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Odorico's Bootstrap of Pseudoscalar Mesons*

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Odorico has derived bootstrap conditions for P (pseudoscalar) meson interactions from the hypothesis that the zeros in PP scattering amplitudes are linear in the Mandelstam plane. We explain the relation between this hypothesis and an earlier bootstrap hypothesis based on duality. If the solution proposed by Odorico for the PP scattering bootstrap is correct, other bootstrap conditions involving virtual P mesons suggest that a tenth member should be associated with the P nonet. If the 1422-MeV E meson is pseudoscalar, it is a possible candidate for the tenth member. It is shown that in some, but not all, hadron-hadron scattering amplitudes, a simple condition based on duality and isotopic-spin invariance is almost sufficient to predict linear zeros.

I. INTRODUCTION

Odorico has proposed a strong bootstrap condition for VPP and TPP interactions, where P denotes a pseudoscalar meson, and V and T Reggeized vector and tensor mesons.[1] His basic hypothesis is that the zeros in $PP \to PP$ amplitudes are straight lines in the Mandelstam plane. The bootstrap condition requires not only an internal symmetry group, but, if the group is SU(3), it also requires the η-X mixing angle to be $\tan^{-1}(1/\sqrt{2})$.[1]

Recently, the author published a paper (to be referred to as C1) extending Odorico's condition to $VP \to PP$ and $\gamma P \to PP$ amplitudes, where γ denotes a photon.[2] Another result of C1 is that the quark-model values of VPP and TPP interactions lead to a solution of the $PP \to PP$ conditions overlooked in Ref. 1.

The purpose of the present paper is to extend the results of C1 in several ways. The first is to explain the relations between various $PP \to PP$ bootstrap conditions. Section II of this paper contains a proof of a statement of C1, of the equivalence of Odorico's condition and a simple set of algebraic equations involving the interaction constants. In addition, several points in Odorico's derivation of his bootstrap condition are clarified.

In Sec. III, the possible solutions to the $PP \to PP$ conditions are considered. It is shown that experimental evidence favors Odorico's solution over the quark-model solution. Furthermore, a more complete set of approximate bootstrap conditions, considered previously by the author, suggests that a tenth P meson should be associated with the P nonet if Odorico's solution is correct.

Finally, in Sec. IV, we discuss the validity of the linear-zero hypothesis. It is shown that approximate linearity of the zeros follows from more basic assumptions for some amplitudes, but not for others. Thus, the significance of experimentally observed linear zeros depends on which type of amplitude is involved.

II. THE BOOTSTRAP CONDITIONS

We consider only $PP \to PP$ amplitudes, interacting with V and T Regge trajectories. The extension to $VP \to PP$ and $\gamma P \to PP$ amplitudes involves only the straightforward modifications described in C1.

Mass differences among the P mesons are neglected, and all the V and T Regge trajectories are taken to be degenerate and linear. The amplitude in the s, t, and u Mandelstam channels is represented by $a+b \to c+d$, $a+\bar{c} \to \bar{b}+d$, and $\bar{c}+b \to \bar{a}+d$, respectively. The residues of the V and T trajectories in the i channel are denoted by V_i and T_i; these are given by[3]

$$\begin{aligned} V_s &= f_{rdc} f_{rab}, \quad & T_s &= d_{rdc} d_{rab}, \\ V_t &= f_{rd\bar{b}} f_{r\bar{c}a}, \quad & T_t &= d_{rd\bar{b}} d_{r\bar{c}a}, \\ V_u &= f_{rd\bar{a}} f_{rb\bar{c}}, \quad & T_u &= d_{rd\bar{a}} d_{rb\bar{c}}, \end{aligned} \tag{1}$$

where f_{rij} and d_{rij} are the VPP and TPP interaction constants, and summation over r is implied. The d and f interaction constants are symmetric and antisymmetric, respectively, in the interchange of the P mesons.

We assume that the residues V_i are all proportional as functions of the energy-squared variable i. Thus, the V_i and f's of Eq. (1) may be defined at any convenient value of i. A similar assumption is made for the T_i and the d's.

It is pointed out in C1 that Odorico's bootstrap condition is closely related to a condition used previously by the author.[4] The earlier condition, which may be derived from duality and a simple proportionality assumption, relates the V and T residues in any pair of Mandelstam channels. The resulting three equations are

$$T_s - V_s = T_t + V_t, \tag{2a}$$

$$T_t - V_t = T_u + V_u, \tag{2b}$$

$$T_u - V_u = T_s + V_s. \tag{2c}$$

The different signs in front of the V_i are related to the orderings of the P-meson indices in Eq. (1); a simple rule is that the sign of V_i is negative in Eq. (2a), (2b), or (2c) if exchanging the last indices on the f's in V_i in Eq. (1) leads to the channel on the other side of the bootstrap condition. One could include a positive proportionality constant in front of the T's in Eqs. (2a)–(2c); our normalization condition for the d's and f's is that this constant may be omitted.

It is stated in C1 that Odorico's bootstrap condition is equivalent to Eqs. (2a)–(2c), plus the additional condition

$$T_i V_i (T_i^2 - V_i^2) = 0, \tag{3}$$

which must be satisfied for each of the three values of the channel index i. We will demonstrate the equivalence later in this section.

Since Odorico's justification of his condition is abbreviated in Ref. 1, we will give the argument in our own words. The zero-width approximation is made, so that resonance poles occur at real energies. If an i-channel resonance exists at $i = m_i^2$, the residue R_i is defined as

$$R_i = \lim_{i \to m_i^2} [(i - m_i^2) A],$$

where A is the invariant amplitude. A resonance on a V or T trajectory may be accompanied by one or more daughters of smaller angular momentum. Thus, each R_i is a sum of a finite number of Legendre polynomials in the scattering angle, and so is a polynomial of finite order in the cross-channel energy. We assume that the relative proportions of daughters (if any are present) is the same for all leading-trajectory resonances of the same spin. Therefore, if two resonances of the same mass on two different s-channel V trajectories are considered, the ratio $R_s(t)/V_s$ at a specific t will be the same for the two resonances. A similar conclusion holds for the T trajectories.

Let m_s and m_t be the masses of an s-channel resonance and a t-channel resonance, not necessarily of the same spin or parity. Odorico considers the intersection of such a pair of resonances in the Mandelstam plane, and makes the following power-series expansion in the vicinity of the intersection[5,6]:

$$(s-m_s^2)(t-m_t^2)A = R_t^0(s-m_s^2)+R_s^0(t-m_t^2) + \text{higher-order terms}, \quad (4)$$

where R_s^0 and R_t^0 are real constants. There is no constant term in this expansion because a residue R_i cannot have a pole in the cross-channel energy. Thus, the quantity $(s-m_s^2)(t-m_t^2)A$ is zero at the intersection. The slope of the path of this zero at the intersection is given by

$$\frac{ds}{dt} = -\frac{R_s^0}{R_t^0}. \quad (5)$$

Odorico's basic hypothesis is that the path of the zero is a straight line. Experimental evidence for linear zeros in the physical region exists.[1,6] The zero line cannot be at constant s or constant t, because this would imply that one of the residues is identically zero. Therefore, the amplitude A is itself zero along the zero line, except at the intersection.

Since each resonance is one of an infinite sequence of resonances of the same parity along a trajectory, each intersection is one of a two-dimensional pattern of intersections in the appropriate corner of the Mandelstam (s, t, u) plane. Odorico assumes that the paths of zeros intersecting a particular trajectory at cross-channel resonances of the same parity are parallel. It may be possible to deduce this condition from other conditions; however, we shall include the condition as part of the "linear-zero hypothesis."

If a zero line crossed a particular resonance line between resonance intersections in the resonance intersection region, then an infinite number of parallel zero lines would cross the resonance between intersections. This would imply that the resonance residue R contained an infinite number of zeros, in violation of the condition that R is a finite polynomial. Thus, a zero line can cross a resonance in the resonance intersection region only at a resonance intersection. This limits severely the possible directions of zero lines. For example, suppose that s- and t-channel resonances of both parities exist, and one considers an intersection in the interior of the intersection region, such as that denoted with a circle in Fig. 1. The zero line must correspond to one of the two dashed lines in the figure.

The direction of the zero line at the intersection determines the residue ratio, by Eq. (5). Because of our proportionality assumption concerning V trajectories, the magnitude of the ratio V_s/V_t is equal to the magnitude of the corresponding R_s^0/R_t^0 at the resonance intersection. A similar statement applies to the tensor residues. However, the signs of the V's depend on the order of indices in Eq. (1). For example, if V and T poles of the s and t channels intersect, and if all the zeros are at constant u, then all the R_s^0/R_t^0 ratios at each intersection are determined from Eq. (5) to be unity. This implies the equalities $T_t = V_t = T_s = -V_s$, the minus sign occurring because the t-channel partners a and c are in different positions in the subscripts of V_s in Eq. (1).

Odorico states that the linear-zero hypothesis requires the pattern of zeros to be one of five types, shown in Fig. 1 of Ref. 1. The residue ratios for the five patterns are

$$\begin{aligned}
&\mathrm{I}(t)\quad T_t = V_t = 0,\quad T_s = V_s = T_u = -V_u,\\
&\mathrm{II}^+(t)\quad V_t = 0,\quad T_s = -V_s = T_u = V_u = \tfrac{1}{2}T_t,\\
&\mathrm{II}^-(t)\quad T_t = 0,\quad T_s = -V_s = -T_u = -V_u = \tfrac{1}{2}V_t, \qquad (6)\\
&\mathrm{III}^+\quad V_t = V_s = V_u = 0,\quad T_t = T_s = T_u,\\
&\mathrm{III}^-(t)\quad V_t = T_s = T_u = 0,\quad T_t = -V_s = V_u.
\end{aligned}$$

The allowed residue ratios are these and the sets that may be obtained from cyclic permutations of the channel indices s, t, and u. The index t following the pattern number is used to distinguish the given ratios from those of the other permutations.

In the rest of this section, we will demonstrate the following two points. *(a) The requirement of one of the patterns of Eq. (6) is equivalent to the algebraic conditions of Eqs. (2a)–(2c) and (3). (b) If the duality conditions of Eqs. (2a)–(2c) are assumed, the linear-zero hypothesis implies one of the patterns of Eq. (6).* This is one form of Odorico's contention of Ref. 1. To my knowledge, this contention has not been demonstrated in the literature.

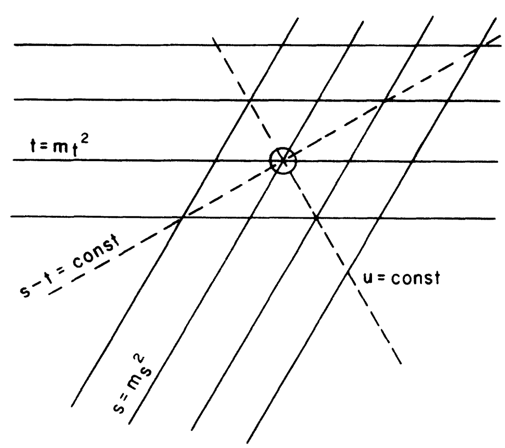

FIG. 1. Resonance intersection region when resonances of both parities exist in s and t channels.

First we consider point (a). One can verify by substitution that each of the five patterns of Eq. (6) satisfies Eqs. (2a)–(2c) and (3). Thus, we need only show that Eqs. (2a)–(2c) and (3) imply one of the five patterns. If T_s, V_s, T_t, and V_t are known, Eqs. (2b) and (2c) can be used to determine T_u and V_u. Hence, we classify the possibilities by the number (N) of these four s- and t-channel residues that are zero. If $N=4$, i.e., all four are zero, it follows from Eqs. (2b) and (2c) that T_u and V_u are zero also. The number N cannot be three, because of Eq. (2a). Therefore, a nontrivial solution requires $N\leq 2$.

We consider the possibility $N=2$. If the two zero residues are T_t and V_t, Eqs. (2a)–(2c) imply the pattern I(t). Similarly, if the zero residues are T_s and V_s, pattern I(s) results. If one of the t-channel residues and one of the s-channel residues are zero, Eqs. (2a)–(2c) imply one of the patterns III$^+$, III$^-$(s), III$^-$(t), and III$^-$(u).

We next consider the possibility $N=1$. If the zero residue is V_t, Eqs. (2a) and (3) require that $T_s=-V_s=\frac{1}{2}T_t$. Then, Eqs. (2b) and (2c) imply pattern II$^+$(t). If the zero residue is T_t, V_s, or T_s, similar arguments lead to pattern II$^-$(t), II$^+$(s), or II$^-$(s), respectively.

The only remaining possibility is $N=0$. In this case, Eqs. (2a) and (3) imply either that $T_s=-V_s=T_t=V_t$ or that $T_s=V_s$ and $T_t=-V_t$. In the first of these two alternatives, Eqs. (2b) and (2c) imply $T_u=V_u=0$, so the pattern is I(u). On the other hand, if $T_s=V_s$ and $T_t=-V_t$, Eqs. (2b) and (2c) imply that $|T_u|\neq|V_u|$. Then, Eq. (3) requires that either T_u or V_u is zero. One of the patterns II$^+$(u) or II$^-$(u) is implied. This completes the proof of point (a).

We next turn to point (b) mentioned above. Because of the equivalence of point (a), it is sufficient to show that Eqs. (2a)–(2c) and the linear-zero hypothesis imply Eq. (3). If one or both the residues T_t and V_t is nonzero for any channel i, it follows from Eqs. (2a)–(2c) that there must be at least one nonzero residue in at least one of the other two channels. Thus, there must be a corner of the Mandelstam plane where resonances intersect. We take this to be the s-t corner, and consider first the case in which T_t, V_t, T_s, and V_s are all nonzero. We call such a corner a dense intersection corner. The intersecting resonances are illustrated in Fig. 1. It has been shown earlier that a line of zeros through a resonance intersection in this case must be in one of the two directions of the dashed lines in Fig. 1. These lines correspond to $ds/dt=1$ and -1, so Eq. (5) implies that $R_s^0/R_t^0=\pm 1$. Since this is true for all parity combinations, the residues must all be of the same magnitude, i.e.,

$$|T_s|=|V_s|=|T_t|=|V_t|. \tag{7}$$

Clearly, Eq. (7) implies that Eq. (3) is satisfied for i equal to s and t. Furthermore, it is easy to show that if Eq. (7) is satisfied, Eqs. (2a)–(2c) imply that one or both of T_u and V_u is zero, so that Eq. (3) is satisfied for all three Mandelstam channels. Therefore, Eq. (3) is satisfied if there is a dense intersection corner in the Mandelstam plane.

We next consider intersection corners that are not dense. If one of the residues (V and T) is zero for each of the three channels, Eq. (3) is satisfied, so the only type of intersection corner we have left to consider is one where exactly one of the four residues is zero. For definiteness, we consider the s-t intersection region, with T_t or V_t equal to zero. In such a case, one can show from Eqs. (2a)–(2c) that neither T_u nor V_u is zero. It follows that the s-u intersection corner is dense, so that Eq. (3) must be satisfied. This completes the proof of point (b).

It is seen from Fig. 1 of Ref. 1 that in the case of patterns II$^+$ and II$^-$ some of the zero lines pass through two intersection regions. It was pointed out in Ref. 1 that in these cases the linear-zero hypothesis cannot be true for arbitrary values of the trajectory parameters, but can be true if $\alpha_s(s)+\alpha_t(t)+\alpha_u(u)=1$. This is equivalent to the condition

$$2K^{-1}=3M^2-4\mu^2, \tag{8}$$

where M and μ are masses of the vector and pseudoscalar mesons, and K is the slope ($d\alpha/ds$) of the trajectories.

III. THE SOLUTIONS FOR *PP* SCATTERING

It was stated in C1 that there are two solutions to Odorico's conditions for PP scattering. In both solutions the VPP and TPP interactions are SU(3)-symmetric, and the η-X mixing angle is $\theta=\tan^{-1}(1/\sqrt{2})$, where

$$\begin{aligned}\eta&=(\cos\theta)\eta_8-(\sin\theta)\eta_1,\\ X&=(\sin\theta)\eta_8+(\cos\theta)\eta_1.\end{aligned} \tag{9}$$

The VPP interaction involves only octets; the over-all interaction constant is denoted by f_{888}. The constant d_{ijk} denotes the interaction of the T trajectory multiplet i with P mesons of multiplets j and k. The ratios of interaction constants in the solutions are

$$\begin{aligned}&f_{888}=(\tfrac{9}{5})^{1/2}d_{888},\quad d_{188}=-(\tfrac{16}{5})^{1/2}d_{888},\\ &d_{818}=r d_{888},\quad d_{111}=-(\tfrac{5}{2})^{1/2}r^2 d_{888}.\end{aligned} \tag{10}$$

The two solutions are the quark-model solution and Odorico's solution. They differ only in the parameter r, which is given by

$$r(\text{quark}) = -(\tfrac{2}{5})^{1/2},$$
$$r(\text{Odorico}) = (\tfrac{1}{10})^{1/2}. \tag{11}$$

The constants $f_{888}{}^2$, $d_{888}{}^2$, and $d_{188}{}^2$ are normalized to be equal to the sum over all PP states coupled to a particular V or T, while d_{818} is equal to $d(T_j P_1 P_j)$, where j is any octet state and P_1 is the P singlet.

In the remainder of this section we discuss some experimental evidence concerning which solution is better, and also a physical interpretation of Odorico's solution.

A. Experimental Evidence

The predicted η-X mixing angle of ~35° in both solutions is quite different than the value of ~10° that follows from the assumption that the Gell-Mann–Okubo mass formula applies to the squares of the P-meson masses. However, since the mass splitting is proportionally much larger in the P multiplet than in any other multiplet, the mass formula does not provide a compelling argument for a small mixing angle.

The Odorico and quark-model solutions differ only in interactions involving the η or the X meson. Some experimental arguments in favor of the Odorico solution are given in Ref. 1. We present here only one additional argument, involving the $\eta\pi/K\bar{K}$ branching ratio of the A_2 meson. Experimentally, this branching ratio is about $2.8 \pm 20\%$.[7] The ratio of the fifth power of the decay momenta is

$$k^5(\eta\pi)/k^5(K\bar{K}) = 2.9.$$

Therefore, the experimental interaction-constant ratio

$$R = [\,d(A_2\eta\pi)/d(A_2 K\bar{K})\,]^2$$

satisfies the relation $R \gtrsim 1$, the approximate equality sign applying if the phase-space factor for the D-wave decays is k^5 and the inequality applying if the phase-space difference is weakened by the inclusion of a finite-range factor. The predicted values of R are $R(\text{quark}) = 0$, $R(\text{Odorico}) = 1$, and $R(\text{no mix}) = \frac{2}{3}$, where (no mix) corresponds to a pure octet assignment for the η. The experimental branching ratio rules out the quark-model solution and is consistent with Odorico's solution. However, one cannot rule out the possibility that the η is nearly a pure octet particle.

B. A Possible Physical Interpretation of Odorico's Solution

If one disregarded Odorico's hypothesis, and looked for solutions of the basic bootstrap equations [Eqs. (2a)–(2c)] for PP scattering, the parameter r of Eq. (10) would be arbitrary.[2] This arbitrariness is present only because the P mesons cannot be internal particles in the amplitudes; i.e., P-meson Regge trajectories do not contribute. In previous references, the author has applied a "complete" set of bootstrap equations to the scattering amplitudes of a hypothetical set of mesons of both particles.[4,8] The set of equations was complete in the sense that the set of internal particles (lowest states on the Regge trajectories) was the same as the set of external particles. There was only one solution for the interaction constants, corresponding to the quark model; i.e., $r = -(\frac{2}{5})^{1/2}$. (This model did not predict the η-X mixing angle, however.)

Because of the complication of particle spin, the correspondence with reality of such a complete set of bootstrap equations is not very clear. Furthermore, the complication of spin has prevented us from applying Odorico's linear-zero hypothesis to amplitudes involving two or more external vector or tensor mesons. However, these considerations suggest that when accurate bootstrap equations are formulated for amplitudes involving P-meson trajectories, the quark-model solution will satisfy the equations, and Odorico's solution will not. Consequently, we attempt to modify Odorico's solution so that it is equivalent to the quark-model solution for processes involving internal P mesons.

We will show that such a modification can be made, but only if a tenth P meson, another SU(3) singlet, exists.[9] This meson is denoted by X' or $(1')$, and does not mix with the η or X. The additional interaction constant ratios, that supplement Eq. (10) with $r = (\frac{1}{10})^{1/2}$, are

$$d_{81'8} = (\tfrac{3}{10})^{1/2} d_{888},$$
$$d_{11'1} = -(\tfrac{3}{40})^{1/2} d_{888}, \tag{12}$$
$$d_{11'1'} = -(\tfrac{9}{40})^{1/2} d_{888}.$$

We now demonstrate that this extension is equivalent to the quark model for processes with internal P mesons, and satisfies the PP scattering conditions of Odorico. One always can add a particle that does not interact with anything to the solution of any set of bootstrap equations. We let $(1)_{\text{in}}$ denote such an inert P meson, and $(1)_q$ denote a P singlet with the interaction constants of the quark model. We let η_1 and X' be orthogonal mixtures of these singlets, i.e.,

$$\eta_1 = (\cos\psi)(1)_q + (\sin\psi)(1)_{\text{in}}\,,$$
$$X' = -(\sin\psi)(1)_q + (\cos\psi)(1)_{\text{in}}. \quad (13)$$

Standard bootstrap equations such as Eqs. (2a)–(2c) or those considered in Refs. 4 and 8 are linear in the sense that if they are satisfied with either of two particles in the role of one of the external mesons, they are satisfied for any linear combination of the two particles. Therefore, such bootstrap equations are satisfied for any value of the angle ψ of Eq. (13). The choice $\cos\psi = 1$ leads to the usual quark model, for which the meson X' is not detectable. The choice $\cos\psi = -\frac{1}{2}$ leads to the modified Odorico solution, with the X' of Eq. (13) identified with the $(1')$ of Eq. (12) and the η_1 of Eq. (13) mixed with the η_8, as in Eq. (9).

We have left to demonstrate only that $PP \to PP$ amplitudes involving the X' in this modified Odorico solution satisfy Eq. (3), the extra condition resulting from the linear-zero hypothesis. This is obvious, however, since the V trajectories do not contribute in any channel if one of the P mesons is a singlet.

The fact that a pure SU(3) singlet satisfies Odorico's bootstrap conditions was not mentioned in C1 or Ref. 1, because in these references two orthogonal singlet-octet combinations were required for a solution. A pure octet η does not satisfy the condition of Eq. (3).

Therefore, if Odorico's solution corresponds to reality, we expect the X' to exist. Most of its properties should be similar to those of the X(958 MeV). There are various experimental indications of neutral, nonstrange mesons of unknown spin and parity in the mass region ~1 BeV/c^2. It is hoped that further measurements will reveal if any of these is a sister of the X.

The 1422-MeV E meson is an isoscalar meson of even G parity; the spin and parity of the E are believed to be either 0^- or 1^+.[10,11] If the 0^- assignment is correct, the mass of the E is smaller than that generally associated with the orbital angular momentum 2, quark-antiquark multiplet of the quark model, and larger than that generally associated with the $l = 0$ ground state. The $l = 2$ assignment would be especially anomalous, since no isoscalar members of the known meson multiplets are very light compared to the average multiplet mass. Hence, the E is a candidate for the predicted tenth member of the ground-state multiplet.

IV. TESTING THE LINEAR-ZERO HYPOTHESIS

The theoretical justification of some bootstrap conditions is better than that of some others. For example, Eqs. (2a)–(2c) follow from duality and a plausible assumption concerning the proportionality of Regge residues.[4] If one of the channels is exotic, the two equations of Eqs. (2a)–(2c) that involve the exotic channel may be derived in several ways, and are especially believable.[12] On the other hand, there is no strong theoretical justification for the linear-zero hypothesis, although it clearly has the virtue of simplicity. Furthermore, this hypothesis could not be valid in the SU(3)-symmetry limit, unless the mass condition of Eq. (8) were satisfied.

In view of these considerations, it is important to test the linear-zero hypothesis experimentally, as has been emphasized by Odorico.[5,6,13] However, it is important to notice that for many amplitudes the linear-zero hypothesis is not needed to predict the residue ratios. For example, if one of the Mandelstam channels is exotic, or if trajectories of only one signature can contribute in each channel, then Eqs. (2a)–(2c) are sufficient to predict that the residue ratios satisfy either pattern I, III^+, or III^- of Eq. (6). These residue ratios are such that the path of a zero as it crosses a resonance intersection is in the direction of a sequence of resonance intersections. Hence, the approximate linearity of the zero is nearly assured by the duality bootstrap conditions. Certainly, it is worthwhile to study such amplitudes experimentally. However, the discovery of linear zeros in these amplitudes is not a compelling reason to expect zeros to be linear in other amplitudes. In order to test Odorico's linear-zero hypothesis, one must study amplitudes for which the approximate linearity does not result from more familiar requirements.

In the case of PP scattering, the duality bootstrap equations for amplitudes with an exotic channel, together with isospin symmetry, determine the residue ratios for all amplitudes that do not involve an η or an X meson. These residue ratios correspond, of course, to the linear-zero patterns of Eq. (6). We demonstrate this for the $\pi K \to \pi K$ amplitudes first. If the two pions are charged, then one of the two strange (πK) channels is exotic and the nonstrange channel is neutral. The duality conditions, Eqs. (2a)–(2c), predict the residue ratios of all four contributing trajectories, i.e., the ρ^0 and f trajectories in the nonstrange channel, and the V and T, K^* trajectories in the nonexotic strange channel. The residue ratios of these trajectories in $\pi K \to \pi K$ amplitudes involving one or two neutral pions then follow from isospin symmetry. In the case of $\pi\pi \to \pi\pi$ amplitudes, the argument is similar, but simpler, since only the ρ^0 and f trajectories are involved. In the case of $K\bar{K} \to K\bar{K}$ amplitudes, there is always an exotic channel.

Odorico has cited experimental evidence for approximately linear zeros in π-π and π-K scattering amplitudes.[6] It is not clear at present if this linearity is a result of duality and isospin symmetry, or if it is a general property of PP amplitudes. As pointed out in Sec. III, the η-X mixing angle and the singlet-octet interaction ratio r of Eq. (10) are not determined from the duality bootstrap conditions. Therefore, these parameters are important for testing the linear-zero hypothesis.

Some of the better tests of the linear-zero hypothesis involve processes other than PP scattering. The prediction of C1, that the VP-photon interaction involves an SU(3)-singlet part of the photon, does not result from duality bootstrap equations alone. Another important test involves pion-nucleon scattering amplitudes, recently studied by Odorico.[13] If the amplitude is $\pi^+p \rightarrow \pi^+p$, no channel is exotic and no trajectory signature is forbidden in any channel. The discovery of linear zeros in πp amplitudes would provide strong support for Odorico's basic hypothesis.

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[1]R. Odorico, Phys. Lett. 38B, 37 (1972).

[2]R. H. Capps, Phys. Rev. Lett. 29, 820 (1972).

[3]The notation used here is slightly different from that of C1. The quantities V_s, V_t, and V_u defined here are the quantities V_{st}, V_{tu}, and V_{us} of C1. We are assuming a representation in which the interaction constants are all real. If more general representations are used, Eq. (1) should be modified, as shown in Ref. 4.

[4]R. H. Capps, Phys. Rev. D 3, 3059 (1971). This paper lists the bootstrap conditions and gives references for the derivation.

[5]R. Odorico, Phys. Lett. 34B, 65 (1971).

[6]R. Odorico, in *Experimental Meson Spectroscopy – 1972* (Third Philadelphia Conference), edited by Kwan-Wu Lai and A. H. Rosenfeld (American Institute of Physics, New York, 1972), pp. 77–90.

[7]Particle Data Group, Phys. Lett. 39B, 1 (1972).

[8]R. H. Capps, Phys. Rev. D 5, 1018 (1972).

[9]The suggestion that a tenth P meson may be associated with Odorico's solution has been made independently by Harry Lipkin (private communication). Lipkin's motivation is to preserve the quark model; his extra P singlet is of the structure $(X\bar{X})$, where X is a fourth quark that does not interact with the V or T trajectory. This meson plays the role of the state denoted by $(1)_{\text{in}}$ here.

[10]B. Lörstad *et al.*, Nucl. Phys. B14, 63 (1969); A. Bettini *et al.*, Nuovo Cimento 62A, 1038 (1969). For a list of other references concerning the E, see Ref. 7.

[11]The possible identification of the E with the X' predicted here was suggested to the author by S. F. Tuan.

[12]A simple derivation of the residue equality that results from duality when one of the two concerned channels is exotic is given by C. B. Chiu and J. Finkelstein, Phys. Lett. 27B, 510 (1968).

[13]R. Odorico, Phys. Lett. 41B, 339 (1972).