag{4.8}$$

$$\mathrm{III}(+-)\quad ik_1=\frac{3}{a_{\underline{10}*}}\{1+[1+\tfrac{1}{2}(9r_{\underline{10}*}+8a_8+2a_{10})/a_{\underline{10}*}]^{1/2}\}^{-1}\ , \tag{4.9}$$

$$\mathrm{IV}(-+)\quad ik_1=\pm[a_{\underline{10}*}(\tfrac{1}{2}r_{\underline{10}*}+a_{10})]^{-1/2}\ . \tag{4.10}$$

In Appendix I we discuss the change in position of the poles in the different sheets as λ varies. The pole on the first sheet moves up into the second sheet, where it meets a shadow pole. They then move away from each other and eventually turn around the branch point corresponding to the $\Sigma\Lambda$ threshold, into the fourth sheet far removed from the physical region [Fig. 2(a)]. At the same time the shadow pole on the third sheet moves upwards, whereas the shadow pole on the upper half of the second sheet moves down. They come together for a small value of λ ($\lambda<0.07$) depending on the value of the scattering length a_{10} at $\lambda=0$. If $a_{10}<4.65$ fm, the two poles meet on the upper half of the second sheet and move in a similar way as that of the first pair of poles. But as they turn around the $\Sigma\Lambda$ branch point they move into the lower half of the second sheet adjacent to the physical region in the interval below the $\Sigma\Lambda$ threshold [Fig. 2(b)]. Therefore this pole becomes a resonance pole just below the $\Sigma\Lambda$ threshold. [Notice that (3.13) has, in this case, solutions in the inter-

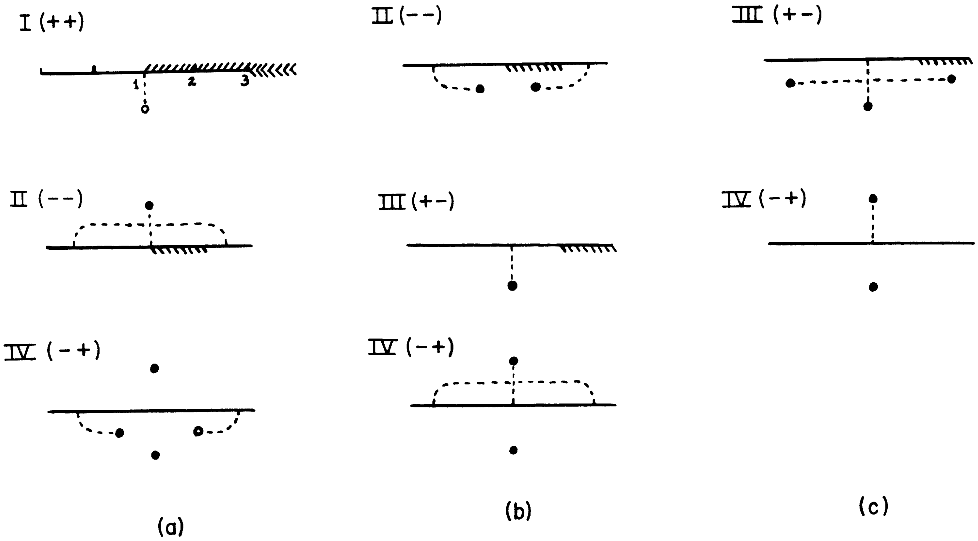

FIG. 2. Position of poles of the scattering matrix for $Y=0$, $I=1$ members of the baryon-baryon $\underline{10}^*$ decuplet, in different sheets of the $k_{\Xi N}$ Riemann surface. Conjectured displacement of poles with SU_3-breaking parameter λ is shown: in (a) for the poles in sheets I and II; in (b) for the poles in sheets III and IV, assuming that they meet in sheet IV and go on to sheet II to become a resonance between the ΞN and $\Sigma\Lambda$ thresholds; in (c) for the poles in sheets III and IV, assuming that they meet in sheet III and become a resonance between the $\Sigma\Lambda$ and $\Sigma\Sigma$ thresholds.

val between the $\Lambda\Sigma$ and $\Sigma\Sigma$ thresholds.]

If, on the other hand, $a_{10} > 4.65$ fm, the two poles meet in the lower half of the third sheet, and, as they move along the real axis, will become a resonance in the interval between the $\Sigma\Lambda$ and $\Sigma\Sigma$ thresholds [Fig. 2(c)]. Based on the position of the poles in the $Y=2$ and $Y=1$ states, namely the deuteron bound state and the ΛN resonance below the ΣN threshold, the Gell-Mann–Okubo mass formula would predict a resonance pole in the $Y=0$ states ΞN and $\Sigma\Lambda$, just below the $\Sigma\Sigma$ threshold. Since nothing is known about the scattering amplitude in states with $Y=0$, $I=1$, our analysis shows that no definite prediction can be made. It indicates that there can be a resonance pole below the $\Sigma\Lambda$ threshold in violation of the Gell-Mann–Okubo mass formula, or below the $\Sigma\Sigma$ threshold, in accordance with that formula. In the latter case a rather large scattering length in the $\underline{10}$ amplitude, $a_{10} > 4.65$ fm, is required. A third possibility is that these poles will not end up in a resonance position. Which of these alternatives is actually realized depends on the dynamical features of the interactions not only in the $\underline{10}^*$ decuplet but in the octet and 10 decuplet as well. It should be pointed out that this dependence on the dynamics is in no way taken into account in the derivation of the Gell-Mann–Okubo mass formula.

In the fourth member of the decuplet the $Y=-1$, $I=\frac{3}{2}$, $\Xi\Sigma$ states the anti-bound pole moves away from threshold. This result is not just specific to the model. In fact, defining $k_{\underline{10}^*}$ as the average momentum for each member of the $\underline{10}^*$ multiplet, then for small λ's the position of the pole is displaced according to the formula

$$\frac{d}{d\lambda}(ik_{\underline{10}^*})_{\text{pole}} = cY . \tag{4.11}$$

Therefore the first and fourth members of the multiplet move in opposite directions. The effect of the anti-bound pole is to produce a rather large *positive* scattering length in the $Y=-1$ amplitude in contrast with the large *negative* scattering length in the triplet s-wave proton-neutron amplitude.

V. SUMMARY OF RESULTS

We have derived a theorem for the K matrix which states that if there is an internal symmetry broken by an interaction proportional to a parameter λ, $(\partial K/\partial\lambda)_{\lambda=0}$ transforms in the same way as the symmetry-breaking Hamiltonian. This theorem allows us to justify the Gell-Mann–Okubo mass formula for compound two-particle states with orbital angular momentum $l \geq 1$. The case of s-wave amplitudes was investigated in detail. It was found that (i) the position of shadow poles in the different sheets of the energy Riemann surface depends on the Clebsch-Gordan coefficients for the reduction of the product of the representations of the scattering particles; (ii) the displacement of these poles as λ varies depend not only on the dynamics of the multiplet to which they initially belong, but also on the dynamics of the states to which that multiplet couples by virtue of the symmetry breaking. This occurs even if dynamical representation mixing is negligible, just as a result of kinematical representation mixing due to the splitting of two-particle thresholds; (iii) in some members of the multiplet which has a pole in the symmetry limit near threshold, the original pole and the shadow poles may move into positions of the energy Riemann surface quite far from the physical region, as conjectured by Oakes and Yang.

ACKNOWLEDGMENTS

The authors wish to thank Professor R. Dashen for a useful discussion and Professor C. Sommerfield for a critical reading of the manuscript.

APPENDIX: DISPLACEMENT OF POLES FOR λ SMALL

In this appendix we shall investigate the displacement of poles for small values of λ in the $Y=0$, $I=1$ amplitudes of triplet states of the baryon-baryon system. The change in position of the pole with λ is given by $(d/d\lambda)(ik_1)\delta\lambda$. This is obtained by differentiating (4.1). Making use of the SU_3 mass relation $2(m_{\Xi}+m_N) = 3m_\Lambda + m_\Sigma$, one obtains at $\lambda=0$:

$$-\{2ik_1[\tfrac{1}{2}r_{\underline{10}^*}+\tfrac{1}{9}(1-\eta_3)^2A_8{}^{-1}+(-\tfrac{1}{3}+\tfrac{1}{2}\eta_2-\tfrac{1}{6}\eta_3)^2A_{10}{}^{-1}]+(\tfrac{1}{3}+\tfrac{1}{2}\eta_2+\tfrac{1}{6}\eta_3)\}\frac{d}{d\lambda}(ik_1)$$

$$+\{-\tfrac{1}{2}(\eta_2+\eta_3)(ik_1)^{-1}+\tfrac{1}{3}\eta_3(1-\eta_3)A_8{}^{-1}-\tfrac{1}{2}(\eta_2-\eta_3)(-\tfrac{1}{3}+\tfrac{1}{2}\eta_2-\tfrac{1}{6}\eta_3)A_{10}{}^{-1}\}(m_\Sigma-m_\Lambda)m_0=0\ . \tag{A1}$$

Therefore, in the different sheets, the initial motion of the pole is given by

$$\text{I}(+,+):\ \frac{d}{d\lambda}(ik_1)=-(m_\Sigma-m_\Lambda)m_0(ik_1)^{-1}(ik_1r_{\underline{10}^*}+1)^{-1}\ , \tag{A2}$$

$$\text{II}(-,-):\ \frac{d}{d\lambda}(ik_1)=(m_\Sigma-m_\Lambda)m_0[(ik_1)^{-1}-\tfrac{2}{3}a_8][ik_1(r_{\underline{10}^*}+\tfrac{8}{9}a_8+\tfrac{8}{9}a_{10})-\tfrac{1}{3}]^{-1}, \tag{A3}$$

$$\text{III}(+,-):\ \frac{d}{d\lambda}(ik_1)=-(m_\Sigma-m_\Lambda)m_0(\tfrac{2}{3}a_8+\tfrac{1}{3}a_{10})[ik_1(r_{\underline{10}^*}+\tfrac{8}{9}a_8+\tfrac{2}{9}a_{10})+\tfrac{2}{3}]^{-1}\ , \tag{A4}$$

$$\text{IV}(-,+):\ \frac{d}{d\lambda}(ik_1)=-(m_\Sigma-m_\Lambda)m_0a_{10}(ik_1)^{-1}(r_{\underline{10}^*}+2a_{10})^{-1}\ . \tag{A5}$$

In the first and second sheets the poles move towards the real axis. If a_{10} is positive, the same happens in the other two sheets. We are interested in the displacement of the poles in the last two sheets. In particular we want to determine under what conditions two poles meet in either of these two sheets. To first order in λ and making use of the baryon octet mass sum rule, the relations between the momenta in the different channels are:

$$\kappa_1{}^2=\kappa_2{}^2-\tfrac{1}{2}\Lambda\ , \tag{A6}$$

$$\kappa_3{}^2=\kappa_2{}^2+\Lambda\ , \tag{A7}$$

$$\Lambda=\lambda m_0(m_\Sigma-m_\Lambda)\ , \tag{A8}$$

with $\kappa_j=ik_j$.

At the position where the poles meet the energy is a simultaneous solution of Eq. (4.1) and its derivative:

$$-r_{\underline{10}^*}-\left(\frac{1}{3}\frac{1}{\kappa_1}+\frac{1}{2}\frac{1}{\kappa_2}+\frac{1}{6}\frac{1}{\kappa_3}\right)-\frac{2}{9}(\kappa_1-\kappa_3)\left(\frac{1}{\kappa_1}-\frac{1}{\kappa_3}\right)a_8-2\left(-\frac{1}{3}\kappa_1+\frac{1}{2}\kappa_2-\frac{1}{6}\kappa_3\right)\left(-\frac{1}{3}\frac{1}{\kappa_1}-\frac{1}{2}\frac{1}{\kappa_2}-\frac{1}{6}\frac{1}{\kappa_3}\right)a_{10}=0\ . \tag{A9}$$

It follows from these equations that unless a_{10} is very large and negative, which would imply the existence of a bound state in the 10 decuplet, the condition for the poles to meet in either of two adjacent sheets is given by

$$\tfrac{1}{3}a_8\kappa_3+(\tfrac{1}{2}\kappa_2-\tfrac{1}{6}\kappa_3)a_{10}\lessgtr\tfrac{1}{2}\ . \tag{A10}$$

The upper and lower signs of inequality correspond to positions of the double pole in the upper and lower k_1 plane ($\kappa_1 \gtrless 0$). Let us determine the value of a_{10} for which the poles meet at $\kappa_1 = 0$, so that

$$\kappa_2^2 = \tfrac{1}{3}\kappa_3^2 = \tfrac{1}{2}\Lambda . \quad \text{(A11)}$$

It will be given by solving the simultaneous equations (4.1) with $\kappa_1 = 0$, $\kappa_2 = (1/\sqrt{2})\Lambda$, $\kappa_3 = \pm(\frac{3}{2})^{1/2}\Lambda$, and

$$\left[\pm\frac{2}{\sqrt{3}} a_8 + \left(1 \mp \frac{1}{\sqrt{3}}\right) a_{10}\right]\frac{\Lambda}{\sqrt{2}} = 1 . \quad \text{(A12)}$$

The upper and lower signs in these equations correspond to occurence of the double pole at $\kappa_1 = 0$ on the first two sheets or the last two sheets, respectively. Taking the values for $a_{\underline{10}*}$, $r_{\underline{10}*}$, and a_8 as given by the model discussed in Sec. IV, one finds for the two cases the solutions,

(i) For double pole at threshold in sheets I-II:

$$a_{10} = 18.15 \text{ fm}, \quad \lambda = 0.0126 . \quad \text{(A13)}$$

(ii) For double pole at threshold in sheets III-IV:

$$a_{10} = 4.65 \text{ fm}, \quad \lambda = 0.02 . \quad \text{(A14)}$$

Then it follows from (A10) that if $a_{10} < 4.65$ fm, the poles will meet on sheet IV, whereas if $a_{10} > 4.65$ fm, they will meet on sheet III. It should be noted that (A13) and (A14) are values for a_{10} at unphysical values of λ. However, since we are considering channels with $Y = 0$, the parameters of the K matrix should have no first-order terms in λ and therefore might not be expected to vary much as we go from the values at $\lambda = 0$ to the actual physical values at $\lambda = 1$.

Assuming $a_8 = a_{10} = 0$, the positions of the double poles and corresponding values of λ are

I-II:

$$\kappa_1 = -0.056 \text{ fm}^{-1}, \quad \kappa_2 = 0.15 \text{ fm}^{-1}, \quad \kappa_3 = 0.25 \text{ fm}^{-1}, \quad \lambda = 0.018; \quad \text{(A15)}$$

III-IV:

$$\kappa_1 = -0.10 \text{ fm}^{-1}, \quad \kappa_2 = 0.30 \text{ fm}^{-1}, \quad \kappa_3 = -0.50 \text{ fm}^{-1}, \quad \lambda = 0.07 . \quad \text{(A16)}$$

---

*Research supported by the U. S. Atomic Energy Commission under contract AT(11-1) 3075.

[1]R. J. Oakes and C. N. Yang, Phys. Rev. Lett. 11, 174 (1963).

[2]R. J. Eden and J. R. Taylor, Phys. Rev. Lett. 11, 516 (1963).

[3]M. Nauenberg and J. C. Nearing, Phys. Rev. Lett. 12, 63 (1964).

[4]Marc H. Ross, Phys. Rev. Lett. 11, 450 (1963).

[5]C. R. Hagen, Phys. Rev. Lett. 12, 153 (1964).

[6]J. Curiale, Yale University report (unpublished).