conservation and good high-energy behavior also emerge for other processes involving gauge bosons and to attempt to apply these ideas to strong interactions.

Note added. After writing this note we learned of an interesting paper by Vainshtein and Khriplovich.[7] These authors show that essentially good high-energy behavior is to be expected in massive gauge theories when there are less than three external gauge particles. However, their approach is very different from ours and they do not discuss asymptotic helicity conservation.

We would like to thank A. P. Balachandran and H. Rupertsberger for helpful discussions.

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*Work supported in part by the U. S. Atomic Energy Commission.

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[3] For simplicity we are suppressing the over-all phase factor which contains the azimuthal (ϕ) dependence of the amplitude by setting $\phi = \frac{1}{2}\pi$.

[4] The ± sign corresponds to different transverse polarization assignments within the given case. Note that the E dependence is not affected by this.

[5] For example, Aachen-Berlin-Bonn-Hamburg-Heidelberg-München Collaboration, Phys. Rev. 175, 1669 (1968); J. Ballam *et al.*, Phys. Rev. Letters 24, 960 (1970).

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Comment on the Partially Conserved Axial-Vector Current Sum Rule for the Anomalous Vertex Functions*

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(Received 19 October 1972)

We investigate the sum rule relating the anomalous vertex function (as a function of virtual-photon squared masses q^2 and k^2) of $\pi^0\gamma\gamma$ to the form factors of the axial-vector–vector–vector (AVV) vertex derived from the hypothesis of partial conservation of axial-vector current including the Bell-Jackiw-Adler anomaly and from the algebra of currents. One of the consequences is that, if the $\pi^0\gamma\gamma$ vertex function decreases at all, a certain combination of the form factors of the AVV vertex should decrease *as fast as* $(q^2+k^2)^{-1}$ when one of q^2 and k^2 increases. An experimental check of the sum rule is suggested.

The $\pi^0\to\gamma\gamma$ decay has been of great theoretical interest since Bell and Jackiw[1] and Adler[1] found that the decay constant provided by a triangle graph does not vanish in the soft-pion limit, in contradiction to a result[2] of partial conservation of axial-vector current[3] (PCAC) and current algebra.[4] In order to see the mechanism more closely, it is desirable to study the vertex

$$\gamma(q)+\gamma(k)\to\pi^0(P)\quad(P=q+k) \tag{1}$$

as a function of three variables, q^2, k^2, and P^2. Gross and Treiman[5] have investigated the vertex function in the Bjorken-Johnson-Low[6] (BJL) limit, as well as in their scaling limit. The present author[7] has shown that the $\pi^0\gamma\gamma$ vertex function, if it decreases at all, should decrease not slower than $(-q^2)^{-1/2}$, as one of the virtual-photon squared masses q^2 increases. In this paper, we investigate the exact sum rule relating the $\pi^0\gamma\gamma$ vertex function to the form factors of the axial-vector–vector–vector (AVV) vertex derived from the PCAC hypothesis,[3] including the Bell-Jackiw-Adler anomaly,[1] and from the algebra of currents.[4] The sum rule holds for arbitrary values of q^2 and k^2. One of the consequences is a simple theorem that, if the $\pi^0\gamma\gamma$ vertex function decreases at all, a certain combination of the AVV form factors should decrease *as fast as* $(q^2+k^2)^{-1}$ when one of

q^2 and k^2 increases.

Let us first define the $\pi^0\gamma\gamma$ vertex function $F(q^2, k^2, P^2)$ by

$$M_{\mu\nu}(q, k) = i\int dx\, e^{-iqx}\langle P|\, T^*(J_\mu(x)J_\nu(0))|0\rangle = \epsilon_{\mu\nu\alpha\beta}q^\alpha k^\beta F(q^2, k^2, P^2), \tag{2}$$

where J_μ is the hadron electromagnetic current. At $q^2 = k^2 = 0$ and $P^2 = m_\pi{}^2$, the vertex function is related to the π^0 lifetime τ_{π^0} by

$$|F(0, 0, m_\pi{}^2)| = (64\pi/e^4 m_\pi{}^3\tau_{\pi^0})^{1/2}. \tag{3}$$

The Bell-Jackiw-Adler anomaly[1] tells us that the PCAC relation[3] should be modified to

$$\partial^\lambda A_\lambda^\pi(x) = f_\pi m_\pi{}^2\phi_\pi(x) + (e^2S/16\pi^2)\epsilon_{\alpha\beta\gamma\delta} : F^{\alpha\beta}(x)F^{\gamma\delta}(x):, \tag{4}$$

where $A_\mu(x)$ is the hadron axial-vector current, f_π ($\simeq$95 MeV) is the pion weak decay constant defined by

$$\langle P|A_\mu^\pi(0)|0\rangle = -iP_\mu f_\pi, \tag{5}$$

and S is the anomalous PCAC constant predicted[8] to be

$$S = \begin{cases} \tfrac{1}{6} \text{ for the original Gell-Mann-Zweig (fractionally charged triplet quark) model;} \\ \tfrac{1}{2} \text{ for the original Sakata (integrally charged triplet), Han-Nambu (integrally charged three triplet), and fractionally charged three-triplet models.} \end{cases} \tag{6}$$

Defining the off-shell amplitude $M_{\mu\nu}(2)$ for $P^2 \neq m_\pi{}^2$ calculated with the reduction formula and using the modified PCAC relation[1] of (4) and the current algebra,[4] one can obtain the Ward-Takahashi identity at $P^2 = 0$:

$$M_{\mu\nu}(q, k)|_{P^2=0} = -(S/2\pi^2 f_\pi)\epsilon_{\mu\nu\alpha\beta}q^\alpha k^\beta + (P^\lambda/f_\pi)i\int dy dx\, e^{iPy-iqx}\langle 0|\, T(A_\lambda{}^\pi(y)J_\mu(x)J_\nu(0))|0\rangle\Big|_{P^2=0}. \tag{7}$$

This identity can be transformed into the relation

$$\begin{aligned} F(q^2, k^2, 0) &= -(S/2\pi^2 f_\pi) + (1/f_\pi)[G_1(q^2, k^2, 0) - G_2(q^2, k^2, 0)] \\ &= -(S/2\pi^2 f_\pi) + (1/f_\pi)[-\tfrac{1}{2}(q^2 + k^2)G_3(q^2, k^2, 0) + k^2G_4(q^2, k^2, 0) \\ &\qquad - q^2G_5(q^2, k^2, 0) + \tfrac{1}{2}(q^2 + k^2)G_6(q^2, k^2, 0)], \end{aligned} \tag{8}$$

where the form factors of the AVV vertex are defined by[1]

$$\begin{aligned} T_{\mu\nu\lambda}(q, k) &= i\int dy dx\, e^{iPy-iqx}\langle 0|\, T(A_\lambda^\pi(y)J_\mu(x)J_\nu(0))|0\rangle \\ &= \epsilon_{\alpha\mu\nu\lambda}q^\alpha G_1(q^2, k^2, P^2) + \epsilon_{\alpha\mu\nu\lambda}k^\alpha G_2(q^2, k^2, P^2) + \epsilon_{\alpha\beta\mu\lambda}q_\nu q^\alpha k^\beta G_3(q^2, k^2, P^2) \\ &\quad + \epsilon_{\alpha\beta\mu\lambda}k_\nu q^\alpha k^\beta G_4(q^2, k^2, P^2) + \epsilon_{\alpha\beta\nu\lambda}q_\mu q^\alpha k^\beta G_5(q^2, k^2, P^2) + \epsilon_{\alpha\beta\nu\lambda}k_\mu q^\alpha k^\beta G_6(q^2, k^2, P^2), \end{aligned} \tag{9}$$

with the gauge-invariance $[q^\mu T_{\mu\nu\lambda}(q, k) = k^\nu T_{\mu\nu\lambda}(q, k) = 0]$ conditions[1]

$$\begin{aligned} G_1(q^2, k^2, P^2) &= q\cdot kG_3(q^2, k^2, P^2) + k^2G_4(q^2, k^2, P^2) \\ \text{and} \\ G_2(q^2, k^2, P^2) &= q^2G_5(q^2, k^2, P^2) + q\cdot kG_6(q^2, k^2, P^2), \end{aligned} \tag{10}$$

and with the symmetry $[T_{\mu\nu\lambda}(q, k) = T_{\nu\mu\lambda}(k, q)]$ properties[1] due to the Bose statistics obeyed by photons

$$\begin{aligned} G_1(q^2, k^2, P^2) &= -G_2(k^2, q^2, P^2), \\ G_3(q^2, k^2, P^2) &= -G_6(k^2, q^2, P^2), \\ \text{and} \\ G_4(q^2, k^2, P^2) &= -G_5(k^2, q^2, P^2). \end{aligned} \tag{11}$$

The sum rule (8) written implicitly in Refs. 1 and 9, which relates the $\pi^0\gamma\gamma$ vertex function to the form factors of the AVV vertex at $P^2 = 0$ through the anomalous constant S, holds for arbitrary values of q^2 and k^2. It is seen trivially that the sum rule reduces to the result obtained by Bell and Jackiw[1] and by Adler[1] at $q^2 = k^2 = 0$:

$$F(0, 0, 0) = -(S/2\pi^2 f_\pi). \tag{12}$$

Several years ago, Cornwall[10] showed that the $\pi^0\gamma\gamma$ vertex function approaches the limit

$$F(q^2, k^2, m_\pi{}^2) \to \tfrac{2}{3}(f_\pi/q^2) \text{ as } q^2 \to \infty \text{ and } q^2/k^2 \to 1 \tag{13}$$

if the BJL theorem[6]

$$M_{\mu\nu}(q, k) \to \frac{1}{Q_0}\int dx\, e^{-iQx}\delta(x_0)\langle P|[J_\mu(\tfrac{1}{2}x), J_\nu(\tfrac{1}{2}x)]|0\rangle + O(1/Q_0{}^2) \tag{14}$$

is valid for $Q=\frac{1}{2}(q-k)$ in the BJL limit of $Q_0\to\infty$ with $\vec{Q}$ fixed, and if the quark model for the space-space component of the equal-time current commutators

$$\delta(x_0)[J_i(x), J_j(0)]=2i\epsilon_{0ijk}A^k_{Q^2}(0)\delta(x) \tag{15}$$

holds [Q in the $A^\mu_{Q^2}(0)$ is the quark charge (matrix)]. Gross and Treiman[5] have predicted the scaling of $F(q^2, k^2, m_\pi{}^2)$

$$P\cdot QF(q^2, k^2, m_\pi{}^2)\to H(\omega)=-\int_{-1}^{1} d\omega' \frac{G(\omega')}{\omega'-\omega-i\epsilon} \tag{16}$$

in their scaling limit, $Q^2\to\infty$ with $\omega=Q^2/P\cdot Q$ fixed, by assuming the gluon quark model for the light-cone current commutator[11]

$$\begin{aligned}[][J_\mu(x), J_\nu(y)]\\ &\simeq\partial^\alpha D(x-y)\{s_{\mu\nu\alpha\beta}[V^\beta_{Q^2}(x,y)-V^\beta_{Q^2}(y,x)]\\ &\qquad+i\epsilon_{\mu\nu\alpha\beta}[A^\beta_{Q^2}(x,y)+A^\beta_{Q^2}(y,x)]\}\\ &\qquad\text{for } (x-y)^2\simeq 0,\end{aligned} \tag{17}$$

$$s_{\mu\nu\alpha\beta}=g_{\mu\alpha}g_{\nu\beta}+g_{\mu\beta}g_{\nu\alpha}-g_{\mu\nu}\,g_{\alpha\beta},$$

and

$$D(x)=\epsilon(x_0)\delta(x^2)/2\pi\,.$$

The absorptive part of the $\pi^0\gamma\gamma$ vertex $G(\omega)$ in Eq. (16), which has been introduced in

$$M_{\mu\nu}(q,k)\to\epsilon_{\mu\nu\alpha\beta}\frac{P^\alpha Q^\beta}{P\cdot Q}\int_{-1}^{1} d\omega' \frac{G(\omega')}{\omega'-\omega-i\epsilon}, \tag{18}$$

is related to the axial-vector bilocal current $A^\mu_{Q^2}(x,y)$ by

$$\begin{aligned}&\langle P|A^\mu_{Q^2}(\tfrac{1}{2}x, -\tfrac{1}{2}x)+A^\mu_{Q^2}(-\tfrac{1}{2}x, \tfrac{1}{2}x)|0\rangle\\ &\qquad=-iP^\mu\int_{-1}^{1} d\omega\, e^{i\omega P\cdot x/2}G(\omega)\,.\end{aligned} \tag{19}$$

In Eqs. (13) and (16), we can see that the vertex function $F(q^2, k^2, m_\pi{}^2)$ decreases as $(q^2)^{-1}$ when both q^2 and k^2 become large. It should be noticed that these results (13) and (16) strongly depend on three assumptions: the validity of the BJL theorem,[6] the nonexistence of the q-number Schwinger term, and the quark model for the space-space (bad-bad) component of the equal-time current commutators. More intuitively, however, we believe that $F(q^2, k^2, m_\pi{}^2)$ decreases as one of the virtual-photon masses squared q^2 (or k^2) increases with the other k^2 (or q^2) fixed, since in this case the vertex function can be interpreted as an ordinary electromagnetic form factor with the external two legs staying nearly on the mass shell.[12] If, in fact, the $\pi^0\gamma\gamma$ vertex function decreases at all, which we shall assume hereafter, then from the sum rule (8) we can arrive at the following remarkable conclusion: A combination of the form factors of the AVV vertex, $G_1(q^2, k^2, 0)-G_2(q^2, k^2, 0)$, should approach a constant $S/2\pi^2$ as one of the virtual-photon masses squared q^2 and k^2 increases, or alternatively another combination of the form factors,

$$\begin{aligned}&-\tfrac{1}{2}G_3(q^2, k^2, 0)+[k^2/(q^2+k^2)]G_4(q^2, k^2, 0)\\ &\qquad-[q^2/(q^2+k^2)]G_5(q^2, k^2, 0)+\tfrac{1}{2}G_6(q^2, k^2, 0),\end{aligned}$$

should decrease *as fast as* $(S/2\pi^2)(q^2+k^2)^{-1}$.

Both the $\pi^0\gamma\gamma$ vertex function and the form factors of the AVV vertex as functions of q^2 and k^2 with $P^2=m_\pi{}^2$ are observable in principle. Of course, we need to assume the smoothness of extrapolation of these functions from $P^2=m_\pi{}^2$ to $P^2=0$ in order to check the sum rule (8) experimentally. An easier measurement of $F(q^2, k^2, m_\pi{}^2)$ can be made by the two-photon process for the π^0 production by $e^\pm e^-$ colliding beams, $e^\pm+e^-\to e^\pm+e^-+\pi^0$. We can find whether the vertex function decreases rapidly by comparing the observed cross section with the cross section calculated exactly by Brodsky, Kinoshita, and the present author[13] for the constant vertex. The total cross section at energies $E\simeq 3$ GeV for each beam is of order 10^{-33} cm², which is large enough to be measured in the near future. It is more desirable to measure directly the vertex function as a function of q^2 and k^2 by detecting both of the scattered leptons (1 and 2) at large angles. The differential cross section is given by[7]

$$\frac{d\sigma^{ee\to ee\pi0}}{dE_1'\,d\cos\theta_1\,dE_2'\,d\cos\theta_2\,d\phi}=128\alpha^4\frac{E^2E_1'E_2'(E-E_1')^2(E-E_2')^2}{(q^2k^2)^2}\delta(P^2-m_\pi{}^2)|F(q^2, k^2, m_\pi{}^2)|^2$$

$$\text{for } m_\pi{}^2\ll-q^2,\quad -k^2\ll 4E^2, \tag{20}$$

where E_i' and θ_i ($i=1$ and 2) are the energy and angle of the scattered leptons, respectively, and ϕ is the coplanarity angle. Although the effective cross section is small for large $-q^2$ and $-k^2$ [~10^{-37} cm² for $E\simeq 3$ GeV and $-q^2$, $-k^2>1$ GeV² if (13) is the case], it is easy to find whether the vertex function decreases as in Eq. (13), or whether it scales as in Eq. (16). On the other hand, we can measure the form factors of the AVV vertex G_i ($i=1,\ldots,6$) by a process such as $e^{\pm}+e^{-}\to e^{\pm}+e^{-}+\nu+\nu$ and $e^{\pm}+e^{-}\to e^{\pm}+e^{-}+\mu^{+}+\mu^{-}$. In the latter process, the AVV vertex appears in the interference term between the α^2 and $\alpha^2 G_F$ (where G_F is the Fermi constant $G_F m_p{}^2 \simeq 10^{-5}$) amplitudes which produces the forward-backward asymmetry and the polarization of the produced muons. Therefore, this proposal is similar to the old one made by Cabibbo and Gatto, and others,[14] for detecting the effect of leptonic neutral weak currents in the process $e^{+}+e^{-}\to\mu^{+}+\mu^{-}$. Of course, a contribution of the interference to the cross section for the processes of our interest is small and of order $G_F s \times 10^{-32}\ \mathrm{cm}^2 \lesssim 10^{-37}\ \mathrm{cm}^2$ for $E\simeq 3$ GeV and $s\simeq 1\ \mathrm{GeV}^2$, where s is the invariant mass squared of the produced lepton pair ($\nu\nu$ or $\mu^{+}\mu^{-}$). Both the $\pi^0\gamma\gamma$ vertex function and the form factors of the AVV vertex will, however, become interesting subjects in high-energy physics because they may provide a best probe to the mechanism of binding the pion and, hopefully, to the interaction between quarks, if any.

In conclusion, it may be worthwhile to note that, from Eqs. (5) and (19), we can easily write down sum rules for the imaginary part of the vertex functions of the various pseudoscalar mesons and two photons[15]:

$$\int_{-1}^{1} d\omega\, G^{\pi}(\omega) = \tfrac{2}{3} f_\pi\,,$$

$$\int_{-1}^{1} d\omega\, G^{\eta}(\omega) = \frac{2}{3\sqrt{3}} f_\eta\,, \tag{21}$$

and

$$\int_{1}^{1} d\omega\, G^{\eta'}(\omega) = \tfrac{4}{9} f_{\eta'}\,,$$

where f_η and $f_{\eta'}$ are the leptonic decay constants of the eighth and the zeroth pseudoscalar meson, respectively. These sum rules, as well as the sum rule (8), correspond to the Adler sum rule[16] for neutrino productions.

ACKNOWLEDGMENT

The author would like to thank M. A. B. Bég, H. Pagels, and A. Pais especially, for reading the manuscript, and S. L. Adler, T. P. Cheng, S. L. Glashow, D. J. Gross, R. Jackiw, Y. Nambu, G. Preparata, and A. Zee, for helpful discussions and private communications.

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*Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-2232.

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[12] There is a serious mistake in the paper by the present author [Rockefeller University Report No. C00-3505-18, 1972 (unpublished)] in which he claimed $F(q^2,k^2,0)$ = constant for $q^2=k^2$. The author thanks many people, including S. L. Adler, for pointing out the error.

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