at High Energies, 1971, edited by N. B. Mistry (Ref. 1).
[5]For a review, see K. Wilson, in *Proceedings of the International Symposium on Electron and Photon Interactions at High Energies, 1971,* edited by N. B. Mistry (Ref. 1).
[6]J. D. Bjorken, Phys. Rev. 179, 1547 (1969).
[7]R. P. Feynman, quoted in Ref. 8 below.
[8]J. D. Bjorken and E. Paschos, Phys. Rev. 185, 1975 (1969).
[9]R. A. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971); Y. Frishman, Phys. Rev. Lett. 25, 966 (1970); H. Fritzsch and M. Gell-Mann, in *Proceedings of the International Conference on Duality and Symmetry in Hadron Physics,* edited by E. Gotsman (Weizmann Science Press, Jerusalem, 1971).
[10]J. C. Polkinghorne, Nuovo Cimento 8A, 592 (1972); R. L. Jaffe, Phys. Lett. 37B, 517 (1971); Phys. Rev. D 5, 2622 (1972).
[11]H. T. Nieh, Phys. Rev. D 1, 3161 (1970). Discussion of the photon-dissociation mechanism in electron-nucleus scattering had been previously made by V. N. Gribov, report, 1969 (unpublished); S. J. Brodsky and J. Pumplin, Phys. Rev. 182, 1794 (1969).
[12]What value of ω should be considered as large will become clear in Sec. IV. See also H. T. Nieh, Ref. 11.
[13]H. T. Nieh, Ref. 11; Ashok suri and D. R. Yennie, Ann. Phys. (N.Y.) 72, 243 (1972).
[14]B. L. Ioffe, Phys. Letters 30B, 123 (1969). See also J. Pestieau, P. Roy, and H. Terazawa, Phys. Rev. Lett. 25, 402 (1970). The latter reference assumes scaling and makes use of the light-cone formalism.
[15]H. T. Nieh, Ref. 11; K. Gottfried, Cornell University report, 1969 (unpublished).
[16]H. T. Nieh, Ref. 11.
[17]S. Drell and T.-M. Yan, Ann. Phys. (N.Y.) 66, 578 (1971).
[18]J. Kuti and V. F. Weisskopf, Phys. Rev. D 4, 3418 (1971).
[19]See, for example, H. Überall, *Electron Scattering From Complex Nuclei* (Academic, New York, 1971).
[20]J. D. Bjorken, in *Proceedings of the International School of Physics "Enrico Fermi," Course XLI,* edited by J. Steinberger (Academic, New York, 1968).
[21]See, for example, S. Drell, in *Proceedings of the International School of Physics "Ettore Majorana," Course VII,* edited by A. Zichichi (Academic, New York, 1971).
[22]J. J. Sakurai, Phys. Rev. Lett. 22, 981 (1969).
[23]J. D. Bjorken and S. F. Tuan, Comments Nucl. Part. Phys. 5, 71 (1972).
[24]S. Adler, Phys. Rev. 143, B1144 (1966).

PHYSICAL REVIEW D VOLUME 7, NUMBER 11 1 JUNE 1973

Kinematics, Gauge Invariance, Helicity Conservation for $\pi V \to \pi V$ and Related Reactions

G. E. Hite and H. J. W. Müller

Fachbereich Physik, Universität Trier-Kaiserslautern, Kaiserslautern, Germany

(Received 30 October 1972)

A detailed discussion of the kinematics and Reggeization of pion and vector-meson scattering and crossed reactions is presented in terms of helicity amplitudes and invariant amplitudes. It is shown that the gauge-invariance condition allows a smooth limit to the case of massless vector mesons, i.e., photons. Particular attention is paid to the importance of kinematic constraints. It is shown how they are necessary in obtaining the pion pole in Compton scattering. The total photoabsorption cross section is considered, and it is seen that in order to obtain a finite, nonzero asymptotic value of the cross section, the introduction of a singularity in the appropriate residue is unavoidable, but compatible with kinematic and gauge-invariance constraints. Finally, the conservation of helicity is investigated. It is found that the factorization of the pion residue functions is a necessary condition for conservation of helicity in the $VV \to \pi\pi$ channel. In Compton scattering, we find that the kinematic constraints imply the conservation of helicity, provided the total photoabsorption cross section approaches a finite and nonzero limit.

I. INTRODUCTION

The e^+e^- annihilation experiments being performed at the Frascati storage ring,[1] and similar experiments being planned at other laboratories have generated considerable interest. The significance of the experiments lies in their capability of determining the structure functions for electroproduction and thus to yield information on the proton and pion form factors. Since the electroproduction processes (including e^+e^- annihilation into hadrons) involve the exchange of virtual offshell photons, the formulation and enforcement of gauge-invariance requirements becomes particularly important in the limit of vanishing photon mass.

Theoretical investigations of electroproduction in various models generally use unitarity for relating the structure functions to the forward amplitude for the scattering of massive photons off the target particle. In the following we confine our discussion of Compton-like scattering to a pion target. This case is of interest beyond the applications just mentioned. For example, the e^+e^- storage-ring experiments[2] enable also a determination of the cross section for the process $\gamma\gamma \to \pi\pi$,[3] which is simply the crossed-channel reaction for Compton scattering. A thorough investigation of πV scattering and crossed reactions is also motivated by the desire to understand the gauge invariance condition in terms of invariant amplitudes, to understand the significance of kinematic constraints on helicity conservation, and to understand the difficulties associated with maintaining a nonzero total photoabsorption cross section when the contributing Pomeranchukon residue contains a $\alpha_P - 1$ factor.

The paper is organized as follows: In Secs. II and III we present the kinematics of two-photon annihilation and Compton-like scattering, which are the basis for the subsequent sections. In Sec. IV, we investigate the gauge invariance in the limit of vanishing photon mass. In Sec. V, we discuss the contribution of the pion Born graphs to the invariant amplitudes. Sections VI and VII deal with the Reggeization of the Compton-like and the crossed channel. The relevant kinematic constraints are derived and used to investigate the limit of vanishing photon mass. In particular, we show that the maintenance of the pion pole in the Compton-like channel in the limit of vanishing photon mass requires a careful consideration of the kinematic constraints obeyed by the helicity amplitudes in that channel. In Sec. VIII, we use the solutions to the kinematic constraint equations to study the mass limit necessary to obtain the total photoabsorption cross section. It is shown that the vanishing of the multiplicative factor $(\alpha_P - 1)$ in the forward direction necessitates the existence of a singularity in the appropriate Pomeranchukon residue. This singularity or pole is compatible with the kinematic constraints even in the limit of vanishing photon mass.

Finally, in Sec. IX, we examine the conditions for conservation of helicity. We find that in $VV \to \pi\pi$ scattering helicity conservation requires the factorization of the pion residues whereas in Compton scattering the conservation of helicity is implied by the kinematical constraints.

Since a great deal of work has been done on the process under discussion we conclude the introduction by referring to related work in the literature. The formulation of gauge invariance has previously been discussed by Ebata and Lassila.[4] However, their conclusions are obtained rather indirectly. We show here that the gauge-invariance conditions are nontrivial and may be discussed in a simpler way. The kinematical constraints of the s-channel helicity amplitudes have not (to our knowledge) been discussed previously. They are necessary in obtaining the pion pole in much the same way as in pion photoproduction, and this problem has been discussed by many authors.[5-7] Questions related to the problem of the fixed pole in Compton scattering have been considered by Abarbanel *et al.* and Arbab and Brower and others.[8] In Sec. VII we obtain the same solution to the kinematical constraints as the latter group of authors; however, our conclusions are different. Some considerations of helicity conservation in Compton scattering, i.e., without a detailed consideration of the kinematical constraints, have been made by Biyajima.[9]

II. TWO-PHOTON ANNIHILATION (t CHANNEL)

We consider first the kinematics of the process of two-photon annihilation with production of a pair of pions, or, more generally, the reactions

$$t:\quad V(p_1)+V(p_2')\to\pi(q_1')+\pi(q_2)\,, \tag{2.1}$$

$$s:\quad V(p_1)+\pi(q_1)\to V(p_2)+\pi(q_2)\,, \tag{2.2}$$

where V and π represent vector and pseudoscalar mesons of mass M and m, respectively. It is convenient to define the quantities

$$P=\tfrac{1}{2}(p_1+q_1')\quad\text{and}\quad Q=\tfrac{1}{2}(p_2+q_2') \tag{2.3}$$

as well as the Mandelstam variables

$$\begin{aligned} s&=-(p_1+q_1)^2\\ &=-(p_1-q_1')^2\,,\\ t&=-(p_1+p_2')^2\\ &=-(p_1-p_2)^2\,,\\ u&=-(p_1+q_2')^2\\ &=-(p_1-q_2)^2\,, \end{aligned}$$

where we have used a prime, e.g., $p_1=-p_1'$ to indicate a time-reversed momentum.

The t-channel center-of-mass momenta, p_t, q_t, and the scattering angle, θ_t, are defined by the vectors

$$
\begin{aligned}
p_1 &= (0, 0, p_t, p_{t0}), \\
q_1' &= (q_t \sin\theta_t, 0, q_t \cos\theta_t, q_{t0}), \\
p_2' &= (0, 0, -p_t, p_{t0}), \\
q_2 &= (-q_t \sin\theta_t, 0, -q_t \cos\theta_t, q_{t0}),
\end{aligned} \tag{2.4}
$$

where

$$p_{t0}^2 = q_{t0}^2 = \tfrac{1}{4} t = p_t^2 + M^2 = q_t^2 + m^2 .$$

The transverse and longitudinal polarization vectors of the incoming vector meson $V(p_1)$ having momentum p_t along the z axis are given by[10]

$$
\begin{aligned}
\epsilon_1^{\mu(\pm)} &= \frac{1}{\sqrt{2}} (\mp 1, -i, 0, 0), \\
\epsilon_1^{\mu(0)} &= \frac{1}{M} (0, 0, p_{t0}, p_t).
\end{aligned} \tag{2.5}
$$

The polarization vectors of the target vector meson $V(p_2')$ are given in the center-of-mass by rotating those of $V(p_1)$ by π radians about the y axis, i.e.,

$$
\begin{aligned}
\epsilon_2^{\mu(\pm)} &= \frac{1}{\sqrt{2}} (\pm 1, -i, 0, 0), \\
\epsilon_2^{\mu(0)} &= \frac{1}{M} (0, 0, -p_{t0}, p_t).
\end{aligned} \tag{2.6}
$$

The S-matrix elements for the t-channel scattering process may be written[11]

$$S_{fi} = i(2\pi)^4 \delta^4(p_1 + p_2' - q_1' - q_2) \epsilon_1^\mu T_{\mu\nu} \epsilon_2^\nu . \tag{2.7}$$

The invariant transition amplitude $\mathfrak{M}_{fi}$ $(=\epsilon_1^\mu \epsilon_2^\nu T_{\mu\nu})$ may be written in terms of the invariant amplitudes A, B, C, D as

$$
\begin{aligned}
\mathfrak{M}_{fi} = {} & (\epsilon_1 \cdot P)(\epsilon_2 \cdot Q) A \\
& + (\epsilon_1 \cdot Q \epsilon_2 \cdot Q + \epsilon_1 \cdot P \epsilon_2 \cdot P) B \\
& + (\epsilon_1 \cdot Q)(\epsilon_2 \cdot P) C + (\epsilon_1 \cdot \epsilon_2) D .
\end{aligned} \tag{2.8}
$$

This expression incorporates explicitly time-reversal invariance. The t-channel helicity amplitudes defined by

$$
\begin{aligned}
f^t_{\lambda\pi_1, \lambda\pi_2; \lambda V_1, \lambda V_2} &\equiv f^t_{\lambda V_1 \lambda V_2} \\
&= \epsilon_1^{\lambda V_1} \cdot T \cdot \epsilon_2^{\lambda V_2}
\end{aligned}
$$

are easily found to be

$$
\begin{aligned}
f^t_{11} &= -\tfrac{1}{8} q_t^2 \sin^2\theta_t (A + 2B + C) - D, \\
f^t_{1\,-1} &= \tfrac{1}{8} q_t^2 \sin^2\theta_t (A + 2B + C), \\
\sqrt{2}\, M f^t_{10} &= \tfrac{1}{4} q_{t_0} q_t \sin\theta_t [(q_t \cos\theta_t - p_t) A + (q_t \cos\theta_t + p_t) B + (q_t \cos\theta_t + 3p_1) C], \\
M^2 f^t_{00} &= -\frac{q_{10}^2}{4} \left[(qt \cos\theta_t - p_t)^2 A + 2(q_t \cos\theta_t - p_t)(q_t \cos\theta_t + 3q_t) B + (q_t \cos\theta_t + 3p_t)^2 C + 4 \left(1 + \frac{p_t^2}{q_{t0^2}}\right) D \right],
\end{aligned} \tag{2.9}
$$

where

$$\cos\theta_t = \frac{s + p_t^2 + q_t^2}{2 p_t q_t} . \tag{2.10}$$

We could also write[4]

$$
\begin{aligned}
\mathfrak{M}_{fi} = {} & \epsilon_1 \cdot \epsilon_2 A_1 + \epsilon_1 \cdot Q' \epsilon_2 \cdot Q' A_2 \\
& + [\epsilon_1 \cdot Q' \epsilon_2 \cdot p_1 - \epsilon_1 \cdot Q' \epsilon_2 \cdot p_2] A_3 \\
& - \epsilon_1 \cdot p_2 \epsilon_2 \cdot p_1 A_4 ,
\end{aligned}
$$

where

$$Q' = q_1 - q_2 .$$

The relationship between the two sets of invariant amplitudes is given by

$$
\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 & 9 & -6 & 1 \\ 0 & 3 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} .
$$

III. COMPTON-LIKE SCATTERING (s and u CHANNELS)

We now consider the kinematics of the Compton-like process (2.2), i.e.,

$$s: \quad V(p_1) + \pi(q_1) \to V(p_2) + \pi(q_2) .$$

The s-channel c.m. momenta, p_s, and scattering angle, θ_s, are defined by the vectors

$$
\begin{aligned}
p_1 &= (0, 0, p_s, p_{s0}), \\
p_2 &= (p_s \sin\theta_s, 0, p_s \cos\theta_s, p_{s0}), \\
q_1 &= (0, 0, -p_s, p_{s0}), \\
q_2 &= (-p_s \sin\theta_s, 0, -p_s \cos\theta_s, p_{s0}),
\end{aligned} \tag{3.1}
$$

where

$$
\begin{aligned}
p_s^2 &= [s - (M+m)^2][s - (M-m)^2]/4s \\
&= p_{s0}^2 - M^2 = q_{s0}^2 - m^2
\end{aligned}
$$

and

$$\cos\theta_s = 1 + \frac{t}{2p_s^2}.$$

The s-channel helicity amplitudes, defined by

$$f^s_{\lambda v_2 \lambda \pi_2;\, \lambda v_1 \lambda \pi_1} = f^s_{\lambda v_2,\, \lambda v_1} = \epsilon_1^{\lambda v_1} \cdot T \cdot \epsilon_2^{\lambda v_2}, \qquad (3.2)$$

where the polarization vectors are given by

$$\begin{aligned}
\epsilon_1^{\pm 1} &= \frac{1}{\sqrt{2}}(\mp 1, -i, 0, 0),\\
\epsilon_1^{0} &= \frac{1}{M}(0, 0, p_{s0}, p_s),\\
\epsilon_2^{\pm} &= (\epsilon_2(p_2')^{\pm})^* \\
&= \frac{1}{\sqrt{2}}(\mp\cos\theta_s, i, \pm\sin\theta_s, 0),\\
\epsilon_2^{0} &= (\epsilon_2(p_2')^{0})^* \\
&= \frac{1}{M}(p_{s0}\sin\theta_s, 0, p_{s0}\cos\theta_s, p_s)
\end{aligned} \qquad (3.3)$$

are easily found to be

$$\begin{aligned}
f^s_{1,1} &= -\tfrac{1}{2}p_s^2\sin^2\theta_s C + \tfrac{1}{2}(1+\cos\theta_s)D,\\
f^s_{1,-1} &= \tfrac{1}{2}p_s^2\sin^2\theta_s C + \tfrac{1}{2}(1-\cos\theta_s)D,\\
\sqrt{2}Mf^s_{1,0} &= \sin\theta_s[p_s^2(p_{s0}-P_0)B \\
&\quad + p_s^2(p_{s0}\cos\theta_s - P_0)C + p_{s0}D],\\
M^2 f^s_{0,0} &= p_s^2(p_{s0}-P_0)A \\
&\quad + 2p_s^2(p_{s0}-P_0)(p_{s0}\cos\theta_s - P_0)B \\
&\quad + p_s^2(p_{s0}\cos\theta_s - P_0)^2 C \\
&\quad + (p_{s0}^2\cos\theta_s - p_s^2)D.
\end{aligned} \qquad (3.4)$$

Owing to conservation of angular momentum, the helicity-flip amplitudes $f^s_{1,-1}$ and $f^s_{1,0}$ vanish in the forward direction and the others are readily seen to be

$$f^s_{1,1} = D$$

and

$$M^2 f^s_{0,0} = p_s^2(p_{s0}-P_0)^2(A+2B+C) + M^2 D.$$

Comparing these expressions to the Lorentz-invariant amplitudes T_1^π and T_2^π, which are used in electroproduction analysis and defined for forward pion Compton scattering[12] by

$$\begin{aligned}
T_{\mu\nu} &= t^*_{\mu\nu}\\
&= \left(q_{1\mu} - \frac{q_1\cdot p_1}{p_1^2}p_{1\mu}\right)\left(q_{1\nu} - \frac{q_1\cdot p_1}{p_1^2}p_{1\nu}\right)\frac{T_2^\pi\gamma}{m}\\
&\quad - \left(g_{\mu\nu} - \frac{p_{1\mu}p_{1\mu}}{p_1^2}\right)mT_1^\pi\gamma,
\end{aligned}$$

where $\gamma = e^2 m/p_{s0}$, we find

$$T_1^\pi m\gamma = -D = f^s_{1,1}(\theta_s = 0)$$

and

$$\begin{aligned}
T_2^\pi 4\gamma/m &= A + 2B + C\\
&= [f^s_{0,0}(\theta_s=0) - f^s_{1,1}(\theta_s=0)]\frac{M^2}{p_s^2(p_{s0}-P_0)^2}.
\end{aligned}$$

IV. GAUGE INVARIANCE FOR $M \to 0$

In this section, we investigate the limit of vanishing vector-meson mass, M, with the restriction that all quantities reduce to those obtained for a physical photon as M goes to zero. This condition is not automatically fulfilled: The relationships between helicity and invariant amplitudes given in Secs. III and IV contain coefficients for the longitudinal vector-meson components which are singular in the limit $M \to 0$.

Gauge invariance[7] means that the total transition amplitude $\mathfrak{M}_{fi}$ remains invariant for vanishing meson mass under the transformation

$$\epsilon_p^\mu \to \epsilon_p^\mu + \lambda p^\mu$$

for arbitrary λ. Since $\mathfrak{M}_{fi} = \epsilon_1^\mu T_{\mu\nu}\epsilon_2^\nu$, we must have

$$T_{\mu\nu}(\lambda_1 p_1^\mu \epsilon_2^\nu + \lambda_2 \epsilon_1^\mu p_2^\nu + \lambda_1\lambda_2 p_1^\mu p_2^\nu) = O(M^2).$$

Since λ_1 and λ_2 are arbitrary, each term in this expression must vanish independently in the limit $M \to 0$, i.e.,

$$\begin{aligned}
T_{\mu\nu}p_1^\mu p_2^\nu &= O(M^2),\\
T_{\mu\nu}p_1^\mu \epsilon_2^\nu &= O(M^2),\\
T_{\mu\nu}\epsilon_1^\mu p_2^\nu &= O(M^2).
\end{aligned} \qquad (4.1)$$

Using the kinematic identities

$$\begin{aligned}
p_1\cdot P &= p_2\cdot Q = \tfrac{1}{4}(s - m^2 - 3M^2),\\
p_1\cdot Q &= p_2\cdot P = \tfrac{1}{4}(2t + s - m^2 - 3M^2),
\end{aligned}$$

the first equation in Eq. (4.1), which takes the form

$$(p_1\cdot P)[A(p_1\cdot P) + B(p_1\cdot Q) - D] + (p_1\cdot Q)[B(p_1\cdot P) + C(p_1\cdot Q) + D] = O(M^2),$$

can be written as

$$(s-m^2)^2 A + 2(s-m^2)(2t+s-m^2)B + (2t+s-m^2)^2 C + 8tD \equiv M^2\alpha(M^2), \qquad (4.2)$$

where $\alpha(M^2)$ is finite as $M \to 0$. The fact that $\alpha(0)$ is not arbitrary will be shown below.

The last two equations in Eq. (4.1) can be written

$$T_{\mu\nu}(\lambda_1 p_1^\mu \epsilon_2^\nu + \lambda_2 p_2^\mu \epsilon_1^\nu)$$
$$= [A(p_1\cdot P) + B(p_1\cdot Q) - D](\lambda_1\epsilon_2\cdot Q + \lambda_1\epsilon_1\cdot P)$$
$$+ [B(p_1\cdot P) + C(p_1\cdot Q) + D](\lambda_1\epsilon_2\cdot P + \lambda_2\epsilon_1\cdot Q)$$
$$= O(M^2)\,.$$

Since Eq. (4.2) implies

$$\frac{A(p_1\cdot P) + B(p_1\cdot Q) - D}{2t + s - m^2} = -\frac{B(p_1\cdot P) + C(p_1\cdot Q) + D}{s - m^2} + O(M^2)\,,$$

we obtain

$$A(p_1\cdot P) + B(p_1\cdot Q) - D = O(M^2)\,,$$
$$B(p_1\cdot P) + C(p_1\cdot Q) + D = O(M^2)$$

or

$$(s - m^2)A + (2t + s - m^2)B - 4D = M^2 O_1(M^2)\,,$$
$$(s - m^2)B + (2t + s - m^2)C + 4D = M^2 O_2(M^2)\,. \quad (4.3)$$

These equations are readily seen to be consistent with Eq. (4.2) provided that

$$\alpha(M^2) = (s - m^2)O_1(M^2) + (2t + s - m^2)O_2(M^2)\,. \quad (4.4)$$

The value of the function $\alpha(M^2)$ at $M^2 = 0$ can be obtained by investigating the above gauge constraints and the helicity amplitudes. One might expect that these constraints alone would ensure that helicity amplitudes with longitudinal spin components will vanish in the limit of $M \to 0$. However, this is not true for arbitrary $\alpha(0)$ and for the helicity amplitudes $f^s_{0,0}$ and $f^t_{0,0}$. Using Eqs. (2.9) and (3.4) and expanding about $M^2 = 0$ we obtain

$$f^s_{0,0} = \frac{\alpha + 16D}{16} + \frac{M^2}{8}\left[O_1 + O_2 - \frac{2s}{(s - m^2)^2}(\alpha - 4tO_2)\right] + O(M^4) \quad (4.5a)$$

and

$$f^t_{0,0} = -\frac{\alpha + 16D}{16} - \frac{M^2}{8t}[(2s + t - 2m^2)(O_1 + O_2) - 4tO_2] + O(M^4)\,. \quad (4.5b)$$

Thus, in order to ensure the vanishing of $f^s_{0,0}$ and $f^t_{0,0}$ in the limit of $M \to 0$, we must have

$$\alpha(0) = -16D\,. \quad (4.6)$$

It is easy to see that for $M = 0$, the t-channel helicity amplitudes are related to the s-channel helicity amplitudes by the expected crossing matrix, i.e.,

$$f^t_{1,1} = -f^s_{1,-1}\,, \quad f^t_{1,-1} = -f^s_{1,1}\,. \quad (4.7)$$

V. PION BORN GRAPHS

It is convenient to discuss the pion Born graphs before we go into Reggeization. The appropriate set of Feynman diagrams which yield gauge-invariant amplitudes are shown in Fig. 1. According to Schweber's Feynman rules,[13] these graphs lead to a transition amplitude[14] of the form

$$\mathfrak{M}^\pi_{fi} = 2g^2[\epsilon_1\cdot\epsilon_2 + 2\epsilon_1\cdot q_1\,\epsilon_2\cdot q_2(s\cdot m^2)^{-1} + 2\epsilon_1\cdot q_2\,\epsilon_2\cdot q_1(u - m^2)^{-1}]\,,$$

where g is the $V\pi\pi$ coupling constant (e.g., see the Hamiltonian given by Schweber[15]). Using the relations

$$q_1 = -2P + p_1 = -(P + Q) + p_2$$

and

$$q_2 = -2Q + p_2 = -(P + Q) + p_1\,,$$

the elementary pion contribution to the invariant amplitudes is found to be

$$A = 4g^2\left(\frac{4}{s - m^2} + \frac{1}{u - m^2}\right),$$
$$B = C = \frac{4g^2}{u - m^2}\,, \quad (5.1)$$
$$D = 2g^2\,.$$

It is easy to verify that these expressions satisfy the gauge-invariance constraints Eq. (4.2) and Eq. (4.3). We note, finally, that pion exchange is possible only in $V\pi^\pm$ scattering.

VI. REGGEIZATION OF s-CHANNEL HELICITY AMPLITUDES

Our next step is to Reggeize the s-channel helicity amplitudes. Our notation is rather standard[16] and differs by a factor of $-16\pi(sp_s/p_s')^{1/2}$ from that of Jacob and Wick[17]:

$$f^s_{cd,ab} = \sum_{j=m}^{\infty}(2j + 1)f^j_{cd,ab}d^j_{\lambda\mu}(\theta_s)\,,$$

where $\lambda \equiv a - b$, $\mu \equiv c - d$, $m = \max(|\lambda|, |\mu|)$, $n = \min(|\lambda|, |\mu|)$. The first step is the construc-

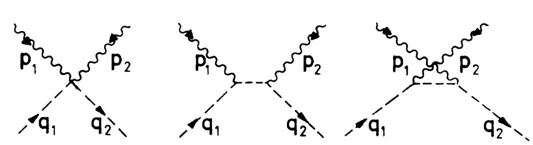

FIG. 1. s-channel pion Born graphs.

tion of amplitudes free of kinematic singularities in t. Since the kinematic singularities in t are known to come from the d functions, they may be removed by defining the amplitudes[18]

$$\bar{f}^{s}_{cd,ab} \equiv \left(\frac{1+\cos\theta_s}{2}\right)^{-|\lambda+\mu|/2} \times\left(\frac{1-\cos\theta_s}{2}\right)^{-|\lambda-\mu|/2} f^{s}_{cd,ab}\,.$$

These amplitudes can then be used to construct the so-called "parity-conserving" amplitudes[16, 19]:

$$f^{\eta}_{cd,ab} \equiv \bar{f}_{cd,ab} + (-1)^{m+\lambda}\eta\eta_c\eta_d(-1)^{S_c+S_d-\nu}\bar{f}_{-c-d,ab}$$
$$= \sum_{j=m}^{\infty}(2j+1)(e^{j+}_{\lambda\mu}f^{j\eta}_{cd,ab} + e^{j-}_{\lambda\mu}f^{j-\eta}_{cd,ab})\,,$$

where η is the spin-parity (i.e., $\eta = P\tau$ = parity × signature) of the amplitudes, η_c and η_d are the intrinsic parities, and S_c and S_d are the intrinsic spins of outgoing particles c and d, respectively; the $e^{j\pm}_{\lambda\mu}$ are the e functions of Gell-Mann *et al.*[19] For the s-channel helicity amplitudes, we then have

$$f^{\pm}_{1,1} = f^{s}_{1,1}\left(\frac{2}{1+\cos\theta_s}\right) \mp f^{s}_{1,-1}\left(\frac{2}{1-\cos\theta_s}\right),$$
$$f^{\pm}_{1,0} = f^{s}_{1,0}\left(\frac{2}{\sin\theta_s}\right)(1\mp 1)\,, \qquad (6.1)$$
$$f^{\pm}_{0,0} = f^{s}_{0,0}(1\mp 1)\,.$$

The Regge-pole contributions are then given by

$$f^{\eta}_{cd,ab} = \sum_{\alpha} \beta^{\eta_\alpha\tau_\alpha}_{\lambda,\mu}\frac{1+\tau_\alpha e^{-i\pi\alpha}}{\sin\pi\alpha} \times\frac{(2\alpha+1)!\,(t/4p_s^{\,2})^{\alpha-m}}{[(\alpha+m)!\,(\alpha-m)!\,(\alpha+n)!\,(\alpha-n)!\,]^{1/2}}\,,$$

where β is the residue of the amplitude at the pole $j=\alpha(s)$. This residue contains the kinematic threshold and pseudothreshold singularities in s as well as kinematical singularities at $s=0$. It is well known that the unitarity condition implies factorization of the residues, i.e.,

$$\beta_{\lambda\mu}(s) = \gamma_\lambda(s)\gamma_\mu(s)\,,$$

or

$$\beta_{\lambda\mu}{}^2(s) = \beta_{\lambda\lambda}(s)\beta_{\mu\mu}(s)\,.$$

In order to demonstrate explicit factorization at the aforementioned singularity $s=0$, a factor s^{δ}, where δ for the "unequal-unequal" mass configuration being considered here is given by n,[20] must be extracted from the residue.

In the sense-choosing mechanism[20] the residue may be written as

$$\beta^{\eta\tau}_{\lambda\mu} = \frac{(s/s_0)^{\delta}K^{\eta}_{\lambda\mu}(s)s_0^{\xi}\pi^{1/2}}{2^{m-n}(\alpha+\frac{1}{2})!}\left[\frac{(\alpha+m)!\,(\alpha+n)!}{(\alpha-m)!\,(\alpha-n)!}\right]^{1/2} \times\gamma_{\lambda\mu}(s)\left(\frac{p_s^{\,2}}{s_0}\right)^{\alpha-m},$$

where $K^{\eta}_{\lambda,\mu}(s)$ contains the kinematic singularities of $f^{\eta}_{cd,ab}$, the arbitrary scale factor s_0 is measured in the same units as s, and ξ is chosen so that $s_0^{\xi}K^{\eta}_{\lambda,\mu}(s)$ is dimensionless. Expressions for $K^{\eta}_{\lambda,\mu}(s)$ for most mass configurations are available in the literature.[20, 21] Since the kinematic singularities are contained in $K^{\eta}_{\lambda,\mu}(s)$, the reduced residue $\gamma_{\lambda,\mu}(s)$ is free of kinematic singularities.

In terms of $\gamma_{\lambda,\mu}$ we have

$$f^{\eta}_{cd,ab} = \sum_{\alpha} K^{\eta}_{\lambda,\mu}s_0{}^{\xi}\left(\frac{s}{s_0}\right)^{\delta}\gamma_{\lambda\mu}\left(\frac{1+\tau e}{\sin\pi\alpha}\right) \times\frac{2^{m+n+1}\alpha!}{(\alpha-m)!\,(\alpha-n)!}\left(\frac{t}{s_0}\right)^{\alpha-m}. \qquad (6.2)$$

Since we are interested in the relation between the invariant amplitudes and the Regge-pole contributions to the helicity amplitudes, it is useful to express the latter in terms of the former. Substituting Eq. (3.4) into Eq. (6.1) we obtain

$$f^{+}_{1,1} = -\frac{\rho^2}{2s}C \equiv \frac{\rho^2}{s}\tilde{f}^{+}_{1,1}\,,$$
$$f^{-}_{1,1} = 2D + \left(t+\frac{\rho^2}{2s}\right)C \equiv \frac{1}{s}\tilde{f}^{-}_{1,1}\,,$$
$$\left(\frac{M}{\sqrt{2}}\right)f^{-}_{1,0} = \frac{1}{\sqrt{s}}\left[\tfrac{1}{4}\rho^2B + \tfrac{1}{4}(\rho^2+2s't)C + s'D\right] \equiv \frac{1}{\sqrt{s}}\tilde{f}^{-}_{1,0}\,, \qquad (6.3)$$
$$\tfrac{1}{2}M^2 f^{-}_{0,0} = \frac{\rho^2}{16}A + \tfrac{1}{8}(\rho^2+2s't)B + \frac{1}{4}\left(\frac{\rho^2}{4}+\frac{s'^2t^2}{\rho^2}+s't\right)C + \left(M^2+\frac{t}{2\rho^2}s'^2\right)D \equiv \frac{1}{\rho^2}\tilde{f}^{-}_{0,0}\,,$$

where $\rho^2 = 4sp_s^{\,2}$ and $s' = s - m^2 + M^2$. We have also define amplitudes $\tilde{f}^{\eta}_{\lambda,\mu}$ by extracting factors of $(M/\sqrt{2})$ and the kinematic factors which can be deduced from the coefficients of the invariant amplitudes in Eq. (6.3) or as expressed by Eq. (26) of Ref. 21. These equations may be inverted to express the invariant amplitudes in terms of the new amplitudes:

$$A=\frac{16}{\rho^4}\tilde{f}_{0,0}^- - \frac{8}{\rho^4}(\rho^2+2ts')\tilde{f}_{1,0}^- + \frac{4}{s\rho^4}[(s'-2M^2)\rho+ts'^2]\tilde{f}_{1,1}^- + \frac{2}{s}[s+2(t-M^2-m^2)]\tilde{f}_{1,1}^+ , \quad (6.4)$$

$$B=\frac{4}{\rho^2}\tilde{f}_{1,0}^- - \frac{2s'}{s\rho^2}\tilde{f}_{1,1}^- - \frac{2(M^2-m^2)}{s}\tilde{f}_{1,1}^+ ,$$

$$C=-2\tilde{f}_{1,1}^+ ,$$

$$D=\frac{1}{2s}[\tilde{f}_{1,1}^- + (\rho^2+2st)\tilde{f}_{1,1}^+] .$$

Since the invariant amplitudes are free of kinematic singularities, we can easily write down the kinematic constraint equations between the $\tilde{f}$ amplitudes. At $s=0$, all of the invariant amplitudes except C result in the same conspiracy relation

$$\tilde{f}_{1,1}^- + (M^2-m^2)^2\tilde{f}_{1,1}^+ = O(s) . \quad (6.5)$$

Near the points $s=s_\pm \equiv (M\pm m)^2$ the following kinematic constraint equations must be satisfied

$$2s\tilde{f}_{1,0}^- - s'\tilde{f}_{1,1}^- = O(\rho^2) \quad (6.6)$$

and

$$\tilde{f}_{0,0}^- - (ts'+\tfrac{1}{2}\rho^2)\tilde{f}_{1,0}^- + \frac{1}{4s}[ts'^2+(s'-2M^2)\rho^2]\tilde{f}_{1,1}^- = O(\rho^4) . \quad (6.7)$$

For the discussion here and considerations of helicity conservation in Sec. IX, it is convenient to consider only the leading Regge-pole contribution and to write

$$\tilde{f}_{0,0}^- = \Gamma_{0,0}t^\alpha ,$$
$$\tilde{f}_{1,0}^- = \Gamma_{1,0}t^{\alpha-1} ,$$
$$\tilde{f}_{1,1}^- = \Gamma_{1,1}t^{\alpha-1} .$$

Then the constraint equations [(6.6) and (6.7)] take the form

$$2s\,\Gamma_{1,0}(s) - s'\Gamma_{1,1}(s) = O((s-s_+)(s-s_-)) , \quad (6.8a)$$

$$4s\,\Gamma_{0,0}(s) - 4ss'\Gamma_{1,0}(s) + s'^2\Gamma_{1,1}(s) = O((s-s_+)^2(s-s_-)^2) . \quad (6.8b)$$

Substituting (6.8a) into (6.8b) yields

$$\Gamma_{0,0}(s) - \left(\frac{s'^2}{4s}\right)\Gamma_{1,1}(s) = O((s-s_+)(s-s_-))$$

or

$$\Gamma_{0,0}(s) - M^2\Gamma_{1,1}(s) = O((s-s_+)(s-s_-)) . \quad (6.8c)$$

Using Eqs. (6.8a) and (6.8c), Eq. (6.8b) may be written as a derivative constraint equation of the form

$$s\Gamma'_{0,0}(s) - ss'\Gamma'_{1,0} + \tfrac{1}{4}s'^2\Gamma'_{1,1} - M^2\Gamma_{1,1} = O((s-s_+)(s-s_-)) . \quad (6.8d)$$

It is interesting to note that the factorization property of the residues $\beta_{\lambda,\mu}$ as expressed in terms of the $\Gamma_{\lambda,\mu}$'s, i.e.,

$$\Gamma_{0,0}(s)\Gamma_{1,1}(s) = [\Gamma_{1,0}(s)]^2 s , \quad (6.9)$$

relates the solutions to Eqs. (6.8a), (6.8b), and (6.8c). In particular, given a solution to Eq. (6.8a), Eq. (6.9) and its derivative then ensure that the other constraints, i.e., Eqs. (6.8b), (6.8c), or (6.8d), are satisfied.

It is also interesting to see how the kinematic constraints ensure that the pion pole, which for $M\neq 0$ occurs only in $f_{0,0}^s$, does not vanish in the limit $M\to 0$, where $f_{0,0}^s$ vanishes.

From Sec. V, we know that only A contains the s-channel pion pole. Looking at the expression for A in Eq. (6.4) and realizing that $\tilde{f}_{0,0}$ and $\tilde{f}_{1,0}$ both vanish and $\rho\to(s-m^2)$ as $M\to 0$, it is clear that the pion pole must come from the coefficient of $f_{1,1}^-$. A factor ρ is of course a kinematic and not a dynamic factor. In fact for $M\neq 0$, the quantity $\rho^{-4}\tilde{f}_{1,1}^-$ vanishes at $s=m^2$ in the sense choosing coupling scheme. The correct procedure for taking the $M\to 0$ limit requires a careful consideration of the threshold and pseudothreshold constraints.

Expressing the kinematic constraint equation (6.8c) in terms of the reduced residues $\gamma_{1,1}$ and $\gamma_{0,0}$ we obtain

$$\gamma_{0,0}(s) - 4M^2 s\alpha_\pi(s)\gamma_{1,1}(s)/s_0^{\,2} = O((s-s_+)(s-s_-)) ,$$

or

$$\frac{1}{M^4}\gamma_{0,0}(s_\pm=(M\pm m)^2) = 4(2m\pm M)^2(s_\pm/s_0^{\,2})(\alpha'_\pi)^2\gamma_{1,1}(s_\pm) ,$$

where we have used $\alpha_\pi(s)=(s-m^2)\alpha'_\pi$. The gauge condition, $f_{0,0}^s=O(M^2)$ as $M\to 0$, ensures that the quantity $\gamma_{0,0}/M^4$ is well behaved as $M\to 0$.

Assuming that the reduced residue $\gamma_{1,1}(s_\pm=(M\pm m)^2)$ is a smooth function of M as $M\to 0$, i.e.,

$$\gamma_{1,1}(s_\pm)=\gamma_{1,1}(m^2)+O(M) ,$$

we have

$$\gamma_{0,0}(s_\pm)/M^4 = s_0^{-2}\,16m^2(\alpha'_\pi)^2\gamma_{1,1}(m^2)+O(M).$$

For M small but nonzero, the dominant pion-pole contribution to A is

$$A=\frac{16s_0^{\,2}}{\pi m^4\alpha'_\pi}\frac{\gamma_{0,0}(m^2)/M^4}{s-m^2}+O(M)$$

$$=\frac{64\,s_0\alpha'_\pi}{\pi}\frac{\gamma_{1,1}(m^2)}{s-m^2}+O(M) .$$

Since both sides of this equation are analytic in M, we may take the limit of $M \to 0$ to obtain

$$A = \frac{64 s_0 \alpha'_\pi}{\pi} \frac{\gamma_{1,1}(m^2)}{s - m^2} \quad (M=0) . \tag{6.10}$$

Comparing this expression with Eq. (5.1), we may identify the residue with the coupling constant $16g^2$:

$$\gamma_{1,1}(m^2) = \frac{\pi g^2}{4 s_0 \alpha'_\pi} \quad (M=0) . \tag{6.11}$$

We see therefore that although the dynamical t-channel pion pole is contained in $f^s_{0,0}$ (as it must to satisfy the angular momentum selection rule for radiative transitions), the kinematic constraints relate its residue to the residue of the "kinematical" pion pole, $-16(ts'/\rho^2)\bar{f}^-_{1,1}$, at vanishing M.

Such interplay between dynamic poles and poles of apparent kinematic origin are not uncommon in reactions involving photons. For example, in the t-channel reaction $\gamma N \to \pi N$, the pion pole for $M_\gamma \neq 0$ is present in all t-channel helicity amplitudes but is present only in $f^s_{00,NN'}$ of the s-channel helicity amplitudes. In the $M_\gamma \to 0$ limit, where this amplitude vanishes, the pion pole is found to come from a "kinematic" pole in the nucleon contributions to the s-channel helicity amplitude, and consequently the pion pole is present for all values of M.

VII. REGGEIZATION OF t-CHANNEL HELICITY AMPLITUDES

Reggeization of the t-channel helicity amplitudes proceeds analogously to those of the s-channel helicity amplitudes as done in Sec. VI, and we mention only the important equations.

The so-called "parity-conserving" amplitudes are given by

$$f^\eta_{cd,ab} = \sum_\alpha K^\eta_{\lambda\mu}(t) s_0{}^\xi \left(\frac{t}{s_0}\right)^\delta \gamma^\eta_{cd,ab}(t) \times \left(\frac{1+\tau e^{-i\pi\alpha}}{\sin\pi\alpha}\right) \frac{2^{m+n+1}\alpha!}{(\alpha-m)!\,(\alpha-n)!}\left(\frac{s}{s_0}\right)^{\alpha-m},$$

where δ for the "equal-equal" mass configuration of the t channel is given by

$$\tfrac{1}{4}\{[1-\eta(-1)^\lambda]+[1-\eta(-1)^\mu]\}-\epsilon ,$$

where ϵ is the power of t in $K^\eta_{\lambda\mu}(t)$.[20] Considering only the dominant Pomeranchukon contribution, we have, in particular,

$$\begin{aligned} f^+_{1\,-1} &= 8\gamma_{1\,-1}(t)(t-4m^2)\alpha_P(\alpha_P-1) \times \left(\frac{1+e^{-i\pi\alpha_P}}{\alpha_P!\,\sin\pi\alpha_P}\right) s^{\alpha_P-2}, \\ f^+_{11} &= 2\gamma_{11}(t)(t-4M^2)^{-1} \times \left(\frac{1+e^{-i\pi\alpha_P}}{\alpha_P!\,\sin\pi\alpha_P}\right) s^{\alpha_P}. \end{aligned} \tag{7.1}$$

From the expression for $f^+_{1\,-1}$, it is clear that $f^+_{1\,-1}$ vanishes at $t=0$, if the intercept of the Pomeranchukon trajectory is one and if $\gamma_{1\,-1}(0)$ is finite. If this were the case for $M=0$, the total photoabsorption cross section would be expected to vanish asymptotically, contrary to experiment. In order to study this problem, we must consider the consequences of both gauge invariance and kinematic constraints on the analytic structure of $f^t_{1\,-1}$ as $M=0$.

Defining kinematic singularity-free amplitudes $\bar{f}_{\lambda\mu}$ by removing the kinematic singularities of the $f^\eta_{\lambda\mu}$ amplitudes as demonstrated by Eq. (2.9), we may write

$$\begin{aligned} \bar{f}_{11} &\equiv 8p_t{}^2 f^t_{11} \\ &= -\frac{e}{4}(A+2B+C) - 2t'D , \\ \bar{f}_{1\,-1} &\equiv \frac{1}{q_t{}^2}\left(\frac{2}{\sin^2\theta_t}\right) f^t_{1\,-1} \\ &= \tfrac{1}{4}(A+2B+C), \\ \bar{f}_{10} &\equiv \left(\frac{4\sqrt{2}\,p_t M}{q_t \sin\theta_t}\right) f^t_{10} \\ &= \tfrac{1}{4}[s'(A+2B+C)+2t'(B+C)], \\ \bar{f}_{00} &\equiv 8p_t{}^2 f^t_{00} \\ &= -\frac{1}{8M^2}[s'^2 tA - 2t(t'^2-u'^2)B + (t'-u')^2 C + 4t'(t+t')D], \end{aligned} \tag{7.2}$$

where

$$\begin{aligned} e &= su-(M^2-m^2)^2 , \\ s' &\equiv s - m^2 + M^2 , \\ t' &\equiv t - 4M^2 , \\ s' &+ u' + t' = 0 . \end{aligned}$$

Because the final state in this channel consists of two identical spinless mesons, all f^η with η negative are zero.

It is interesting to see how the gauge conditions as expressed by Eqs. (4.1) and (4.2) modify the kinematic structure of the t-channel amplitudes in the limit $M \to 0$. Using Eq. (2.9), the amplitudes of interest are given by

$$f^{+}_{11} = -\frac{1}{4t'}[e(A+2B+C)+8t'D],$$
$$f^{+}_{1\,-1} = \tfrac{1}{4}t'(A+2B+C). \qquad (7.3)$$

As emphasized by Ader *et al.*,[7] one does not obtain the correct kinematic structure by taking the limit of $M \to 0$, in just the kinematic coefficients. One must consider the gauge constraints on the invariant amplitudes as expressed by Eq. (4.2), i.e.,

$$s'(A+B)+2tB-4D = O(M^2),$$
$$s'(B+C)+2tC+4D = O(M^2).$$

Using these constraints, we obtain for $M=0$

$$f^{+}_{11} = -\frac{t}{2(s-m^2)}[s(B+C)-2(s-m^2)C] \equiv t\hat{f}_{11},$$
$$f^{+}_{1\,-1} = \frac{-t(t-4m^2)}{2(s-m^2)}(B+C) \equiv t(t-4m^2)\hat{f}_{1\,-1}.$$

Thus it is seen that the kinematic structure in t is not what would be expected just from the kinematic coefficient in Eq. (7.3). These expressions also show how the "pion" pole factor $(s-m^2)^{-1}$ is introduced into s-channel helicity amplitudes by the gauge constraints. As mentioned before, this phenomenon has its analogy in $\gamma N \to \pi N$, where gauge invariance introduces a "pion" pole factor into the coefficient of the appropriate invariant amplitudes. Equation (7.2) can be inverted to give

$$A = \frac{-2M^2}{tt'^2}\tilde{f}_{00} - \frac{2(t'-u')}{t'^2}\tilde{f}_{10} + \frac{t'+t}{2tt'^2}\tilde{f}_{11} + \left(\frac{t'+t}{2tt'^2}e + \frac{(t'-u')^2}{t'^2}\right)\tilde{f}_{1\,-1},$$
$$B+C = \frac{2}{t'}(\tilde{f}_{10} - s'\tilde{f}_{1\,-1}),$$
$$C = \frac{-2M^2}{tt'^2}\tilde{f}_{00} - \frac{2s'}{t'^2}\tilde{f}_{10} + \frac{t'+t}{2tt'^2}\tilde{f}_{11} + \left(\frac{t'+t}{2tt'^2}e + \frac{s'^2}{t'^2}\right)\tilde{f}_{1\,-1},$$
$$D = -\frac{1}{2t'}(\tilde{f}_{11} + e\tilde{f}_{1\,-1}). \qquad (7.4)$$

To continue our investigation of the effect of kinematic and gauge constraints on the Regge-pole contributions to the s-channel reactions as $M \to 0$, we write

$$\tilde{f}^{+}_{00} = \Gamma_{00}s^{\alpha}, \quad \tilde{f}^{+}_{10} = \Gamma_{10}s^{\alpha-1},$$
$$\tilde{f}^{+}_{11} = \Gamma_{11}s^{\alpha}, \quad \tilde{f}^{+}_{1\,-1} = \Gamma_{1\,-1}s^{\alpha-2}.$$

Considering only the leading contributions in s, the threshold $(t=4M^2)$ and pseudothreshold $(t=0)$ constraints result in

$$\Gamma_{10}(t) - \Gamma_{1\,-1}(t) = O(t-4M^2),$$
$$\Gamma_{00}(t) + 2\Gamma_{1\,-1}(t) = O(t-4M^2),$$
$$\Gamma_{11}(t) - \Gamma_{1\,-1}(t) = O(t-4M^2),$$
$$2M^2[\Gamma'_{00}(t) + 4\Gamma'_{10}(t) - \Gamma'_{11}(t) - \Gamma'_{1\,-1}(t)] - \Gamma_{1\,-1}(t) = O(t-4M^2),$$
$$\Gamma_{00}(t) + \Gamma_{11}(t) - \Gamma_{1\,-1}(t) = O(t). \qquad (7.5)$$

These five constraint equations agree with those previously obtained by Arbab and Brower.[8] Expanding the amplitudes about $t=0$ and writing

$$\Gamma_{\lambda\lambda'}(t) = a_{\lambda\lambda'}(M^2) + t b_{\lambda\lambda'}(M^2) + O(t^2),$$

the solution to the kinematic constraints Eq. (7.5) are found to be

$$\Gamma_{00} = 4M^2(b_{11} - b_{1\,-1}) + b_{00}t + O(t^2, M^4),$$
$$\Gamma_{10} = 2M^2(b_{1\,-1} - 2b_{11} - b_{00}) + \tfrac{1}{2}b_{11}t + O(t^2, M^4),$$
$$\Gamma_{11} = 2M^2(b_{1\,-1} - 3b_{11} - b_{00}) + b_{11}t + O(t^2, M^4),$$
$$\Gamma_{1\,-1} = -2M^2(b_{1\,-1} + b_{11} + b_{00}) + b_{1\,-1}t + O(t^2, M^4). \qquad (7.6)$$

With the realization that we have absorbed a factor of M in our definition of Γ_{10}, it is clear that this is the same as the solution obtained by Arbab and Brower.[8] However, our conclusions with regard to the nonvanishing of the total photoabsorption cross section will be opposite to theirs, since these authors did not take into account gauge invariance, which is an essential property of amplitudes for photonic reactions.

From our earlier discussion, we know that gauge invariance implies for small M, but arbitrary values of t, that

$$f^{s}_{10} = O(M) \quad \text{and} \quad f^{s}_{00} = O(M^2),$$

which in turn implies

$$\Gamma_{10} = O(M^2) \quad \text{and} \quad \Gamma_{00} = O(M^2)$$

for all t. Thus the solutions for Γ_{11} and $\Gamma_{1\,-1}$ to the kinematic and gauge constraints in the limit of $M \to 0$ are

$$\Gamma_{11} = 2M^2 b_{1\,-1} + b'_{11}M^2 t + O(t^2, M^4),$$
$$\Gamma_{1\,-1} = -2M^2 b_{1\,-1} + b_{1\,-1}t + O(t^2, M^4), \qquad (7.7)$$

where we set $b_{11} = b'_{11}M^2$ as demanded by $\Gamma_{10} = O(M^2)$. The significance of these solutions will be discussed in Sec. VIII.

VIII. POMERANCHUKON EXCHANGE AND THE TOTAL PHOTOPRODUCTION CROSS SECTION

We now wish to calculate the total photoproduction cross section for hadrons in the s channel, i.e., $\gamma\pi \to$ hadrons. With our normalization of amplitudes and the unitarity relation, the total cross section is given by

$$\sigma^{\text{tot}}(\gamma\pi \to \text{hadrons}) = \frac{\text{Im}[\mathfrak{M}_{fi}(s, t=0)]}{s - m^2}. \tag{8.1}$$

Since $\mathfrak{M}_{fi} = f^s_{1,1}$ in the forward direction, we need to consider

$$\lim_{t\to 0} f^s_{1,1} = \lim_{t\to 0} D = \lim_{t\to 0} (\tilde{f}^+_{11} + e\tilde{f}^+_{1-1})/8M^2, \tag{8.2}$$

where we used Eq. (7.4). Retaining only the dominant Pomeranchukon contribution and using the $t=0$ constraint equation we have

$$\lim_{t\to 0} f^s_{1,1} = \frac{\Gamma_{11}(0) - \Gamma_{1-1}(0)}{8M^2} s^{\alpha_P(0)} = -\frac{\Gamma_{00}(0)}{8M^2} s^{\alpha_P(0)}. \tag{8.3}$$

Using the solutions to the kinematic constraint equations given in Eq. (7.7), we have

$$\lim_{t\to 0} f^s_{1,1} = \tfrac{1}{2} b_{1,-1}(M^2) s^{\alpha_P(0)} + O(M^2).$$

Consequently, in order to obtain a finite nonzero total photoproduction cross section for physical photons, we must have

$$b_{1-1}(M^2) = O(M^2) \neq 0. \tag{8.4}$$

From Eq. (7.7), the expression for Γ_{1-1}, i.e.,

$$\Gamma_{1-1} = (t - 2M^2)b_{1-1} + O(t^2, M^4),$$

it is clear that Γ_{1-1} cannot vanish as $\alpha_P(t) - 1 = \alpha'_P t$ as $t \to 0$ for all M, as suggested by Eq. (7.1) without b_{1-1} being zero. Consequently the zero due to $[\alpha_P(t) - 1]$ must be canceled by a pole in $\gamma_{1-1}(t)$. Such a zero could be removed either by the residue of the Regge pole having a t^{-1} singularity or by the existence of a multiplicative fixed pole at $j=1$ in the partial-wave amplitude, which would result in the Regge-pole residue being proportional to $[\alpha_P(t) - 1]^{-1}$.[8]

IX. DOMINANCE OF HELICITY-CONSERVING AMPLITUDES

We now ask under what conditions the helicity-conserving amplitudes dominate over helicity-nonconserving amplitudes. There are two cases. In the case of the physical t channel, helicity conservation at high energy, i.e., $t \to \infty$ means that[22]

$$|f^t_{00}| \gg |f^t_{11}, f^t_{10}, f^t_{1-1}| \tag{9.1}$$

(recall that the pion helicities are not explicitly written). Assuming the dominance of the pion Regge pole, we will now show that the conditions (9.1) imply that the pion residue functions must satisfy two equations, which are equivalent, assuming factorization of the pion residues, and which are valid at least at two kinematical points outside the physical region. From Eq. (6.4), we see that the invariant amplitudes may have the following behavior as $t \to \infty$:

$$A \sim t^{\alpha_\pi},\ B \sim t^{\alpha_\pi - 1},\ C \sim t^{\alpha_\pi - 2},\ D \sim t^{\alpha_\pi - 1}.$$

The first two equations of Eq. (2.9) then tell us that in order to satisfy (9.1), we must have

$$A \sim O(t^{\alpha_\pi - 1}). \tag{9.2a}$$

Using the third equation in Eq. (2.9) and this property of A, we similarly find

$$B \sim O(t^{\alpha_\pi - 2}). \tag{9.2b}$$

Using Eq. (6.4), the conditions (9.2a), (9.2b) may be written as

$$\begin{aligned} A &= \frac{16}{\rho^4}\left(\tilde{f}^-_{0,0} - ts'\tilde{f}^-_{1,0} + \frac{ts'^2}{4s}\tilde{f}^-_{1,1}\right) \sim O(t^{\alpha_\pi - 1}), \\ B &= \frac{4}{\rho^2}\left[\tilde{f}^-_{1,0} - \frac{s'}{2s}\tilde{f}^-_{1,1}\right] \sim O(t^{\alpha_\pi - 2}). \end{aligned} \tag{9.3}$$

If these conditions are satisfied, then the last equation of Eq. (2.9) and the expressions for C and D in Eq. (6.4), ensure helicity conservation, i.e.,

$$M^2 f^t_{0,0} \simeq -\frac{t}{4s}\tilde{f}_{1,1} \sim O(t^{\alpha_\pi}),$$

while all other t-channel helicity amplitudes are of order $t^{\alpha - 1}$.

In order to understand the significance of condition (9.3) we consider the Γ's defined in Sec. IV. Since the conspiracy relation [Eq. (6.5)] must be satisfied, we assume that $\Gamma_{1,1}(s)$ vanishes at $s=0$,[23] i.e., the evasive solution, and thus avoid the necessity of a pion conspirator or other more complicated contribution to $f^+_{1,1}$. Thus we have

$$\begin{aligned} &\Gamma_{0,0}(s) - \tfrac{1}{2}s'\Gamma_{1,0}(s) = 0, \\ &s\Gamma_{1,0}(s) - \tfrac{1}{2}s'\Gamma_{1,1}(s) = 0. \end{aligned} \tag{9.3'}$$

These two equations are readily seen to be equivalent to the factorization property of the residues by Eq. (6.9) and one other condition which we take to be

$$\Gamma_{0,0}(s) - \left(\frac{s'^2}{4s}\right)\Gamma_{1,1}(s) = 0 . \tag{9.4}$$

Comparing this to Eq. (6.8c) or to the equation immediatly preceding Eq. (6.8c), it is clear that this condition is satisfied at least at the points $s = s_\pm = (M \pm m)^2$. For $M = m$, the point s_- moves to the boundary of the physical region. Thus, in conclusion, we find at high energies that helicity conservation in $VV \to \pi\pi$ does hold at the cross channel normal- and pseudo-threshold points in the unphysical region, but are unable to predict it for the physical scattering.

In the case of the physical s channel, i.e., $V\pi \to V\pi$, the problem is similar. Here one is interested in finding under what conditions the conservation of helicity, i.e.,

$$|f^s_{1,1}|, \; |f^s_{0,0}| \gg |f^s_{1,-1}|, \; |f^s_{1,0}|, \tag{9.5}$$

is satisfied for $s \to \infty$ and t fixed. For simplicity we consider in detail only the case of physical photons, i.e., $M = 0$. With the use of Eq. (3.4) the leading inequality of (9.5) is

$$|tC + 2D| \gg \left|tC + \frac{2t}{s} D\right|,$$

which by using Eq. (7.4) can be written as a condition on Γ_{11} and Γ_{1-1} of the form

$$|S^\alpha \Gamma_{1-1}(t)| \gg |s^\alpha \Gamma_{11}(t) + t(t - 2m^2)s^{\alpha-2}\Gamma_{1-1}(t)| . \tag{9.5$'$}$$

This inequality would be satisfied if

$$|\Gamma_{11}(t)| \ll |\Gamma_{1-1}(t)| . \tag{9.6}$$

We see, therefore, that the helicity-conserving amplitude dominates in the high-energy limit of Compton scattering off a pseudoscalar meson provided the Pomeranchukon decouples from the amplitude f^t_{11}, i.e., $\Gamma_{11} = 0$. However, the condition (9.6) must necessarily hold in the region of small t due to the kinematic constraints. This can be seen from Eq. (7.7) according to which

$$\Gamma_{11} = O(t^2)$$

and

$$\Gamma_{1-1} = t b_{1-1} + O(t^2)$$

for $M \to 0$ and from the fact that b_{1-1} must be nonzero to ensure a nonzero asymptotic total photoproduction cross section as shown in Sec. VIII.

Thus the kinematical constraints which determine these relations ensure the conservation of helicity for small values of t. The situation here is analogous to that in vector-meson photoproduction discussed by Maor[5] in a different context. In the case of $V\pi$ scattering, i.e., $M \neq 0$, we expect the other, more complicated inequalities (9.5) to exist due to the solutions [Eq. (7.6)] of the constraint equations, but refrain from a more detailed investigation.

---

[1]B. Bartoli *et al.*, Nuovo Cimento 70A, 615 (1970); Phys. Rev. D 6, 2374 (1972).

[2]For a summary of feasible e^+e^- storage-ring experiments see D. Schildknecht, H. J. Willutzki, and G. Wolf, DESY Report No. 71/28, 1971 (unpublished).

[3]Low-energy investigations of this reaction have been performed by D. H. Lyth, Nucl. Phys. B30, 173 (1971); G. Schierholz and K. Sundermeyer, *ibid.* B40, 125 (1972); C. E. Carlson and Wu-Ki Tung, Phys. Rev. D 6, 147 (1972).

[4]T. Ebata and K. E. Lassila, Phys. Rev. 185, 1717 (1969). For related work see T. Akiba, M. Sakuraoka, and T. Ebata, Nucl. Phys. B31, 381 (1971).

[5]U. Maor, University of California (Berkeley) report 1970 (unpublished).

[6]E. Gotsman and U. Maor, Phys. Rev. 171, 1495 (1968); P. D. Mannheim and U. Maor, *ibid.* 186, 1703 (1969).

[7]General aspects of gauge invariance for photoproduction amplitudes have also been discussed by N. Dombey, Nuovo Cimento 32, 1696 (1964); J. S. Ball and M. Jacob, *ibid.* 54A, 620 (1968); J. P. Ader, M. Capdeville, and P. Salin, Univ. of Bordeaux Report No. PTB-33, 1970 (unpublished).

[8]H. D. I. Abarbanel and S. Nussinov, Phys. Rev. 158, 1462 (1967); F. E. Low, I. J. Muzinich, S. Nussinov, and J. H. Schwarz, *ibid.* 160, 1329 (1967); H. D. I. Abarbanel, in Proceedings of the Regge Pole Conference, Irvine, California, 1969 (unpublished); H. F. Jones, M. D. Scadron, and F. D. Gault, Nuovo Cimento 66A, 424 (1970); P. D. B. Collins and F. D. Gault, *ibid.* 10A, 189 (1972); F. Arbab and R. C. Brower, Phys. Rev. 181, 2124 (1969); P. V. Landshoff and J. C. Polkinghorne, Phys. Rev. D 5 2056 (1972).

[9]M. Biyajima, Progr. Theoret. Phys. (Kyoto) 47, 1047 (1972).

[10]See, for instance, S. U. Chung, CERN Report No. 71-8, 1971 (unpublished).

[11]We are using the relativistic normalization used, for instance, by P. D. B. Collins and E. J. Squires, *Regge Poles in Particle Physics* (Springer, Berlin, 1968), pp. 8, 10.

[12]J. W. Moffat and V. G. Snell, Phys. Rev. D 4, 1452 (1971).

[13]S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row, New York, 1961).

[14]Note that with Schweber's one-particle state normalization (p. 485) our

$$i(2\pi)^4 \delta^4 \mathfrak{M}_{fi} \equiv \alpha$$

would have to be replaced by

$\alpha/[4(2\pi)^6(p_{10}q_{10}p_{20}q_{20})^{1/2}]$.

[15]Reference 13, p. 482.

[16]G. E. Hite, Rev. Mod. Phys. 41, 669 (1969).

[17]M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).

[18]L. L. Wang, Phys. Rev. 142, 1187 (1966).

[19]M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. 133, B145 (1964).

[20]P. D. B. Collins, Phys. Reports 1C, 103 (1971).

[21]J. D. Jackson and G. E. Hite, Phys. Rev. 169, 1248 (1968). Note the method used in this paper and thus the results cannot be applied to the pseudo-threshold point $m-m'$ as it approaches zero, i.e., the equal-mass case.

[22]Similar arguments have been used by K. Itabashi, Tohoku University report, 1970 (unpublished); H. J. W. Müller and D. Stocker, Nuovo Cimento Letters 2, 1031 (1971).

[23]See Ref. 20 or L. Bertocchi, in *Proceedings of the International Conference on Elementary Particles, Heidelberg, Germany, 1967,* edited by H. Filthuth (North-Holland, Amsterdam, 1968).

PHYSICAL REVIEW D VOLUME 7, NUMBER 11 1 JUNE 1973

Bounds on the Moments of Absorptive Parts of Elastic Scattering Amplitudes*

A. P. Balachandran, Maurice L. Blackmon, and Cassio Sigaud†
Physics Department, Syracuse University, Syracuse, New York 13210
(Received 26 January 1973)

A rigorous set of bounds on the moments of the absorptive part of elastic scattering amplitudes and its derivatives with respect to the cosine of the scattering angle is presented for pion-pion and pion-nucleon scattering. These bounds are similar to unitarity bounds found by Singh and Roy, and involve integrals over absorptive parts and their derivatives in the physical scattering region only.

I. INTRODUCTION

Singh and Roy[1] have derived several bounds on the absorptive part $A(s,z)$ of, say, the π-π elastic scattering amplitude, which follow from a judicious use of unitarity and analyticity. The following are typical results. (1) Given the total cross section and the elastic cross section, an upper bound on $A(s,z)$ is found at a physical point $s \geq 4m_\pi^2$, $|z|<1$ (s is the square of the center-of-mass energy, z is the cosine of the scattering angle, and m_π is the pion mass). (2) Given elastic and total cross sections, a lower bound on $A(s,z)$ is found for $z>1$. (3) Given elastic and total cross sections, lower bounds on the derivatives with respect to z of $A(s,z)$ for $z=1$ are found.

The bounds in (2) and (3) are conveniently calculated, but the more interesting bound (1) is difficult to evaluate (except near $z=1$) due to the difficulty of finding the order by decreasing size of the sequence of Legendre polynomials $P_l(z)$ for $|z|<1$. For result (2), by contrast, $P_{l+1}(z)>P_l(z)$ for $z>1$, and the Legendre polynomials are trivially ordered in magnitude by their order l.

In this paper, we point out that appropriately chosen moments of Legendre polynomials, their derivatives, and simple Jacobi polynomials (appropriate for nonzero helicity) have straightforward ordering properties. Therefore, we are able to construct rigorous bounds on the absorptive parts of elastic scattering amplitudes in the physical region which are easier to evaluate than result (1) of Singh and Roy.[1] Our bounds, however, have the disadvantage of involving integrals over all physical z, whereas the result (1) of Singh and Roy is a point statement good for any $|z|<1$.

In Sec. II we present our results for pion-pion scattering, and in Sec. III we consider pion-nucleon scattering.

II. BOUNDS ON PION-PION AMPLITUDES

We give the normalization of the scattering amplitude $F(s,z)$ in terms of the differential cross section by

$$\frac{d\sigma}{d\Omega}=\frac{|F(s,z)|^2}{s}, \tag{2.1}$$

where s is the square of the center-of-mass energy and z is the cosine of the scattering angle. The amplitude $F(s,z)$ has the partial-wave expansion

$$F(s,z)=\frac{s^{1/2}}{k}\sum_{\substack{l\geq 0\\ l\text{ even}}}^{\infty}(2l+1)[2f_l(s)]P_l(z), \tag{2.2}$$

where k is the center-of-mass momentum. Here