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Higher-Order Weak Processes Without Quadratic Divergences*

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If the interaction kernel is energy-independent, the Bethe-Salpeter equation can be transformed to a version wherein the kernel is replaced by the zero-energy scattering amplitude. This transformed equation is then used in order to unitarize lepton-lepton scattering processes. Quadratic divergences are eliminated in this approach. All lepton-lepton scattering amplitudes and other higher-order weak phenomena, e.g., the neutrino charge radius, can be described in terms of a single unknown parameter.

I. INTRODUCTION

The correct prescription for the calculation of high-energy and higher-order weak-interaction processes has long been one of the major unsolved problems in theoretical physics. The current-current interaction Hamiltonian, originally suggested by Fermi,[1]

$$\mathcal{H}_w = \frac{G}{2\sqrt{2}}\{J_\lambda, J_\lambda^\dagger\}, \qquad (1)$$

where

$$J_\lambda = \bar{\psi}_e\gamma_\lambda(1+\gamma_5)\psi_{\nu_e} + \bar{\psi}_\mu\gamma_\lambda(1+\gamma_5)\psi_{\nu_\mu}$$

is known to provide a good first-order description of weak leptonic and semileptonic processes at low energies and momentum transfers.[2] However, it cannot be correct at all energies, since it predicts leptonic cross sections for certain reactions which are proportional to E^2, E being the c.m. energy. These cross sections are known to violate unitarity for $E \geqslant 300$ GeV. Moreover, if the Hamiltonian in (1) is used to calculate higher-order weak processes according to standard field-theory procedures, serious divergences are encountered.[3] Second-order diagrams diverge quadratically. Third- and higher-order diagrams have stronger divergences. These divergences are, of course, a reflection of the fact that the field theory defined by the current-current interaction is not renormalizable. To put the problem another way, an infinite number of phenomenological constants must be introduced into the theory in order to obtain finite results.

Recently Weinberg and others have proposed models for weak interactions which may be renormalizable and would then eliminate the problems mentioned above.[4] However, in this paper we examine an alternate approach which does not involve the introduction of hitherto unobserved particles and currents. Our basic philosophy is closer to that of Appelquist and Bjorken, who used unitarity to estimate higher-order weak processes.[5] Like them we restrict ourselves to purely leptonic processes. Our attitude is that the current-current Hamiltonian is properly viewed as an effective Hamiltonian and, as such, represents the *exact* low-energy weak-interaction amplitude. We then develop a calculational technique which unitarizes the zero-energy amplitude T_0 to yield an amplitude T_E which is valid at all energies.

Connell has described a reformulation of the Bethe-Salpeter equation in which the interaction kernel is replaced by the scattering amplitude at zero energy.[6] This transformed Bethe-Salpeter equation is ideally suited to situations in which the low-energy amplitude is known, but the fundamental interaction is not. In this context the Bethe-Salpeter equation should be viewed as a convenient method for unitarizing a zero-energy amplitude. Undoubtedly it would be possible to develop an equivalent approach based on dispersion relations. Following Connell, we write the integral equation for T_E in the form

$$T_E = V + VG_E T_E \ . \tag{2}$$

The exact interaction kernel V is the sum of all two-particle irreducible diagrams, and G_E is the exact propagator for two particles. In general V will depend on the c.m. energy E. We make the admittedly strong assumption that V is independent of energy. The effect of including energy-dependent terms in V will be discussed briefly. Formally (2) can be rewritten in the form

$$\begin{aligned} V^{-1} &= T_E^{-1} + G_E \\ &= T_0^{-1} + G_0 \ , \end{aligned} \tag{3}$$

where we have used the energy independence of V to write the second equality. From (3) we have

$$T_E = T_0 + T_0(G_E - G_0)T_E \ . \tag{4}$$

This expression for T_E constitutes our basic working equation. Its virtues are readily apparent. It involves T_0, which is at least approximately known, rather than the unknown interaction V. Secondly, the subtracted Green's function removes quadratic divergences from the theory, leaving only logarithmic ones. If the assumption that V is energy-independent is relaxed, Eq. (4) is replaced by

$$T_E = T_0 + T_0(G_E - G_0)T_E - T_0 \Delta V_E T_E \ , \tag{5}$$

where $\Delta V_E = V_E^{-1} - V_0^{-1}$. To use this equation we need information about ΔV_E which is unavailable. However, for a reasonable V_E, the integrals that appear in an iterative solution of (5) are more convergent than those encountered in conventional weak-interaction calculations.

The basic program of this paper is as follows. We assume that the exact zero-energy scattering amplitude is given by the matrix elements of (1). In other words, there are no neutral currents or intermediate vector bosons in our model. In addition the Fermi term is presumed to include the effects of all internal electromagnetic interactions. Free particle propagators are used for G_E. The use of physical masses for the particles takes into account a large class of higher-order effects.

Equation (4) is solved with T_0 given by the Fermi interaction for all possible leptonic interactions. Except for complications due to spin, solutions are trivial since the equation with constant T_0 is separable. Two approximations are involved in using the current-current amplitude for T_0. First there is the assumption that this amplitude, although experimentally determined with all particles on the mass shell, represents T_0 with a pair of particles off shell, since the Bethe-Salpeter equation requires a knowledge of T_0 with two legs off the mass shell. This assumption is justified *a posteriori* by the fact that the solutions of (4) have the same decomposition into invariant amplitudes as T_0. The other approximation involves the fact that T_0 in (4) is really $T(E^2=0, t)$, where t is the momentum transfer, while the Fermi term is $T(E^2=0, t=0)$. This deficiency is corrected in the third section of the paper by crossing the solutions obtained with Fermi input in order to generate a better approximation to T_0. The resulting integral equations are of the Fredholm type and are, in principle, soluble. Rather than actually solve these equations with crossed input, we use them to derive a crossing-symmetric amplitude $T(s, t, u)$ which vanishes for large values of any of its variables and which satisfies elastic unitarity to second order in each channel. This amplitude still contains a logarithmic dependence on cutoffs introduced to define the integrals in each channel. These cutoffs are arbitrary parameters. We then use the first iteration of the equations with crossed-channel input for T_0 in order to fix the cutoffs in each channel in terms of a single dimensionless parameter Γ. The cutoff parameter Λ^2 in each channel has the form $\beta(G\ln\Gamma)$, where β is a known channel-dependent number and G is the weak-interaction coupling constant. Scattering amplitudes depend on $\ln(\ln\Gamma)$. In the final section we calculate the charge radius of the neutrino in terms of the same parameter.

What are the results of our calculations? We obtain scattering amplitudes for all leptonic processes which are crossing-symmetric and approximately unitary. The unitarity limits on cross sections are satisfied. The amplitudes for about thirty different leptonic processes depend on a single parameter in addition to the coupling constant G. This should be compared to the work of Appelquist and Bjorken, where there are a large number of unknown parameters even in second order. Although our results are reliable only to second order in G, their high-energy behavior may be qualitatively correct. Thus we have a model of high-energy and higher-order weak interactions in which all amplitudes are finite and well behaved. However, we stress the fact that the possibility of obtaining these forms is a feature of the rather strong assumptions used in setting up our model.

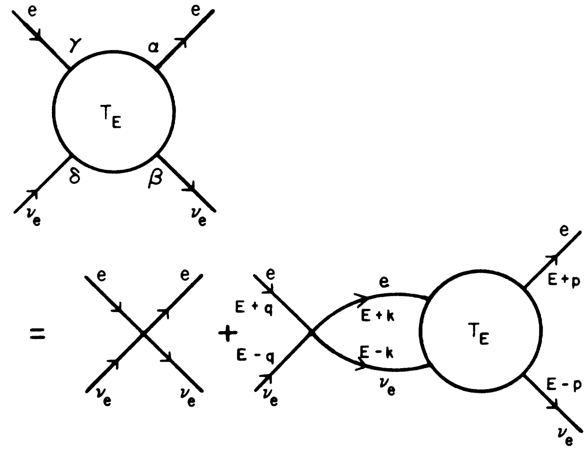

FIG. 1. Kinematics used in solution of the Bethe-Salpeter equation for the s-channel process $e\nu_e \to \nu_e e$. Greek letters α, β, γ, δ refer to spinor indices.

II. LEPTON-LEPTON SCATTERING FERMI INPUT

In this section we address the problem of unitarizing the lepton-lepton scattering amplitudes. μ-e universality is assumed, and the zero-energy amplitude is described by the current-current effective Hamiltonian. We use the transformed Bethe-Salpeter equation to carry out unitarization. Although neutral currents could in principle be included in our discussions, we assume they are absent. This is consistent with the requirement that the Fermi interaction correctly describes low-energy, low-momentum-transfer scattering.

There are three basic types of processes to be considered. They can be characterized by $N = N_e + N_\mu$ and Q, where N_e (N_μ) is the electron (muon) number and Q is the electric charge:

s channel: $N = \pm 2$, e.g., $e\nu_e \to \nu_e e$, $e\nu_\mu \to \nu_e \mu$;

t channel: $N = 0$, $Q = \pm 1$, e.g., $e\bar{\nu}_e \to \bar{\nu}_e e$, $e\bar{\nu}_e \to \bar{\nu}_\mu \mu$;

u channel: $N = 0$, $Q = 0$, e.g., $e\bar{e} \to \nu_e \bar{\nu}_e$, $e\bar{\mu} \to \nu_e \bar{\nu}_\mu$.

These reactions are related by crossing. In a weak-interaction theory with charged intermediate vector bosons, s and u channels involve boson exchange, while t-channel reactions contain one-boson intermediate states.

Consider first a typical s-channel process $e\nu_e \to \nu_e e$, where the explicit form of the transformed Bethe-Salpeter equation in the c.m. frame is

$$T^E_{\alpha\beta;\gamma\delta}(p,q)u^e_\gamma(E+q)u^\nu_\delta(E-q) = T^0_{\alpha\beta;\gamma\delta}(p,q)u^e_\gamma u^\nu_\delta + i\int \frac{d^4k}{(2\pi)^4}\, T^E_{\alpha\beta;\sigma\theta}(p,k)[\,G^{e\nu}_E(k) - G^{e\nu}_0(k)]_{\sigma\tau;\theta\rho} T^0_{\tau\rho;\gamma\delta}(k,q)u^e_\gamma u^\nu_\delta\,, \tag{6}$$

where u^e and u^ν are spinors and

$$G^{e\nu}_E(k)_{\sigma\tau;\theta\rho} = \left(\frac{i}{\gamma\cdot(E+k) - m_e}\right)_{\sigma\tau}\left(\frac{i}{\gamma\cdot(E-k)}\right)_{\theta\rho}.$$

In the Fermi input approximation with $\Gamma^\mu = \gamma^\mu(1+\gamma_5)$, T_0 is given by

$$T^0_{\alpha\beta;\gamma\delta} = \frac{G}{\sqrt{2}}\, g_{\mu\lambda}\Gamma^\mu_{\alpha\delta}\Gamma^\lambda_{\beta\gamma}\,, \tag{7}$$

when all particles are on the mass shell. The kinematic variables p, q, and E are defined in Fig. 1. Spinor indices are denoted by $\alpha\beta\gamma\delta$. $T^E_{\alpha\beta;\gamma\delta}(p,q)$ is the scattering amplitude with all particles off the mass shell. The kernel of the integral equation needs T_0 with the two final-state particles off the mass shell. We use the Fermi amplitude for this amplitude.

Solution of (6) is simplified by making a Fierz transformation on T_0:

$$g^{\mu\lambda}[\gamma_\mu(1+\gamma_5)]_{\alpha\delta}[\gamma_\lambda(1+\gamma_5)]_{\beta\gamma} = -2(\gamma^0\gamma^2+i\gamma^3\gamma^1)_{\alpha\beta}(-\gamma^0\gamma^2+i\gamma^3\gamma^1)_{\gamma\delta} \ . \tag{8}$$

This rearrangement allows us to isolate the spinor indices for the final-state particles and separate the problem of Dirac algebra from that of solving the integral equation. We find then that $T^E_{\alpha\beta;\gamma\delta}$ is given by

$$T^E_{\alpha\beta;\gamma\delta} = g_{\mu\lambda}\Gamma^\mu_{\alpha\delta}\Gamma^\lambda_{\beta\gamma}t_E(p,q) \ , \tag{9}$$

where

$$t_E(p,q) = t_0(p,q) - 16i\int\frac{d^4k}{(2\pi)^4}\, t_0(p,k)\left(\frac{E^2-k^2}{(E-k)^2[(E+k)^2-m^2]}+\frac{1}{k^2-m^2}\right)t_E(k,q) \ . \tag{10}$$

Use of the Fermi approximation $t_0 = G/\sqrt{2}$ gives

$$t_E(p,q) = \frac{G/\sqrt{2}}{1+(G/\sqrt{2})I(E)} \ , \tag{11}$$

with

$$I(E) = 16i\int\frac{d^4k}{(2\pi)^4}\left(\frac{E^2-k^2}{(E-k)^2[(E+k)^2-m^2]}+\frac{1}{k^2-m^2}\right) . \tag{12}$$

Since we are interested in high-energy scattering processes, in which higher-order weak processes should become important, we neglect all lepton masses. Then the integral in (12) can be cast in the form

$$I(E) = \frac{2E^2}{\pi^2}\int_0^\infty \frac{dk^2}{E^2-k^2} \ . \tag{13}$$

$I(E)$ diverges logarithmically. Standard calculations of the same process would lead to a quadratically divergent integral at this point. If the integration is cut off at $k^2 = \frac{1}{4}\Lambda^2$, we find

$$I(s) = \frac{s}{2\pi^2}\ln\left(-\frac{s}{\Lambda^2}\right) , \tag{14}$$

where $s = 4E^2$. The cutoff Λ^2 is a parameter which we would like to take to infinity, except for the divergence of the integral. In the next section we develop constraints on Λ^2. Our conclusion is that for the s-channel process $e\nu_e \to \nu_e e$, the unitarized scattering amplitude is just the Fermi amplitude with the coupling constant $G/\sqrt{2}$ multiplied by an energy-dependent factor which is unity at low energies and vanishes like $(s\ln s)^{-1}$ at high energies:

$$t_0 = \frac{G}{\sqrt{2}} \to t_E = \frac{G}{\sqrt{2}}\Big/\left[1+\frac{G}{2\sqrt{2}\,\pi^2}\, s\ln\left(-\frac{s}{\Lambda^2}\right)\right]. \tag{15}$$

The solution contains the free parameter Λ^2. To second order it agrees with the results of Appelquist and Bjorken for s-channel unitarization. Solutions for other s-channel amplitudes are obtained in exactly the same way. There is a slight additional complication for the processes $e\nu_\mu \to e\nu_\mu$, $e\nu_\mu \to \nu_e\mu$, and $\mu\nu_e \to \mu\nu_e$, which are coupled to each other. Only $e\nu_\mu \to \nu_e\mu$ has a nonvanishing T_0 amplitude. The coupled-channel problem is trivial in the limit of zero lepton mass and μ-e universality. All s-channel amplitudes are displayed in Table I.

For a typical t-channel process such as $e\bar{\nu}_e \to \bar{\nu}_e e$, a Fierz rearrangement is unnecessary. The T_0 amplitude is

$$T^0_{\alpha\beta;\gamma\delta}(p,q) = \frac{G}{\sqrt{2}}\, g_{\mu\lambda}\Gamma^\mu_{\alpha\beta}\Gamma^\lambda_{\delta\gamma} \ . \tag{16}$$

The initial- and final-state particles are arranged so that the appropriate integral equation is

$$\begin{aligned} T^E_{\alpha\beta;\gamma\delta}(p,q)\bar{v}^\nu_\delta(E-q)u^e_\gamma(E+q) &= T^0_{\alpha\beta;\gamma\delta}(p,q)\bar{v}^\nu_\delta u^e_\gamma \\ &+ i\int\frac{d^4k}{(2\pi)^4}T^E_{\alpha\beta;\sigma\theta}(p,k)[G^{e\bar{\nu}}_E(k)-G^{e\bar{\nu}}_0(k)]_{\sigma\tau;\theta\rho}T^0_{\tau\rho;\gamma\delta}(k,q)\bar{v}^\nu_\delta u^e_\gamma \ . \end{aligned} \tag{17}$$

The kinematics for this process are shown in Fig. 2(a). There are coupled channels in this case. If channel 1 is $e\bar{\nu}_e$, then channel 2 is $\mu\bar{\nu}_\mu$. In channel space T_0 is given by the 2×2 matrix

$$T_0=\frac{G}{\sqrt{2}}\begin{pmatrix}1&1\\1&1\end{pmatrix}.$$

The t-channel solutions are of the form

$$^{ij}T^{E}_{\alpha\beta;\,\gamma\delta}=(\Gamma_\lambda)_{\alpha\beta}(\Gamma_\eta)_{\delta\gamma}\,{}^{ij}t_E^{\lambda\eta}\,, \tag{18}$$

where the ij superscripts are channel indices and the ${}^{ij}t_E^{\lambda\eta}$ satisfy the equations

$$^{ij}t_E^{\lambda\eta}=\frac{G}{\sqrt{2}}\,g^{\lambda\eta}+\frac{G}{\sqrt{2}}\,J^{\lambda\tau}(E)\sum_{k=1}^{2}{}^{kj}t_E^{\tau'\eta}g_{\tau\tau'}\,. \tag{19}$$

In the limit of massless leptons and μ-e universality, ${}^{12}t_E={}^{21}t_E$, ${}^{22}t_E={}^{11}t_E$, and the integral $J^{\lambda\tau}(E)$ is given by

$$J^{\lambda\tau}(E)=\frac{2}{3}\frac{E^2}{\pi^2}\int_0^\infty\frac{dk^2}{E^2-k^2}\left(g^{\lambda\tau}-\frac{E^\lambda E^\tau}{E^2}\right). \tag{20}$$

In the c.m. frame E has only a time component, and $J^{\lambda\tau}(E)$ is a diagonal matrix. We need only consider J^{ii}, $i=1,2,3$, which is independent of i. To see this, we note that

$$\bar{v}\Gamma_0 u=\frac{1}{E}\bar{v}(E-q)\gamma\cdot E(1+\gamma_5)u(E+q)=0$$

for massless particles. Using this result, we can drop the $E^\lambda E^\tau/E^2$ term in (20). Hence, with $t=4E^2$, we find

$$^{11}t_E^{\lambda\eta}={}^{21}t_E^{\lambda\eta}=g^{\lambda\eta}\frac{G}{\sqrt{2}}\Big/\left[1-\frac{G}{\sqrt{2}}\,\frac{t}{3\pi^2}\ln\left(-\frac{t}{\Lambda^2}\right)\right]. \tag{21}$$

Again the solutions for the scattering amplitude have the form of the Fermi input modified by a denominator function. The results for the various t-channel amplitudes are collected in Table I.

For a u-channel process like $e\bar{e}\to\nu_e\bar{\nu}_e$, it is necessary both to make a Fierz rearrangement *and* to treat

TABLE I. All lepton-lepton scattering reactions are catalogued except for those which are related to other reactions by T or CP. Reactions which are related by μ-e universality and the replacement $e\longleftrightarrow\nu_e$, $\mu\longleftrightarrow\nu_\mu$ are grouped together in column 2. In column 3 are listed the corresponding unitarized scattering amplitudes. Here $g=G/\sqrt{2}$ and $J(s,a)=(as/\pi^2)\ln(-s/\Lambda^2)$. Column 4 gives the constant C_n, in terms of which the cutoff parameters Λ^2 are given by $\Lambda^2=4\pi^2/(gC_n\ln\Gamma)$, where Γ is defined in (40).

n	Reaction	Scattering amplitude (Fermi input)	C_n
1	$ee\to ee$ $\nu_e\nu_e\to\nu_e\nu_e$ $\mu\mu\to\mu\mu$ $\nu_\mu\nu_\mu\to\nu_\mu\nu_\mu$	0	...
2	$e\mu\to e\mu$ $\nu_e\nu_\mu\to\nu_e\nu_\mu$	0	...
3	$e\nu_e\to e\nu_e$ $\mu\nu_\mu\to\mu\nu_\mu$	$g[1+gJ(s,\frac{1}{2})]^{-1}$	0.86
4	$e\nu_\mu\to e\nu_\mu$ $\mu\nu_e\to\mu\nu_e$	$g^2J(s,\frac{1}{2})[1-g^2J^2(s,\frac{1}{2})]^{-1}$	0.86
5	$\mu\nu_e\to e\nu_\mu$ $e\nu_\mu\to\mu\nu_e$	$g[1-g^2J^2(s,\frac{1}{2})]^{-1}$	0.86
6	$e\bar{\nu}_\mu\to e\bar{\nu}_\mu$ $\mu\bar{\nu}_e\to\mu\bar{\nu}_e$	0	...
7	$e\bar{\mu}\to e\bar{\mu}$ $\nu_e\bar{\nu}_\mu\to\nu_e\bar{\nu}_\mu$	$g^2J(s,\frac{1}{6})[1-g^2J^2(s,\frac{1}{6})]^{-1}$	1.17
8	$e\bar{\mu}\to\nu_e\bar{\nu}_\mu$	$g[1-g^2J^2(s,\frac{1}{6})]^{-1}$	1.17
9	$e\bar{\nu}_e\to e\bar{\nu}_e$ $\mu\bar{\nu}_\mu\to\mu\bar{\nu}_\mu$	$g[1-2gJ(s,\frac{1}{6})]^{-1}$	1.07
10	$e\bar{\nu}_e\to\mu\bar{\nu}_\mu$	$g[1-2gJ(s,\frac{1}{6})]^{-1}$	0.92
11	$e\bar{e}\to e\bar{e}$ $\nu_e\bar{\nu}_e\to\nu_e\bar{\nu}_e$ $\mu\bar{\mu}\to\mu\bar{\mu}$ $\nu_\mu\bar{\nu}_\mu\to\nu_\mu\bar{\nu}_\mu$	$g^2J(s,\frac{1}{6})[1-g^2J^2(s,\frac{1}{6})]^{-1}$	0.73
12	$e\bar{e}\to\nu_e\bar{\nu}_e$ $\mu\bar{\mu}\to\nu_\mu\bar{\nu}_\mu$	$g[1-g^2J^2(s,\frac{1}{6})]^{-1}$	0.73
13	$\nu_e\bar{\nu}_e\to\mu\bar{\mu}$ $\nu_e\bar{\nu}_e\to\nu_\mu\bar{\nu}_\mu$ $e\bar{e}\to\nu_\mu\bar{\nu}_\mu$ $e\bar{e}\to\mu\bar{\mu}$	0	...

a coupled-channel problem. The Fierz rearrangement is

$$[\gamma^\mu(1+\gamma_5)]_{\alpha\gamma}[\gamma_\mu(1+\gamma_5)]_{\delta\beta}=-[\gamma^\mu(1+\gamma_5)]_{\alpha\beta}[\gamma_\mu(1+\gamma_5)]_{\delta\gamma}\ .$$

Solutions have the form [see Fig. 2(b)]

$$^{ij}T^{E}_{\alpha\beta;\,\gamma\delta}=-\Gamma^{\lambda}_{\alpha\beta}\Gamma^{\eta}_{\delta\gamma}\ {}^{ij}t_E^{\lambda\eta}\,, \tag{22}$$

where

$$\begin{aligned} {}^{11}t_E^{\lambda\eta}&=\frac{G}{\sqrt{2}}\,J^{\lambda\tau}(E)\,{}^{21}t^E_{\tau\sigma}g^{\sigma\eta}\,,\\ {}^{21}t_E^{\lambda\eta}&=\frac{G}{\sqrt{2}}\,g^{\lambda\eta}+\frac{G}{\sqrt{2}}\,J^{\lambda\tau}(E)\,{}^{11}t^E_{\tau\sigma}g^{\sigma\eta}\,. \end{aligned} \tag{23}$$

Channel 1 refers to $e\bar{e}$ and channel 2 to $\nu_e\bar{\nu}_e$. $J^{\lambda\tau}(E)$ is given in (20). The $\lambda=\eta=0$ components of $t_E^{\lambda\eta}$ do not contribute if the initial-state particles are on their mass shells. Thus, the solution for ${}^{12}t_E$ is just ${}^{12}t_0$ divided by a denominator factor, while ${}^{11}t_E$ has an extra factor in the numerator, expressing the fact that it is a second-order process:

$$\begin{aligned} {}^{12}t_E^{\lambda\eta}&=\frac{G}{\sqrt{2}}\,g^{\lambda\eta}\Big/\left\{1-\left[\frac{G}{\sqrt{2}}\,\frac{u}{6\pi^2}\ln\left(-\frac{u}{\Lambda^2}\right)\right]^2\right\},\\ {}^{11}t_E^{\lambda\eta}&=\left(\frac{G}{\sqrt{2}}\right)^2 g^{\lambda\eta}\,\frac{u}{6\pi^2}\ln\left(-\frac{u}{\Lambda^2}\right)\Big/\left\{1-\left[\frac{G}{\sqrt{2}}\,\frac{u}{6\pi^2}\ln\left(-\frac{u}{\Lambda^2}\right)\right]^2\right\}. \end{aligned} \tag{24}$$

The amplitudes for all u-channel processes also appear in Table I.

The most convenient way to describe our solutions for lepton scattering is to use the fact that in every case the amplitudes have the spinor structure of the Fermi amplitude. Thus, we have invariant functions t_s, t_t, and t_u which reduce to the Fermi coupling constant $G/\sqrt{2}$ (or zero) in the absence of higher-order corrections. The cross sections in each channel are as follows:

$$\begin{aligned} &s\text{ channel:}\quad \frac{d\sigma}{d\Omega}=\frac{s}{\pi^2}\,|t_s|^2\,,\\ &t\text{ channel:}\quad \frac{d\sigma}{d\Omega}=\frac{t^2}{\pi^2 s}\,|t_t|^2\,,\\ &u\text{ channel:}\quad \frac{d\sigma}{d\Omega}=\frac{u^2}{\pi^2 s}\,|t_u|^2\,. \end{aligned} \tag{25}$$

From Table I we see that the Bethe-Salpeter equation with Fermi input leads to cross sections which satisfy the unitarity bound. On the other hand, the amplitudes are not crossing-symmetric. Crossing symmetry tells us, for example, that the same invariant amplitude $A(s,t,u)$ should describe the three processes $e\nu_e\to\nu_e e$, $e\bar{\nu}_e\to\bar{\nu}_e e$, and $e\bar{e}\to\nu_e\bar{\nu}_e$. From Table I we have

$$\begin{aligned} &s\text{ channel:}\quad A(s,t,u)=t_s(s)=\frac{G}{\sqrt{2}}\Big/\left[1+\frac{G}{\sqrt{2}}\,\frac{s}{2\pi^2}\ln\left(-\frac{s}{\Lambda_s^{\,2}}\right)\right],\\ &t\text{ channel:}\quad A(s,t,u)=t_t(t)=\frac{G}{\sqrt{2}}\Big/\left[1-\frac{G}{\sqrt{2}}\,\frac{t}{3\pi^2}\ln\left(-\frac{t}{\Lambda_t^{\,2}}\right)\right],\\ &u\text{ channel:}\quad A(s,t,u)=t_u(u)=\frac{G}{\sqrt{2}}\Big/\left\{1-\left[\frac{G}{\sqrt{2}}\,\frac{u}{6\pi^2}\ln\left(-\frac{u}{\Lambda_u^{\,2}}\right)\right]^2\right\}. \end{aligned} \tag{26}$$

The fact that the s-channel representation of $A(s,t,u)$ has no t or u dependence is a reflection of the fact that the Fermi amplitude is really $A(s=0,t=0,u=0)$. In the next section we argue that a better approximation to t_s can be obtained by using t_u and t_t as input rather than $t_0=G/\sqrt{2}$. A separable approximation to the s-channel equation with the t-channel solution for t_0 suggests that t_s has the form of a product of t- and s-channel solutions. Extending this argument to include the u channel, we conclude that

$$A(s,t,u)=\frac{G}{\sqrt{2}}\Big/\left[1+\frac{G}{\sqrt{2}}\,\frac{s}{2\pi^2}\ln\left(-\frac{s}{\Lambda_s^{\,2}}\right)\right]\left[1-\frac{G}{\sqrt{2}}\,\frac{t}{3\pi^2}\ln\left(-\frac{t}{\Lambda_t^{\,2}}\right)\right]\left\{1-\left[\frac{G}{\sqrt{2}}\,\frac{u}{6\pi^2}\ln\left(-\frac{u}{\Lambda_u^{\,2}}\right)\right]^2\right\}\ . \tag{27}$$

This amplitude is manifestly crossing-symmetric. It satisfies unitarity to second order in each channel and vanishes when any of the invariants becomes large. The product form of the invariant amplitude is also suggested by other arguments. The s-, t-, and u-channel amplitudes should not be added, since then the Fermi term would be counted three times. Moreover, an additive amplitude would not vanish when any one of the invariants becomes large. Yet an amplitude which is unitary in each channel must vanish in this limit.

Similar crossing-symmetric invariant amplitudes can be formed for any set of processes connected by crossing. Expanded to second order, our expressions agree with those of Appelquist and Bjorken. In third and higher orders our invariant amplitudes contain terms which have simultaneous cuts in two variables. In other words, they have Mandelstam double spectral functions. The double cut terms represent an approximation to the contributions of diagrams with cuts in two variables. However, given our basic assumption about the current-current interaction being the exact low-energy scattering amplitude, the diagrammatic analogy cannot be pushed too far. A second difference between our amplitudes and those of Appelquist and Bjorken relates to the number of unknown constants. A second-order expansion of $A(s, t, u)$ contains two unknown constants; an expansion to arbitrary order has only three. Appelquist and Bjorken have three constants in second order and still more in higher orders. In the next section, we argue that all the processes in Table I can be described in terms of just one unknown parameter.

The proper procedure for forming crossing-symmetric invariant amplitudes is not so obvious when one of the channels is empty in first order. In the absence of other criteria, a simple separable approximation to the Bethe-Salpeter equation with crossed-channel input can be used to suggest a form for the invariant amplitude.

Finally we comment on the assumptions made in this section in order to obtain the solutions in Table I. First is the assumption that went into the derivation of the transformed Bethe-Salpeter equation – the energy independence of the basic, unknown interaction. If we allow for energy dependence of V, then in a single-channel problem with $T_0 = G/\sqrt{2}$ Eq. (5) becomes

$$T_E(p, q) = \frac{G}{\sqrt{2}} + \frac{G}{\sqrt{2}} \int [G_E(k) - G_0(k)] T_E(k, q) - \frac{G}{\sqrt{2}} \int \Delta V_E(k) T_E(k, q) .$$

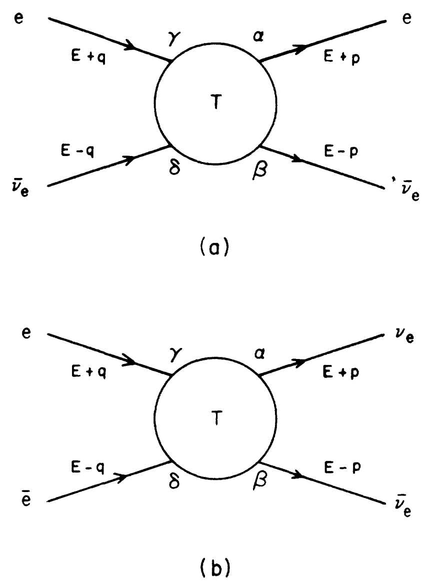

FIG. 2. Kinematics for the (a) t-channel process $e\bar{\nu}_e \to \bar{\nu}_e e$, (b) u-channel process $e\bar{e} \to \nu_e \bar{\nu}_e$. Greek letters α, β, γ, δ refer to spinor indices.

We suppress the dependence on spinor indices. This equation can be solved to give

$$T_E = \frac{G}{\sqrt{2}} \Big/ \left(1 - \frac{G}{\sqrt{2}} \int [G_E(k) - G_0(k)] + \frac{G}{\sqrt{2}} \int \Delta V_E(k)\right).$$

The extra integral over ΔV_E in the denominator represents the unknown energy dependence of the interaction term. It does not have the two-particle elastic unitarity cut. It vanishes as $E \to 0$. It would be reasonable to hope both that ΔV_E is small and that its integral converges rapidly enough so that $\int \Delta V_E$ vanishes as $E \to \infty$. Our solutions might then dominate at low and high energies.

The second assumption made in this section is that the Fermi term represents the correct T_0 amplitude to be used in the transformed Bethe-Salpeter equation. There are two points to be considered. The choice of a $V-A$ representation for the off-shell amplitude is consistent in the sense that it leads to a T_E with the $V-A$ structure. Moreover, CP invariance, the fact that the equation involves T_0 with two legs on shell, and the requirement that off-shell terms have to vanish on shell severely constrain the off-shell behavior of T_0. The second point is that $T(E=0, t) = T(E=0, t=0)$ is a strong assumption. A better approximation to T_0 is the appropriate combination of crossed-channel amplitudes. It is just this improvement which leads to crossing-symmetric amplitudes. We consider the effect of crossed-channel input in the next section.

III. LEPTON-LEPTON SCATTERING: CROSSED-CHANNEL INPUT

The results of Sec. II have the distasteful feature of depending on unknown cutoffs Λ^2. We needed a cutoff for the integral in (15) because, by using the Fermi term for $T_0(t)$, we did not have any momentum-transfer dependence to damp the contributions from the large-k^2 region. To correct this defect, we propose to use the crossed-channel solutions for T_0. For instance, in the s-channel reaction $e\nu_e \to \nu_e e$, we use the unitarized amplitudes for $e\bar{\nu}_e \to \bar{\nu}_e e$ and $e\bar{e} \to \nu_e \bar{\nu}_e$ in the form

$$t_0(p,q)=\frac{G}{\sqrt{2}}\left[1-\frac{G}{\sqrt{2}}\frac{(p+q)^2}{3\pi^2}\ln\left(-\frac{(p+q)^2}{\Lambda_t^{\,2}}\right)\right]^{-1} \times\left\{1-\left[\frac{G}{\sqrt{2}}\frac{(p-q)^2}{6\pi^2}\ln\left(-\frac{(p-q)^2}{\Lambda_u^{\,2}}\right)\right]^2\right\}^{-1} \tag{28}$$

The choice of a product of t- and u-channel solutions is based on arguments given in the previous section for the construction of a crossing-symmetric invariant amplitude. In addition, to obtain (28) we have analytically continued mass-shell amplitudes in the crossed channels to the s channel.

The Bethe-Salpeter equation can be solved if we use a separable approximation to $t_0(p,q)$,

$$t_0(p,q)\cong\frac{t_0(p,0)t_0(0,q)}{t_0(0,0)}\ . \tag{29}$$

Such a separable approximation has been shown to lead to reasonable solutions in other applications of the Bethe-Salpeter equation.[7] With this approximation we find

$$t_E(p,q)=\frac{t_0(p,0)t_0(0,q)}{t_0(0,0)[1+(G/\sqrt{2})I'(E)]} \cong\frac{t_0(p,q)}{1+(G/\sqrt{2})I'(E)}\ , \tag{30}$$

where the integral in the denominator is given by

$$I'(E)=\left(\frac{\sqrt{2}}{G}\right)^2\frac{2E^2}{\pi^2}\int_0^\infty\frac{dk^2}{E^2-k^2}\,[t_0(k,0)]^2\,. \tag{31}$$

Since this integral is convergent, it need not be cut off at large values of k^2. In this separable limit $t_E(p,q)=A(s,t,u)$, where $A(s,t,u)$ is defined in (27). In this approximation $A(s,t,u)$ emerges naturally as a product of direct and crossed-channel amplitudes.

However, rather than calculating the solutions to (6) with T_0 given by (28), we use the fact that the integrals are now convergent in order to fix the cutoff parameter Λ^2. To be more specific, we assume that the crossing-symmetric solutions of the previous section have the correct form. We expand them to second order in the coupling constant and compare the result at $p=q=0$ with the first iteration of the equation for $t_E(p,q)$ with $t_0(p,q)$ given by (28). Since integrals converge, we are able to fix Λ^2. Consider the process $e\nu_e \to \nu_e e$. The integral equation has the form

$$t_E(p,q)=t_0(p,q) +\int t_0(p,k)[G_E(k)-G_0(k)]t_E(k,q)\,. \tag{32}$$

The spinor decomposition of the amplitude is the same as in the previous section. The difference $G_E(k)-G_0(k)$ appears in (10). If we iterate (32) and then set $p=q=0$, we find

$$t_E(0,0)=t_0(0,0) +\int t_0(0,k)[G_E(k)-G_0(k)]t_0(k,0)+\cdots\,. \tag{33}$$

Now $t_0(0,0)$ is just the Fermi term $G/\sqrt{2}$, while $t_0(0,k)\equiv t_0(k^2)$ is given in (28). Combining the predictions of (33) and (27), we find

$$t_E(0,0)=\frac{G}{\sqrt{2}}\Big/\left[1+\frac{G}{\sqrt{2}}\frac{s}{2\pi^2}\ln\left(-\frac{s}{\Lambda_s^{\,2}}\right)\right] =\frac{G}{\sqrt{2}}-\frac{s}{2\pi^2}\int_0^\infty\frac{dk^2}{E^2-k^2}\,[t_0(-k^2)]^2+\cdots\,. \tag{34}$$

Thus, we make the identification

$$\left(\frac{G}{\sqrt{2}}\right)^2\ln\left(-\frac{s}{\Lambda_s^{\,2}}\right)=\int_0^\infty\frac{dk^2}{E^2-k^2}\,[t_0(-k^2)]^2\,. \tag{35}$$

In the Appendix we discuss the steps necessary to write the integral in (33) in the form appearing in (35).

If we use (28) for $t_0(k^2)$, the resulting integrals are nontrivial. Moreover, they depend on $\Lambda_t^{\,2}$ and $\Lambda_u^{\,2}$, the crossed-channel cutoffs. These are parameters which we would like to let become as large as possible. In the Appendix we evaluate the integral in (35) in the limit $\Lambda_t^{\,2}=\Lambda_u^{\,2}\to\infty$. In this limit the integrals are sensitive to values of $k^2\ll\Lambda^2$, the region where the form of $t_0(k^2)$ should be most reliable.

For the process $e\bar{\nu}_e \to \bar{\nu}_e e$, the iteration scheme is similar, although there is the problem of coupled channels. In this case we find

$$2\left(\frac{G}{\sqrt{2}}\right)^2 \ln\left(-\frac{t}{\Lambda_t^{\,2}}\right) = \int \frac{dk^2}{E^2-k^2}\{[{}^{11}t_0(-k^2)]^2 + [{}^{12}t_0(-k^2)]^2\} , \tag{36}$$

where

$${}^{11}t_0(-k^2) = t_0(s=u=-k^2,\ e\bar{\nu}_e \to \bar{\nu}_e e)$$

and

$${}^{12}t_0(-k^2) = t_0(s=u=-k^2,\ e\bar{\nu}_e \to \bar{\nu}_\mu \mu) .$$

In the case of $e\bar{e} \to \nu_e\bar{\nu}_e$, the situation is somewhat different in that the lowest-order correction to the Fermi amplitude is, in fact, third-order in G. This correction arises then both from a double iteration of $t_0(e\bar{e}\to\nu_e\bar{\nu}_e)$, which is first-order in G, and from a single iteration of $t_0(e\bar{e}\to e\bar{e})$, which is second-order in G, and $t_0(e\bar{e}\to\nu_e\bar{\nu}_e)$. It would be very difficult to evaluate the double-iteration integral in a reliable way. Thus, we consider $e\bar{e}\to e\bar{e}$ instead and assume that for this coupled-channel problem, the cutoffs should be the same. In this case

$$\begin{aligned}{}^{11}t_E(p=q=0) &= \left(\frac{G}{\sqrt{2}}\right)^2 \frac{u}{6\pi^2}\ln\left(-\frac{u}{\Lambda_u^{\,2}}\right) \Big/ \left\{1 - \left[\frac{G}{\sqrt{2}}\frac{u}{6\pi^2}\ln\left(-\frac{u}{\Lambda_u^{\,2}}\right)\right]^2\right\} \\ &= \int {}^{12}t_0(0,k)[G_E(k) - G_0(k)]\,{}^{21}t_0(k,0) + \cdots ,\end{aligned}$$

and

$$\left(\frac{G}{\sqrt{2}}\right)^2 \ln\left(-\frac{u}{\Lambda_u^{\,2}}\right) = \int \frac{dk^2}{E^2-k^2}[{}^{12}t(-k^2)]^2 , \tag{37}$$

which fixes the cutoff for $e\bar{e}\to e\bar{e}$ and (by the above assumption) for $e\bar{e}\to\nu_e\bar{\nu}_e$.

In order to actually determine the cutoff in each channel, we need to evaluate integrals of the form

$$I_m(\tau,\lambda,\Lambda^2) = \int_0^\infty \frac{dz}{z+\tau}\frac{1}{[1+\lambda z\ln(z/\Lambda^2)]^m} , \quad m=1,2 \tag{38}$$

in the limit $\Lambda^2\to\infty$. The specific integrals in each case can be put in this form by use of partial fractions. In the Appendix we show that

$$I_m(\tau,\lambda,\Lambda^2) \underset{\Lambda^2\to\infty}{\sim} \ln\left(\frac{1}{|\lambda|\,\tau e^{m-1}\ln(|\lambda|\,\Lambda^2)}\right) , \quad m=1,2 . \tag{39}$$

Using this result, we find for the nth channel in Table I

$$\ln\left(\frac{1}{\Lambda_n^{\,2}}\right) = \ln\left(\frac{GC_n\ln\Gamma}{4\sqrt{2}\,\pi^2}\right) , \tag{40}$$

where $\Gamma = G\Lambda^2$. The constant C_n is given in the last column of Table I. We have taken all crossed-channel cutoffs to be the same in obtaining these results. However, the values of C_n are independent of this assumption, since

$$\begin{aligned}\ln[\ln(\alpha\Lambda^2)] &= \ln(\ln\Lambda^2) + \ln\left(1+\frac{\ln\alpha}{\ln\Lambda^2}\right) \\ &\cong \ln(\ln\Lambda^2)\end{aligned}$$

in the limit $\Lambda^2\to\infty$. Thus we have managed to express all lepton scattering amplitudes in terms of one unknown parameter $\ln(\ln\Gamma)$. To the extent that $\ln\Gamma$ is truly large, one expects deviations from the predictions of the simple Fermi amplitude to set in considerably below the unitarity limit – i.e., near

$$s \sim \frac{\pi^2}{G}\frac{1}{\ln(\ln\Gamma)} \sim \frac{(1000\ \text{GeV})^2}{\ln(\ln\Gamma)} .$$

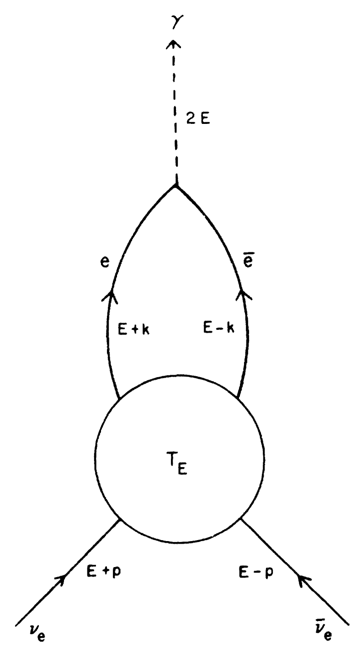

FIG. 3. Graph employed in calculation of the neutrino charge radius.

However, there are reasons to believe that $\ln\Gamma$ is bounded from above.[8]

Finally, we note that this procedure for determining all cutoff parameters in terms of a single unknown would not work if we had not used the transformed Bethe-Salpeter equation or an equivalent procedure, for then the cutoff would be necessary to give meaning to quadratically divergent integrals. The integrals $I_n(\tau, \lambda, \Lambda^2)$ would be replaced by

$$J_n(\tau, \lambda, \Lambda^2) = \int_0^\infty \frac{z\,dz}{z+\tau} \frac{1}{[1+\Lambda^2\lambda + \lambda z \ln(z/\Lambda^2)]^n} ,$$

which either diverge or go to zero as $\Lambda^2 \to \infty$.

IV. CHARGE RADIUS OF THE NEUTRINO

Other usually divergent weak parameters can also be calculated in terms of $\ln G\Lambda^2$, if our invariant amplitude $A(s, t, u)$ is used for the weak-interaction four-point function. As an example, we consider the electromagnetic interactions of the neutrino generated by the diagram in Fig. 3.[9] Using the kinematics shown in the figure, we find

$$A_\lambda \cong \bar{v}_\nu(E+q)\gamma_\lambda(1+\gamma_5)u_\nu(E-q)\frac{i4G}{\sqrt{2}} \int \frac{d^4k}{(2\pi)^4} \frac{E^2 - k_0^2 + \frac{1}{3}\vec{k}^2}{[(E+k)^2 - m^2][(E-k)^2 - m^2]}$$

$$\times\left(\left[1+\frac{G}{\sqrt{2}}\frac{k^2}{2\pi^2}\ln\left(-\frac{k^2}{\Lambda_s^2}\right)\right]\left[1-\frac{G}{\sqrt{2}}\frac{k^2}{3\pi^2}\ln\left(-\frac{k^2}{\Lambda_t^2}\right)\right]\left\{1-\left[\frac{G}{\sqrt{2}}\frac{q^2}{6\pi^2}\ln\left(-\frac{q}{\Lambda_u^2}\right)\right]^2\right\}\right)^{-1}, \quad (41)$$

where $q^2 = 4E^2$ is the momentum transfer at the vertex. In (41) we used (27) for the $\nu_e\bar{\nu}_e \to e\bar{e}$ amplitude. After making a Wick rotation and carrying out the angular part of the integration (see the Appendix), we subtract the resulting integral at $q^2 = 0$ to ensure the vanishing of the neutrino charge. In the limit $q^2 \gg m^2$, A_λ becomes

$$A_\lambda = \bar{v}_\nu\gamma_\lambda(1+\gamma_5)u_\nu\frac{G}{\sqrt{2}}\frac{q^2}{12\pi^2}$$

$$\times\int_0^\infty \frac{dk^2}{\frac{1}{4}q^2 - k^2}\left(1-\frac{G}{\sqrt{2}}\frac{k^2}{2\pi^2}\ln\frac{k^2}{\Lambda^2}\right)^{-1}\left(\left(1+\frac{G}{\sqrt{2}}\frac{k^2}{3\pi^2}\ln\frac{k^2}{\Lambda^2}\right)\left\{1-\left[\frac{G}{\sqrt{2}}\frac{q^2}{6\pi^2}\ln\left(-\frac{q^2}{\Lambda^2}\right)\right]^2\right\}\right)^{-1}$$

$$\simeq \bar{v}_\nu\gamma_\lambda(1+\gamma_5)u_\nu\frac{G}{\sqrt{2}}\frac{q^2}{12\pi^2}\ln\left(\frac{G}{\sqrt{2}}\tfrac{1}{4}q^2\, 0.43\ln\Gamma\right). \quad (42)$$

In order to obtain the charge radius, (41) is evaluated in the limit $q^2 \ll m^2$. In this case the q^2 in the logarithm in (42) is replaced by m^2. We can hope that the charge radius might be measured, say by scattering from a high-Z nucleus.[10] This would determine the unknown parameter $\ln G\Lambda^2 = \ln\Gamma$ and allow definite predictions to be made concerning the various leptonic scattering reactions.

APPENDIX

In this appendix we discuss the evaluation of two types of integrals. The first has the form

$$I(E) = \int \frac{d^4k\, F(k^2)}{[(E-k)^2 - m^2][(E+k)^2 - m^2]} , \quad (A1)$$

where $F(k^2)$ is an arbitrary function of $k^2 = k_0^2 - |\vec{k}|^2$. If $0 < E < 2m$, the k_0 integration contour may be rotated from the real to the imaginary axis. After this Wick rotation, we have an integration over a four-dimensional Euclidean region. If E is purely timelike, $k^2 \to -k^2$, and $k_0 = k\cos\beta$, $|\vec{k}| = k\sin\beta$, we have

$$I(E) = 4\pi i\int_0^\infty k^3 dk\, F(-k^2)\int_0^\pi \frac{\sin^2\beta\, d\beta}{[(E^2 - k^2 - m^2)^2 + 4E^2k^2\cos^2\beta]} . \quad (A2)$$

The integral over β gives

$$I(E) = \pi^2 i\int_0^\infty \frac{k\,dk}{E^2} F(-k^2)\left(\frac{[(k^2+m^2-E^2)^2 + 4E^2k^2]^{1/2}}{k^2+m^2-E^2} - 1\right) . \quad (A3)$$

This integral and the one with $F(k^2)$ replaced by $k_0^2F(k^2)$ are used in going from (33) to (35). The k_0^2 factor introduces $-\cos^2\beta$ in the numerator of the β integral in (A2) and causes no difficulty. Both integrals can be continued to all values of E^2 and to $m=0$.

The second type of integral appears when we try to fix the cutoff parameters. We must evaluate integrals of the type

$$I_n(\lambda,\tau,\Lambda^2)=\int_0^\infty \frac{dx}{x+\tau}\frac{1}{[1+\lambda x\ln(x/\Lambda^2)]^n}\ ,\quad n=1,2\ . \tag{A4}$$

The constant λ can be either positive or negative. We want the value of the integral in the limit $\lambda\to 0$, $\lambda\Lambda^2\to\infty$, $\lambda\ln\Lambda^2\to 0$.

If λ is negative and $\gamma=|\lambda|\Lambda^2$, we have

$$\begin{aligned}I_1(\tau,\lambda,\Lambda^2)&=\int_0^\infty \frac{dx}{x+\tau}\frac{1}{1-\lambda x\ln(x/\Lambda^2)}\\&=\int_0^\infty\frac{dx}{x+\tau/\Lambda^2}\frac{1}{1-\gamma x\ln x}\ ,\end{aligned}$$

which we write as

$$I_1=\int_0^1\frac{dx}{x+\tau/\lambda^2}+\int_0^1\frac{dx\,x}{x+\tau/\Lambda^2}\frac{\gamma\ln x}{1-\gamma x\ln x}+\int_1^\infty\frac{dx}{x+\tau/\Lambda^2}\frac{1}{1-\gamma x\ln x}\ . \tag{A5}$$

In the limit $\Lambda^2\to\infty$ we may drop τ/Λ^2 in the second and third integrals. The latter integral contains a pole at $x=x_0$ with $x_0\ln x_0=1/|\lambda|\Lambda^2=1/\gamma$. Since I_1 must be real, it is treated as a principal value integral. This pole, and the ones in the case λ is positive, represent distant left-hand singularities in the amplitude $A(s,t,u)$. See (27). However, these poles do not contribute to the large-Λ^2 limit of the integrals. The last integral in (A5) can be broken up into the ranges from 1 to $1+\delta$ to ∞, where $\delta\ll 1$, but $\gamma\delta\to\infty$. Then we find

$$\begin{aligned}\mathrm{P}\int_1^\infty\frac{dx}{x(1-\gamma x\ln x)}&=\mathrm{P}\int_1^{1+\delta}\frac{dx}{x(1-\gamma x\ln x)}\\&\simeq\int_0^\delta\frac{dz}{1-\gamma z}=-\frac{1}{\gamma}\ln(\gamma\delta-1)\to 0\ .\end{aligned} \tag{A6}$$

The integral from $1+\delta$ to ∞ is dropped, since it is easily seen to vanish as $\gamma\to\infty$.

The remaining nontrivial integral in (A5) can be written in the form

$$\begin{aligned}\int_0^1\frac{dx\gamma\ln x}{1-\gamma x\ln x}&=-\int_0^\infty dz\left(\gamma e^{-z}-\frac{d}{dz}(\gamma z e^{-z})\right)\Big/(1+\gamma z e^{-z})\\&=-\gamma\int_0^\infty\frac{dz}{e^z+\gamma z}\underset{\gamma\to\infty}{\sim}-\ln(\gamma\ln\gamma)\ .\end{aligned}$$

The last step is not obvious, and we proved it by numerical integration. Thus, we find

$$\begin{aligned}I_1(\lambda,\tau,\Lambda^2)&=\ln(\Lambda^2/\tau)-\ln[|\lambda|\Lambda^2\ln(|\lambda|\Lambda^2)]\\&=-\ln[|\lambda|\tau\ln(|\lambda|\Lambda^2)]\ .\end{aligned} \tag{A7}$$

$I_2(\lambda,\tau,\Lambda^2)$ with λ negative has a double pole. We evaluate the double pole as the limit of two single poles which approach each other. This definition is consistent with the definition of $t_E(0,0)$ in (33) as the limit of $t_E(p,0)$ as $p\to 0$; $t_E(p,0)$ does not have a double pole. We find

$$\begin{aligned}J(\lambda,\tau,\Lambda^2,\alpha)&=\int_0^\infty\frac{dx}{x+\tau}\frac{1}{1+\lambda x\ln(x/\Lambda^2)}\frac{1}{1+\lambda\alpha x\ln(x/\Lambda^2)}\\&=\frac{1}{1-\alpha}\int_0^\infty\frac{dx}{x+\tau}\left(\frac{1}{1+\lambda x\ln(x/\Lambda^2)}-\frac{\alpha}{1+\lambda\alpha x\ln(x/\Lambda^2)}\right)\\&\underset{\Lambda^2\to\infty}{\sim}\frac{1}{1-\alpha}\{\alpha\ln[|\lambda|\alpha\ln(|\lambda|\alpha\Lambda^2)]-\ln[|\lambda|\tau\ln(|\lambda|\Lambda^2)]\}\ .\end{aligned}$$

In the limit $\alpha\to 1$,

$$\lim_{\alpha\to 1} J(\lambda, \tau, \Lambda^2, \alpha) = I_2(\lambda, \tau, \Lambda^2)$$
$$= -\ln[|\lambda|\,\tau \ln(|\lambda|\Lambda^2)] - 1$$
$$= -\ln[|\lambda|\,\tau e \ln(|\lambda|\Lambda^2)] \,. \tag{A8}$$

If λ is positive, evaluation is more difficult, since there exist two poles in the interval $0 < x < 1$. Again defining $\gamma = \lambda\Lambda^2$, we have in this case

$$I_1 = \int_0^{1/e} \frac{dx}{x + (\tau/\Lambda^2)(1 + \gamma x \ln x)} + \int_{1/e}^{1} \frac{dx}{x(1+\gamma x \ln x)} + \int_1^{\infty} \frac{dx}{x(1+\gamma x\ln x)} \,. \tag{A9}$$

The first two integrals have poles. We drop τ/Λ^2 where possible. The third integral vanishes since

$$\int_1^{\infty} \frac{dx}{x(1+\gamma x \ln x)} < \int_1^{e} \frac{dx}{x(1+\gamma x\ln x)} + \int_e^{\infty} \frac{dx}{x(1+\gamma x)} = \int_1^{e} \frac{dx}{x(1+\gamma x \ln x)} + \ln\left(1 + \frac{1}{\gamma e}\right) \underset{\gamma\to\infty}{\sim} 0 \,. \tag{A10}$$

The second integral in (A9) can be shown to vanish in the same way as the integral in (A6) vanished. It contains a single pole which moves to $x = 1$ in the limit $\gamma \to 1$. We are left with

$$I_1 = \left(\int_e^{x_0 - \delta} + \int_{x_0+\delta}^{\infty} \right) \frac{dx}{(1 + \tau x/\Lambda^2)(x - \gamma \ln x)} \,, \tag{A11}$$

where $x_0 = \gamma \ln x_0$. In the region $x < x_0$, we write

$$x - \gamma \ln x = x - \gamma \ln x_0 - \gamma \ln\left(1 - \frac{x_0 - x}{x_0}\right)$$
$$= (x - x_0)\left(1 - \gamma \sum_{n=1}^{\infty} \frac{(x_0 - x)^{n-1}}{n x_0^{\,n}}\right) \,. \tag{A12}$$

For $x > x_0$ we have

$$x - \gamma \ln x = x - \gamma \ln x_0 + \gamma \ln\left(1 - \frac{x - x_0}{x}\right)$$
$$= (x - x_0)\left(1 - \gamma \sum_{n=1}^{\infty} \frac{(x - x_0)^{n-1}}{n x^{n}}\right) \,. \tag{A13}$$

By using more and more terms of these series in the integrand of (A11), we generate a set of better and better approximations to the integral, the first few of which can be done by hand. The first and second approximations, in fact, give the same answer for I_1, the answer which appears in (A7). $I_2(\lambda, \tau, \Lambda^2)$ for positive λ is defined in the same way as for negative λ and has the same value.

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[10] See Arbuzov, Ref. 3.