Renormalization of Gauge Theories, W Decay, and μ Decay*

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We present a renormalization program for spontaneously broken gauge theories in the unitary formalism. It is compared with the renormalization program for renormalizable gauges and necessary constraints are discussed. We then explicitly formulate this program for the Weinberg SU(2)×U(1) theory of leptons and show how it provides the basis for the dispersive calculations of higher-order corrections to μ decay reported in a previous paper. As we pointed out there, certain symmetry-breaking effects are finite and calculable. In this paper, we calculate the breaking of μ-e universality in order α. We also explicitly verify the consistency of our renormalization procedure on the one-loop level. There are certain subtleties in the treatment of infrared divergences in gauge models, particularly in dispersive calculations. We give a method for handling this problem, calculate the rate for W decay, and complete the calculation of the photonic contributions to μ decay in the Weinberg SU(2)×U(1) model. Our predictions for the decay rate and the electron energy spectrum in μ decay for the Weinberg model are experimentally indistinguishable from those of the old current-current theory of weak interactions. Finally we make some comments about other gauge models and other physical processes.

I. INTRODUCTION

The spontaneous breaking of a gauge symmetry allows the incorporation of massive intermediate vector bosons in the construction of renormalizable field theories of weak and electromagnetic interactions. This approach was first proposed by Weinberg[1] in the context of an SU(2)×U(1) model of the weak and electromagnetic interactions of leptons, and it has recently been pursued vigorously as a result of the work of 't Hooft[2] and others.[3]

In a previous paper (I),[4] we have presented a calculation of higher-order weak contributions to μ decay in the Weinberg model. The calculation was done in the U formalism,[3-5] in which the propagator of the massive vector mesons $W^{\pm}$ and Z takes the canonical form $i(k_\mu k_\nu/M^2 - g_{\mu\nu})/(k^2 - M^2)$, and only one real scalar field ϕ appears in the Lagrangian. A dispersive approach was employed. Despite the bad behavior of the vector propagator at high energy, it was shown that if the absorptive parts corresponding to all the W-Z, W-γ, and W-ϕ exchange graphs are separately added, the asymptotically growing terms cancel. The dispersion relations then converge with only the usual subtractions, corresponding to mass and wave-function renormalization of the propagator and to renormalization of the W-lepton vertices. All finite contributions, except those coming from the W-γ cut, were calculated explicitly in I. It was concluded that these contributions to the μ-decay rate were less than 1% (except for singular limits of the model) and that they would have negligible effect on the shape of the final electron spectrum. In addition, certain aspects of the dispersive method of calculation, including the complications due to spin and to the instability of the W boson, were discussed in some detail in I.

In the present paper, we discuss our renormalization procedure for calculations in the U formalism; and we complete the calculation of the W-γ contributions to μ decay in Weinberg's model, including a detailed treatment of infrared divergences. As we explained in I, individual Feynman graphs are badly ultraviolet-divergent in the U formalism, and Green's functions remain divergent even after the usual renormalizations have been performed. When the graphs corresponding to physical S-matrix elements are summed, however, the divergences cancel as a consequence of the Higgs *et al.*[6] mechanism: the spontaneous breaking of the gauge symmetry. The divergent nature of Green's functions in the U formalism, and the necessity of respecting constraints imposed by the underlying gauge symmetry, require that care be taken in the renormalization of the field theory. We present here an explicit procedure for fixing the free renormalization constants in the U formalism and we check that essential constraints are fulfilled. Our discussion here thus supplements that in paper I by verifying the consistency of the calculational procedure adopted there.

As always, the burden of renormalization is to enable the calculation of physical amplitudes in terms of physically measurable parameters: masses and coupling constants. We have chosen the following parameters as fundamental in the Weinberg model: the electric charge e, the charged-intermediate-vector-boson mass M_W, and a coupling constant g which is related in a simple way to the on-mass-shell $(Wl\nu_l)$ couplings g_l $(l=e, \mu)$.[7] The quantity R, used to parametrize the Weinberg model in I, is given in terms of e^2 and g^2 as follows:

$$R \equiv M_W{}^2/M_Z{}^2 = (1 - e^2/g^2)^{-1} + O(e^2) . \quad (1)$$

The massive neutral-intermediate-boson mass is thus determined in the Weinberg model in terms of the parameters already specified.

We note here some of the special features of our renormalization procedure: Renormalization is performed starting from the bare U-formalism Lagrangian, obtained after shifting the neutral scalar field to remove its vacuum expectation value, and eliminating the three remaining unphysical components of the (complex) scalar doublet by means of a gauge transformation. Consequently, we have – and must use – the freedom to rescale the scalar field, its vacuum expectation value, and the gauge couplings all independently. The left- and right-handed fermion fields are separately renormalized. The mass and wave-function renormalizations of all particles except the neutral vector boson Z are performed on the mass shell in the canonical fashion. The Z mass M_Z is then determined in the Weinberg model in terms of M_W and the ratio of electromagnetic and weak coupling constants e and g, as we have noted in Eq. (1). The Z wave-function renormalization can then be performed in the usual manner, aside from the trivial complication of including Z-γ mixing, and this determines the last free renormalization constant. The subtractions to be made for all remaining vertices in the theory are then determined. If the theory is renormalizable and our subtraction procedure is consistent, these subtractions will remove all remaining infinities in physical S-matrix elements expressed in terms of physical coupling constants and masses. We have verified Eq. (1) and have also verified[4] that our procedure yields a finite physical $(We\nu_e)$ coupling g_e which differs from g_μ only by ultraviolet-finite terms of order e^2. These calculations were performed using the gauge-invariant regulator method of 't Hooft and Veltman.[2] The consistency of our U formalism renormalization procedure has also been verified more fully at the one-loop level in a simplified Abelian model.[8]

Infrared divergences are a well-known feature of calculations of electromagnetic radiative corrections, and they appear with a vengeance in a dispersive treatment such as ours. This is because all renormalizations are done on the mass shell, which for example introduces infrared divergences into g_l, the $(Wl\nu_l)$ coupling constant. We can easily understand this physically: The process $W \to l + \nu_l + \gamma$ has an infrared divergence as the photon energy goes to zero, and it is only the sum of this rate with the nonradiative decay rate that is infrared-finite in order $g^2\alpha$. We discuss the detailed treatment of infrared divergences in spontaneously broken gauge (SBG) theories of weak and electromagnetic interactions, and show how to introduce a fictitious photon mass δ in the Weinberg model so that the infrared divergences can be handled in the usual fashion. These considerations are illustrated by the explicit calculation of the contributions to μ decay from W-γ intermediate states in the Weinberg model of leptons.

The paper is organized as follows. In the next section, II, we discuss the renormalization of spontaneously broken gauge models in the U formalism, and verify for the Weinberg model that certain crucial constraints are satisfied. Formal details, including the explicit forms of bare and renormalized Lagrangians, are relegated to Appendix A, and calculational details are placed in Appendix B. In Sec. II we largely ignore the problem of infrared divergence. Section III remedies this omission. There we introduce a photon "mass" δ as an effective infrared cutoff parameter, and discuss the separation of infrared-divergent from nondivergent terms in the context of a calculation of radiative corrections to $W \to l + \nu_l$, $l = e, \mu$. The formalism is then applied in Sec. IV to complete the calculation of electromagnetic corrections to μ decay. In Sec. V we discuss other processes briefly and conclude with some comments on the import of our results for other SBG models.

II. RENORMALIZATION

The starting point for any SBG model is a renormalizable Lagrangian $\mathcal{L}_I$ involving a set of massless gauge fields coupled to a set of complex scalar fields and possibly other fields [e.g., Eq. (A1)]. Spontaneous breaking of the gauge symmetry gives masses to some of the gauge fields and can be introduced in such a way that the resulting Lagrangian $\mathcal{L}_R$ or $\mathcal{L}_U$ possesses manifest renormalizability or unitarity, respectively. The unitary Lagrangian $\mathcal{L}_U$ for the Weinberg model is displayed in Appendix A [Eqs. (A2), (A3)].

The first step in a renormalization program is a regularization procedure. For the dispersive calculations of I we did not need to use a regulator, but in examining the subtraction constants it is necessary to define such a procedure. Furthermore we are very much constrained as to the method of regularization we may use, as we must maintain not only the explicit quantum-electrodynamic (QED) gauge invariance of our Lagrangian, but also other formal relationships between different amplitudes which are a remnant of the larger gauge invariance of the initial Lagrangian. The only regularization procedure which we know that satisfies these conditions and is strong enough to regulate the U formalism is the dimensional continuation scheme of 't Hooft and Veltman.[2,9] All our results are to be understood in the context of this regularization prescription.

The gauge-symmetric theory is renormalized by adding to $\mathfrak{L}_I$ a set of gauge-invariant counterterms. It is then claimed[2,3,5] that this set of counterterms is sufficient to remove the divergences of the spontaneously broken theory expressed in an R formalism.

If one chooses instead to work in the unitary formalism, the above set of counterterms is *not* sufficient. That is, it is not sufficient to generate the counterterms for $\mathfrak{L}_U$ by adding gauge-invariant counterterms to $\mathfrak{L}_I$ and then transforming to the unitary formalism. This is not surprising if it is recalled that the only possible gauge invariance remaining in $\mathfrak{L}_U$ is electromagnetic. The divergence cancellations which lead to a finite S matrix are due to the many relations between the coupling constants and masses of $\mathfrak{L}_U$ (which are of course a result of the full original gauge symmetry of $\mathfrak{L}_I$). The important thing then is to preserve these relations when generating the counterterms, and this is assured if the counterterms are generated by a separate multiplicative renormalization (rescaling) of each of the *independent* parameters and fields of $\mathfrak{L}_U$.

The difference between renormalizing in the U formalism and an R formalism can readily be understood. The gauge-independent counterterms are sufficient to perform an intermediate renormalization in the R formalism. Certain finite but gauge-dependent shifts must be added to define conventionally renormalized on-shell quantities.[2,5]

TABLE I. Renormalization constants. The subscript 0 designates the unrenormalized fields and parameters of $\mathfrak{L}_U$ [Eq. (A3)]. Vector indices have been dropped and $l_L=\frac{1}{2}(1-\gamma_5)l$, $l_R=\frac{1}{2}(1+\gamma_5)l$, $l=e$ or μ.

$W_0=\sqrt{Z_W}\,W$	$e_0=\dfrac{1}{\sqrt{Z_A}}\,e$
$\begin{pmatrix}A_0\\Z_0\end{pmatrix}=\begin{pmatrix}\sqrt{Z_A} & \sqrt{Z_M}\\0 & \sqrt{Z_Z}\end{pmatrix}\begin{pmatrix}A\\Z\end{pmatrix}$	$g_0=\dfrac{Z_g}{(Z_W Z_{\nu_\mu} Z_{\mu L})^{1/2}}\,g_\mu$
$\phi_0=\sqrt{Z_\phi}\;\phi$	$\lambda_0=\sqrt{Z_\lambda}\;\lambda$
$(\nu_l)_0=\sqrt{Z_{\nu_l}}\;\nu_e$	$\mu_0^2=\dfrac{Z_\mu}{Z_\phi}\,\mu^2$
$(l_L)_0=\sqrt{Z_{lL}}\;l_L$	$h_0=\dfrac{Z_h}{Z_\phi^2}\,h$
$(l_R)_0=\sqrt{Z_{lR}}\;l_R$	$(G_l)_0=\dfrac{Z_{Gl}}{(Z_\lambda Z_{lR} Z_{lL})^{1/2}}\,G_l$

These pieces can be different even when the intermediate counterterms are the same – for example, the wave-function renormalization subtractions for different members of an isotopic multiplet. In going to the U-formalism treatment, these additional pieces become divergent and thus the relevant counterterms for the U formalism differ by infinite quantities. Even this expanded set of independent counterterms is fewer than the number of quantities to be subtracted; but if the theory is to be renormalizable, these counterterms must be sufficient to produce a finite S matrix.

The details of this program will now be described for the Weinberg model. The unrenormalized Lagrangian is exhibited in Appendix A. Note that we write it in terms of e_0 and g_0. Then

$$(M_W)_0=\frac{\lambda_0 g_0}{\sqrt{2}}$$

and

$$(M_Z)_0=\frac{\lambda_0}{\sqrt{2}}\,\frac{g_0^2}{(g_0^2-e_0^2)^{1/2}}\,.$$

The rescalings and our notation for the renormalization constants are displayed in Table I. The following points should be noted:

(i) The usual quantum-electrodynamic Ward identity has been incorporated so that $e_0\to(1/\sqrt{Z_A})e$.

(ii) The scalar field and its vacuum expectation value are rescaled independently.

(iii) The left- and right-handed fermion fields are rescaled independently.

(iv) The renormalization of the neutral vector fields A and Z involves a mixing which preserves the masslessness of the physical photon.

The complete set of counterterms generated in this way is written in Appendix A [Eq. (A7)]. The next step is to physically define the renormalized parameters of the theory by adjusting the counterterms properly. The eventual finiteness of the theory depends upon being able to remove enough of the singular pieces using the counterterms so that the divergences remaining in the Green's functions cancel when calculating the on-mass-shell S matrix. In this paper we examine in detail only the parts of this program which are relevant to a μ-decay calculation on the one-loop level.

We begin by doing conventional on-shell mass and wave-function renormalization subtractions for the leptons and charged vector boson so that

$$\frac{\lambda g_\mu}{\sqrt{2}}=M_W=\text{physical } W \text{ mass}\,,$$

$$\frac{\lambda G_e}{\sqrt{2}}=m_e=\text{physical electron mass}\,, \tag{2.1}$$

$$\frac{\lambda G_\mu}{\sqrt{2}}=m_\mu=\text{physical muon mass}\,.$$

This fixes the renormalization constants Z_W, Z_{lL}, Z_{lR}, Z_{ν_l}, Z_{G_l} and the combination $Z_\lambda Z_g{}^2$ which enters the mass counterterm for the W. The ϕ mass and wave-function renormalization are also performed conventionally.

The wave-function renormalization counterterms for the charged particles are, by gauge invariance, also the electromagnetic vertex counterterms for each of these particles. They subtract the vertices at $q^2=0$ and so by adjusting Z_A to effect the photon wave-function subtraction at $q^2=0$, we ensure that the renormalized electric charge e is the physical (on-shell) coupling of the photon to each of the charged particles.

We next define the renormalized weak coupling constant g_μ to be the on-shell $(W\mu\nu_\mu)$ coupling. This fixes Z_g since $Z_g - 1$ is the counterterm for the $(W\mu\nu_\mu)$ vertex correction. Because of electromagnetic radiative corrections, the renormalization constants Z_W, Z_{lL}, Z_{lR}, and Z_g will be infrared divergent, as will

$$g_\mu=[(Z_W Z_{\nu_\mu} Z_{\mu L})^{1/2}/Z_g]g_0\,.$$

This infrared divergence is expected, as we noted in the Introduction, since W decay will become infrared convergent only by including the radiation of real photons. The problems associated with inserting an infrared cutoff and the separation of the infrared pieces will be discussed in Sec. III.

We have now expressed the theory in terms of the physical parameters e, M_W, and g_μ. We are then not free to perform the Z mass renormalization conventionally, however, since all the renormalization constants which enter the Z-mass-renormalization counterterm [see Eq. (A7)] have been previously determined. If the theory is to be renormalizable, this counterterm must be sufficient to give a finite M_Z:

$$\begin{aligned} M_Z{}^2 &= (\text{physical } Z \text{ mass})^2 \\ &= \frac{\lambda^2 g_\mu{}^2}{4(1-e^2/g_\mu{}^2)} + (\text{ultraviolet-finite terms of order } \alpha)\,. \end{aligned} \tag{2.2}$$

The finiteness of the order α $(=e^2/4\pi \approx \frac{1}{137})$ corrections is explicitly verified in Appendix B.

The Z wave-function renormalization constant Z_Z is now defined in the usual way. The A-Z mixing renormalization constant Z_M is defined such that the $A\to Z$ amplitude vanished for $q^2=M_Z{}^2$. (Note that ordinary electromagnetic gauge invariance already implies that it vanishes at $q^2=0$.)

The other subtraction which is important for the μ-decay calculation is the $(We\nu_e)$ vertex. The renormalization counterterm for this vertex is

$$\{Z_g[(Z_{\nu_e} Z_{eL})/(Z_{\nu_\mu} Z_{\mu L})]^{1/2} - 1\}\,,$$

which has also been previously fixed. This counterterm must leave the $(We\nu_e)$ vertex correction finite on the mass shell and yield a physical $(We\nu_e)$ coupling given by

$$g_e = g_\mu[1 + (\text{ultraviolet-finite piece of order } \alpha)]\,. \tag{2.3}$$

This has also been checked on the one-loop level and was reported in paper I. (See also Appendix B.) The finiteness of the deviation from electron-muon universality as expressed by Eq. (2.3) is necessary if the theory is to be renormalizable, since the two vertices cannot be renormalized independently. The absence of independent counterterms in SBG theories also implies the finiteness of other symmetry-breaking effects, for example certain mass splittings.[10]

There are many more pieces of the subtraction program to be checked. For example, the subtraction constants for the three-vector and four-vector vertices are already determined. Such a complete treatment is, however, beyond the scope of this paper. A full investigation of the consistency of our U-formalism renormalization scheme on the one-loop level and its connection to the R-formalism method for an Abelian theory will be presented in a future publication.[8]

III. INFRARED DIVERGENCES

It is natural, in a dispersive calculation, to calculate in terms of the on-mass-shell $(Wl\nu_l)$ coupling constants g_l. These coupling constants contain infrared divergences, however, as can easily be seen by considering the decay $W\to l+\nu_l$ (Fig. 1), a process which will of course be observed if the W is discovered. A finite prediction for this partial decay rate is obtained only when the emission of real soft photons (with energies below some detection threshold energy k) is included to any given order in α, and the infrared-divergent part of g_l is explicitly kept to that same order. It is convenient to express the resulting finite answer in terms of an infrared-finite coupling constant g, thus defining g as a measurable quantity, at least in principle.

In order to keep track of infrared divergences in a consistent manner, we introduce a photon "mass" δ into $\mathcal{L}$. This can be done without destroying the renormalizability of the Weinberg model by simply adding the term $\frac{1}{2}\tilde{\delta}^2 B_\mu B^\mu$ to the initial Lagrangian Eq. (A1). This term breaks the Abelian gauge invariance but preserves renormalizability since B just becomes a massive neutral vector field coupled to a conserved current. (It is also possible to introduce a photon mass term without destroying the underlying gauge invariance, at the cost of adding an additional scalar field to the Lagrangian.[11]) After transforming to the U formalism, there results a photon mass term $\frac{1}{2}\delta^2 A_\mu A^\mu$, corrections of order δ to the Z-boson mass, and a Z-A mixing of order δ, where $\delta^2 = \tilde{\delta}^2 g_0^2/(g_0^2 - e_0^2)$. These additional pieces, some of which are ultraviolet-divergent or contain divergences as $\delta \to 0$, complicate the renormalization program. However, we have checked that these extra divergences cancel, so that we are free to take the limit $\delta \to 0$ everywhere except in the denominator of the photon propagator in calculating S-matrix elements.

We now define an infrared-finite coupling constant g by means of the equation

$$g_\mu = g(1 + \alpha B_\mu)\,, \tag{3.1}$$

where the quantity B_μ is defined to include all the infrared divergences that are present in g_μ. We specify in addition that B_μ also contains all terms which are divergent as $m_\mu/M_W \to 0$. Then, to order α,

$$B_\mu = \frac{1}{\pi}\left[-\left(\ln\frac{M_W}{m_\mu}\right)\ln\frac{M_W}{\delta} + \frac{1}{2}\left(\ln\frac{M_W}{m_\mu}\right)^2 + \ln\frac{M_W}{m_\mu} + \frac{1}{2}\ln\frac{M_W}{\delta} + \frac{1}{2}\ln\frac{m_\mu}{\delta} - \frac{1}{4}\ln\frac{M_W}{m_\mu}\right]. \tag{3.2}$$

The first three terms come from the photon contribution to the $(W\mu\nu_\mu)$ vertex renormalization Fig. 1(b); the fourth, from W wave-function renormalization Fig. 1(c); and the last two, from muon wave-function renormalization Fig. 1(d). The nonelectromagnetic contributions to coupling-constant renormalization are not infrared-divergent and also are finite in the limit $m_\mu/M_W \to 0$, and are therefore by definition not included in B_μ. Such terms are combined with those of Figs. 1(b), 1(c), and 1(d) to define an ultraviolet-finite g_μ. Thus g of Eq. (3.1) is free of all divergences, even in the limit $m_\mu/M_W \to 0$. [In writing Eq. (3.2) we omit infrared-divergent terms of order $(m_\mu^2/M_W^2)\ln(\delta/M_W)$, which become negligible when soft-photon emission is included. This is consistent with the omission of such terms in paper I, and below.]

From the universality result Eq. (2.3), it follows that, aside from terms of order $\alpha m_\mu^2/M_W^2$,

$$g_e = g(1 + \alpha B_e)\,, \tag{3.3}$$

where $B_\mu \to B_e$ upon the substitution $m_\mu \to m_e$.

The partial decay rate for the process $W \to l\nu_l$ is

$$\Gamma_l = \frac{1}{48\pi} M_W g_l^2 \qquad (l = e \text{ or } \mu)\,. \tag{3.4}$$

Including soft-photon emission and using (3.1) and (3.2) then gives the infrared-finite result

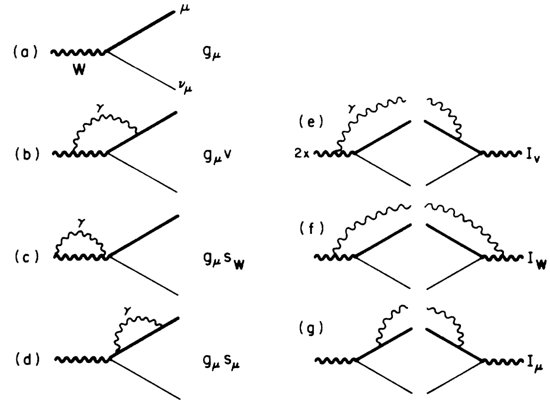

FIG. 1. Infrared-divergence cancellations in W decay. The quantities $(g_\mu^2 v + I_v)$, $(g_\mu^2 s_e + I_e)$, and $(g_\mu^2 s_W + I_W)$ are each independent of the photon mass. Diagrams (e), (f), and (g) represent the contributions to the rate from decay with emission of a real photon.

$$\Gamma_l = \frac{1}{48\pi} M_W g^2 \left\{1 + \frac{\alpha}{\pi}\left[-2\left(\ln\frac{M_W}{m_l}\right)\ln\frac{M_W}{2k} + \ln\frac{M_W}{2k} + \ln\frac{m_l}{2k} + \frac{3}{2}\ln\frac{M_W}{m_l} + O(1)\right] + O(\alpha^2)\right\}, \tag{3.5}$$

where k is the maximum energy of the emitted undetected soft photon.

From (3.5), the ratio $R = \Gamma_e/\Gamma_\mu$ can be read off. The order-α correction can be quite large (~10%) due to the lepton mass factors and photon energy resolution factors in the logarithms.[12] The fact that this ratio is calculable at all – that is, that μ-e universality is broken only by a finite amount – is a consequence of the renormalizability of spontaneously broken gauge theories.

Finally, let us return to a question left open in the preceding section. The expression Eq. (2.2) for M_Z is not manifestly infrared-finite. If, however, we substitute into this equation the definition $\lambda^2 g_\mu^2/4 = M_W^2$ and explicitly exhibit the infrared part of g_μ using Eq. (3.1), the infrared divergences cancel, and we obtain the desired result

$$M_Z = \frac{M_W}{(1 - e^2/g^2)^{1/2}} + (\text{ultraviolet- and infrared-finite corrections of order } \alpha). \tag{3.6}$$

IV. μ DECAY–THE W-γ CUT CONTRIBUTION

For μ decay it is simplest to present our result by comparing the various contributions with the results of the current-current theory.[13] With this

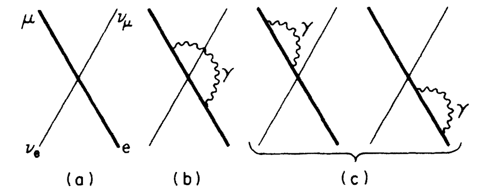

FIG. 2. Photonic corrections to μ decay in the current-current theory.

in mind we first outline the contributions of the various one-loop graphs in that theory, making a separation of terms which will be convenient for this comparison. The contributions are shown in Fig. 2. The Born graph, Fig. 2(a), gives

$$A_{2a}=\frac{G}{\sqrt{2}}\times 4\left(\bar{e}\gamma_\alpha\frac{1-\gamma_5}{2}\nu_e\right)\left(\bar{\nu}_\mu\gamma^\alpha\frac{1-\gamma_5}{2}\mu\right)$$

$$=\frac{4G}{\sqrt{2}}J^{(e)}\cdot J^{(\mu)\dagger}, \tag{4.1}$$

where G is the bare coupling constant appearing in the current-current Lagrangian. The graph of Fig. 2(b) yields[14]

$$A_{2b}=\frac{4G}{\sqrt{2}}\left[J^{(e)}\cdot J^{(\mu)\dagger}\frac{\alpha}{4\pi}\ln\left(\frac{\Lambda^2}{m_\mu{}^2}\right)+\chi\right], \tag{4.2}$$

where χ is some known quantity, of order α, which is independent of the ultraviolet cutoff Λ. We note that χ includes an infrared-divergent part. We also note that it is not directly proportional to the Born-graph contribution so that it causes a change in the shape of the μ-decay spectrum from that predicted by the Born graph. The external-line corrections, Fig. 2(c), give

$$A_{2c}=\frac{4G}{\sqrt{2}}J^{(e)}\cdot J^{(\mu)\dagger}$$

$$\times\left[-\frac{\alpha}{4\pi}\ln\left(\frac{\Lambda^2}{m_e m_\mu}\right)+\frac{\alpha}{2\pi}\ln\left(\frac{m_e m_\mu}{\delta^2}\right)-\frac{5}{8}\frac{\alpha}{\pi}\right]. \tag{4.3}$$

Combining these and including the effect of real radiated photons, one can calculate the total decay rate. One finds[13]:

(i) The ultraviolet cutoff-dependent terms in A_{2b} and A_{2c} cancel.

(ii) There are no terms involving logarithms of lepton masses in the resulting expression for the *total* decay rate.

(iii) The important corrections to the Michel shape parameter ρ away from its Born amplitude value ($\rho=0.75$) arise from χ and from real-photon emission.

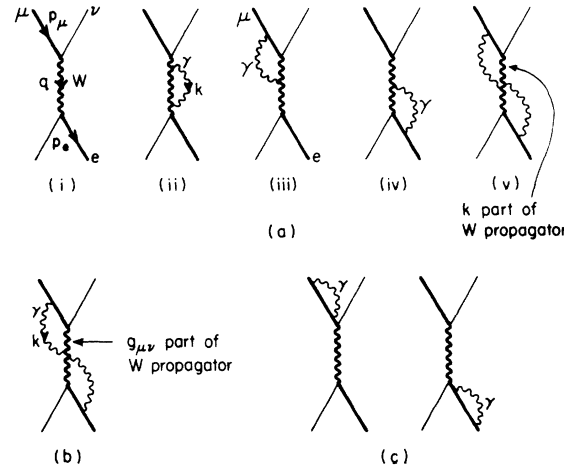

FIG. 3. Photonic corrections to μ decay in the Weinberg model.

The photonic corrections to the μ-decay amplitude in the Weinberg model are displayed in Fig. 3. The contributions of Fig. 3(c) correspond exactly to those of Fig. 2(c) for the current-current theory, but here they are necessarily absorbed into the renormalization of the coupling constants. This was discussed in the previous section, where the infrared and lepton-mass-dependent parts of g_e and g_μ were exhibited explicitly [Eq. (3.2)]. By contrast, in the current-current theory, the logarithmically divergent part of these contributions is canceled so that the result can be expressed in terms of the "bare" Fermi constant G.

Figure 3(b), representing the contribution from the $g_{\mu\nu}$ part of the W vector-boson propagator in this Feynman graph, can readily be compared with the contribution of Fig. 2(b) in the current-current theory

$$A_{3b}=\frac{g_e g_\mu}{2M_W{}^2}\left[J^{(e)}\cdot J^{(\mu)\dagger}\frac{\alpha}{4\pi}\ln\left(\frac{M_W{}^2}{m_\mu{}^2}\right)+\chi\right]$$

$$=\frac{g_e g_\mu}{8M_W{}^2}\frac{\sqrt{2}}{G}A_{2b}(\Lambda=M_W). \tag{4.4}$$

That is, $A_{2b}\to A_{3b}$ if the weak coupling $G/\sqrt{2}$ is replaced by $g_e g_\mu/8M_W{}^2$ and the ultraviolet cutoff Λ is set equal to the vector-boson mass M_W.

The remaining terms, Fig. 3(a), can be written as

$$A_{3a}=\frac{g_e g_\mu}{2M_W{}^2}J^{(e)}\cdot J^{(\mu)\dagger}(1+\alpha Y), \tag{4.5}$$

where the 1 term corresponds to the Born graph (i) and

$$Y = -\frac{M_W^2}{\pi}\int_{(M_W+\delta)^2}^{\infty}\frac{dq^2}{q^2}\,\frac{|\vec{k}|}{8\sqrt{q^2}}\left[\frac{16M_W^2}{(q^2-M_W^2)^2}+\frac{8(-\frac{4}{3}|\vec{k}|^2)}{(q^2-M_W^2)^2} - \frac{2M_W^2}{(q^2-M_W^2)|\vec{k}|}\left(\frac{1}{|\vec{p}_e|}\ln|f_e|+\frac{1}{|\vec{p}_\mu|}\ln|f_\mu|\right)+\frac{4k_0}{q^2-M_W^2}\left(\frac{1}{|\vec{p}_e|}+\frac{1}{|\vec{p}_\mu|}\right)\right], \tag{4.6}$$

where

$$f_l = \frac{\delta^2 - 2k_0p_0 - 2|\vec{k}||\vec{p}_l|}{\delta^2 - 2k_0p_0 + 2|\vec{k}||\vec{p}_l|}$$

and the momentum components are evaluated in the center-of-mass frame $q=(q_0, 0)$. The various momenta are labeled in Fig. 3(a). The method used to obtain Eq. (4.6) was discussed in detail in I. We notice that Y is an infrared-divergent quantity even though the diagrams of Fig. 3(a) appear to be infrared-finite.

We have introduced spurious infrared divergences into the amplitudes for Figs. 3(a ii), 3(a iii), and 3(a iv) by subtracting the vector propagator and weak vertex with the W boson on the mass shell. These divergences are of course exactly canceled by terms in the product $g_e g_\mu$. The full expression of Eq. (4.5), when written in terms of the infrared-finite coupling constant g defined in Sec. III, contains only the infrared-divergent pieces corresponding to Fig. 2(c) [Eq. (4.3)] which will be made finite by the inclusion of soft-photon emission in the expression for any measured rate. Integration of Eq. (4.6) yields

$$A_{3a} = \frac{g^2}{2M_W^2}J^{(e)}\cdot J^{(\mu)\dagger}\left\{1+\frac{\alpha}{\pi}\left[-\frac{1}{4}\ln\left(\frac{M_W^2}{m_e m_\mu}\right)+\frac{1}{2}\ln\left(\frac{m_e m_\mu}{\delta^2}\right)+\frac{5}{6}+\frac{\pi^2}{12}\right]\right\}. \tag{4.7}$$

Thus, combining all contributions,

$$A = A_{3a}+A_{3b}+A_{3c} = \frac{g^2}{2M_W^2}\left(J^{(e)}\cdot J^{(\mu)\dagger}\left\{1+\frac{\alpha}{\pi}\left[\frac{1}{4}\ln\left(\frac{m_e}{m_\mu}\right)+\frac{1}{2}\ln\left(\frac{m_e m_\mu}{\delta^2}\right)+\frac{5}{6}+\frac{\pi^2}{12}\right]\right\}+\chi\right), \tag{4.8}$$

whereas for the current-current theory

$$A_{cc} = A_{2a}+A_{2b}+A_{2c} = \frac{4G}{\sqrt{2}}\left(J^{(e)}\cdot J^{\mu)\dagger}\left\{1+\frac{\alpha}{\pi}\left[\frac{1}{4}\ln\left(\frac{m_e}{m_\mu}\right)+\frac{1}{2}\ln\left(\frac{m_e m_\mu}{\delta^2}\right)-\frac{5}{8}\right]\right\}+\chi\right). \tag{4.9}$$

We caution the reader that our definition of g is somewhat arbitrary. The difference between Eq. (4.8) and (4.9) could be absorbed into a redefinition of g. Equation (4.8) is only meaningful in conjunction with Eq. (3.2) or (3.5), which define g.

To summarize our results for μ decay, we must include the contributions from other cuts calculated in paper I. Denoting the summed contributions of the W-Z, W-ϕ, and lepton cuts by $\frac{1}{2}\xi(g, M_\phi)$, we recall that ξ is a number of order α except at extreme values of the parameters g and M_ϕ. Combining ξ with Eq. (4.8) and including real-photon emission, one can calculate the physical total rate for the μ decay. In fact, this rate can be obtained with no further calculation by using the similarity of Eqs. (4.8) and (4.9). We know that the μ-decay rate in the current-current theory is[13]

$$\Gamma_{cc} = \frac{G^2 m_\mu^5}{192\pi^3}\left[1-\frac{\alpha}{2\pi}(\pi^2-\tfrac{25}{4})\right], \tag{4.10}$$

where experimentally $G = 1.026\times 10^{-5}(m_{proton})^{-2}$.

Thus, from Eqs. (4.8) and (4.9) plus the results of paper I, we find in Weinberg's model

$$\Gamma_{total} = \left(\frac{g^2}{8M_W^2}\right)^2\left[1+\xi+\frac{\alpha}{6\pi}(\pi^2+\tfrac{45}{2}) - \frac{\alpha}{2\pi}(\pi^2-\tfrac{25}{4})\right]\frac{m_\mu^5}{96\pi^3}. \tag{4.11}$$

Furthermore the Michel parameter ρ is unchanged up to terms of order $\alpha m_\mu^2/M_W^2$. We can use Eq. (4.11) as a way to fix some combination of the parameters of Weinberg's model. We find, comparing Eq. (4.10) with Eq. (4.11),

$$\frac{g^2}{8M_W^2}\left[1+\tfrac{1}{2}\xi+\frac{\alpha}{12\pi}(\pi^2+\tfrac{45}{2})\right] = \frac{G}{\sqrt{2}}. \tag{4.12}$$

Precise computation and measurement of any other weak process would provide a test of the model, but unfortunately there are no other observed decays which are not complicated by the presence of strong interactions.

V. CONCLUSION

A. Including Hadrons

If hadrons are included in the Lagrangian in a way which preserves the underlying gauge symmetry and thus the renormalizability of the model, then β decay and all other semileptonic and hadronic weak processes will be finite. This is to be contrasted with the situation in the current-current theory, where the radiative corrections to μ decay are finite [because of the possibility of rewriting the $(V-A)^2$ interaction in charge-retention form] but the radiative corrections to β decay are not if the strong interactions provide no damping. Unfortunately, the presence of strong interactions complicates the actual calculation of weak processes involving hadrons. In β decay, for example, one must include the effect of the anomalous magnetic moment of the neutron and proton. This cannot be done in strong-interaction perturbation theory, of course. Yet, in a gauge theory, if we try to do it phenomenologically, introducing an explicit anomalous magnetic moment term, we destroy the renormalizability. Even if a finite second-order calculation of β decay could be made in some gauge models, it appears from the results of our μ-decay calculation that comparison of μ and β decay will not provide a sensitive test of the model.

B. Comparison with R-Formalism Calculation

Ross[5] has performed a calculation of the second-order corrections to μ decay in the Weinberg SU(2)×U(1) model, in the renormalizable 't Hooft gauge.[2,15] In this gauge, the W propagator has the form $-ig_{\mu\nu}/(k^2-M_W{}^2)$, for example, and there is also a fictitious scalar propagator $1/(k^2-M_W{}^2)$ which serves to cancel out the wrong-metric scalar piece included because of the $g_{\mu\nu}$ in the W propagator. A general discussion of the renormalization of gauge theories in the R formalism has been given by Lee and Zinn-Justin,[3] and an explicit renormalization program for the Weinberg model in the 't Hooft gauge has been given by Ross and Taylor.[5] We have discussed in Sec. II the differences between renormalizing in the R and U formalisms. The most important difference is that rescaling the gauge-symmetric Lagrangian $\mathcal{L}_0$ [Eq. (A1)] provides enough counterterms to perform an intermediate renormalization in an R gauge, but not enough to renormalize the U formalism. Consequently, a U-formalism renormalization program must be based on a rescaling of the shifted Lagrangian [Eqs. (A3) and (A4)].

Ross[5] chose e, M_W, and M_Z as his fundamental parameters in the Weinberg model; we chose e, g, and M_W. Either choice is equally "physical," since all of these parameters are in principle measurable. In order to compare our explicit results with those of Ross, however, it would be necessary to evaluate the terms of order α in Eq. (2.2). Another difficulty in making a direct comparison is that Ross does not present his answer in a very convenient form.

A remark is perhaps in order on the relative ease of such calculations in the 't Hooft gauge and in the U formalism. The complications in the U formalism arise from the many ultraviolet divergences and, particularly in our dispersive calculation, from the infrared divergences. The infrared divergences have a real physical origin, of course, and consequently are present in any formalism; however, by renormalizing off the mass shell, Ross and Taylor avoid the appearance of infrared divergences in coupling constants. The price for this convenience is the necessity to calculate and include renormalization constants on external lines. The R-formalism intermediate renormalization program is also complicated by the presence of the "ghost" particles,[16] whose function is to preserve unitarity. Thus, in the μ decay calculation at least, there unfortunately appears to be a "conservation of difficulty."

C. Summary

In this paper we have set forth a renormalization program for spontaneously broken gauge theories in the U formalism. The program begins by generating the renormalization constants with a separate rescaling of each independent parameter and field of the unrenormalized U-formalism Lagrangian. Using the 't Hooft-Veltman[2] method of dimensional continuation as a regulator prescription, subtraction constants and physical parameters are defined in a conventional fashion. The particular feature of SBG theories which makes this program unusual is that the number of independent counterterms is quite limited. Once a certain number of physical parameters have been defined – in this case the coupling constants e and g_μ and the masses of the leptons, charged vector boson, and Higgs scalar particle – all remaining subtractions are fixed and the remaining physical parameters are determined as functions of these quantities.

We have in this paper also shown how to treat the infrared-divergence problems associated with on-mass-shell subtractions, calculated the W-decay rate, and completed the calculation of μ decay in Weinberg's SU(2)×U(1) model begun in paper I. We find that the shape of the final electron spectrum in μ decay is essentially the same in Weinberg's model as in the current-current theory.

Neglecting terms of order (lepton mass)/M_W, this will also be true in other such models, for example those of Georgi and Glashow[17] and Lee, Prentki, and Zumino.[18] The reason for this is simply understood. These theories are constructed so that, up to terms of order m_l^2/M_W^2, the Born terms for μ decay are identical. Up to terms of order $\alpha m_e^2/M_W^2$, the change in the Michel parameter ρ, which describes the spectrum shape, arises only (a) from the contribution corresponding to Fig. 3(b) where a photon is exchanged between the muon and electron and only the $g_{\mu\nu}$ part of the W propagator is included, and (b) from the emission of real photons from the external lines. These contributions are model-independent for the class of models under discussion. All other terms yield contributions which have the same shape as the Born term, or are negligible. As well as these effects, in the Georgi-Glashow and Lee, Prentki-Zumino models there are also corrections to ρ which are as large as $(\alpha/\pi)(m_x/M_W)^2$, where m_x is a heavy lepton mass. In the Georgi-Glashow model the ratio m_x/M_W is at most $\frac{1}{10}$,[19] so that such terms are undetectable. The calculation of the *total* rate for μ decay in any of these models also yields a result experimentally indistinguishable from that of the current-current theory.

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APPENDIX A. LAGRANGIAN COUNTERTERMS FOR THE WEINBERG MODEL

Before symmetry breaking the Lagrangian is

$$\mathcal{L}_I=-\tfrac{1}{4}(\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu+g\vec{A}_\mu\times\vec{A}_\nu)^2-\tfrac{1}{4}(\partial_\mu B_\nu-\partial_\nu B_\mu)^2+\sum_{l=e,\mu}\overline{L}_l(i\not\partial+\tfrac{1}{2}g\vec{\tau}\cdot\vec{\not A}+\tfrac{1}{2}g'\not B)L_l$$

$$+\sum_{l=e,\mu}\overline{R}_l(i\not\partial+g'\not B)R_l+|(\partial_\mu+\tfrac{1}{2}ig\vec{\tau}\cdot\vec{A}_\mu-\tfrac{1}{2}ig'B_\mu)\hat{\phi}|^2-\sum_{l=e,\mu}G_l(\overline{L}_l\hat{\phi}R_l+\overline{R}_l\hat{\phi}^\dagger L_l)-\mu^2\hat{\phi}^\dagger\hat{\phi}-h(\hat{\phi}^\dagger\hat{\phi})^2. \quad \text{(A1)}$$

All parameters and fields above are unrenormalized. For simplicity we delete the subscript 0 used in Sec. II to denote unrenormalized quantities. The Lagrangian of the U formalism is obtained by the substitutions

$$A_\mu^1\pm iA_\mu^2=\sqrt{2}W_\mu^\pm,\quad A_\mu^3=(g^2+g'^2)^{-1/2}(gZ_\mu-g'A_\mu),\quad B_\mu=(g^2+g'^2)^{-1/2}(g'Z_\mu+gA_\mu), \quad \text{(A2)}$$

$$L_l=\tfrac{1}{2}(1-\gamma_5)\begin{pmatrix}\nu_l\\ l\end{pmatrix}=\begin{pmatrix}\nu_l\\ l_L\end{pmatrix},\quad R_l=\tfrac{1}{2}(1+\gamma_5)l=l_R,\quad \hat{\phi}=\frac{1}{\sqrt{2}}(\lambda+\phi)\begin{pmatrix}0\\1\end{pmatrix}.$$

We express $\mathcal{L}_U$ in terms of the parameters g, $e\,[=gg'/(g^2+g'^2)^{1/2}]$, λ, μ^2, h, and G_l. Then the free and interacting parts of $\mathcal{L}_U$ are

$$\mathcal{L}_0=-\tfrac{1}{2}|\partial_\mu W_\nu-\partial_\nu W_\mu|^2+\frac{\lambda^2g^2}{4}|W_\mu|^2-\tfrac{1}{4}(\partial_\mu Z_\nu-\partial_\nu Z_\mu)^2+\tfrac{1}{2}\frac{\lambda^2g^2}{4}\frac{g^2}{g^2-e^2}Z_\mu{}^2$$

$$-\tfrac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2+\tfrac{1}{2}(\partial_\mu\phi)^2-\tfrac{1}{2}(\mu^2+3h\lambda^2)\phi^2+\sum_l\left[\bar{l}\left(i\not\partial-\frac{G_l\lambda}{\sqrt{2}}\right)l+\nu_l i\not\partial\nu_l\right] \quad \text{(A3)}$$

and

$$\mathcal{L}_I=\sum_l\left\{\frac{g}{\sqrt{2}}(\bar{\nu}_l\gamma^\mu l_LW_\mu^++\bar{l}_L\gamma^\mu\nu_lW_\mu^-)+\frac{g}{2}\left(\frac{g^2}{g^2-e^2}\right)^{1/2}\bar{\nu}_l\gamma^\mu\nu_lZ_\mu\right.$$

$$\left.+\frac{g}{2}\left(\frac{g^2}{g^2-e^2}\right)^{1/2}\left[\left(\frac{2e^2}{g^2}-1\right)\bar{l}_L\gamma^\mu l_L+2\frac{e^2}{g^2}\bar{l}_R\gamma^\mu l_R\right]Z_\mu+e\bar{l}\gamma^\mu lA_\mu-\frac{G_l}{\sqrt{2}}\bar{l}l\phi\right\}$$

$$+\frac{g^2}{4}|W_\mu|^2\phi(2\lambda+\phi)+\frac{g^2}{8}\frac{g^2}{g^2-e^2}\phi(2\lambda+\phi)-h\lambda\phi^3-\tfrac{1}{4}h\phi^4-\frac{g}{2}(\partial_\mu\vec{A}_\nu-\partial_\nu\vec{A}_\mu)\cdot(\vec{A}_\mu\times\vec{A}_\nu)-\frac{g^2}{4}(\vec{A}_\mu\times\vec{A}_\nu)^2, \quad \text{(A4)}$$

where, from (A2),

$$A_\mu^3 = \frac{(g^2-e^2)^{1/2}}{g} Z_\mu - \frac{e}{g} A_\mu .$$

We now imagine putting the subscript 0 back on the above fields and parameters and rescaling according to Table I. The result is

$$\mathcal{L}_U = \tilde{\mathcal{L}}_0 + \tilde{\mathcal{L}}_I + \mathcal{L}_c , \tag{A5}$$

where, having expressed everything in terms of renormalized fields and parameters ($M_W = \lambda g_\mu/2$, $m_l = \lambda G_l/\sqrt{2}$)

$$\tilde{\mathcal{L}}_0 = -\tfrac{1}{2}|\partial_\mu W_\nu - \partial_\nu W_\mu|^2 + M_W^2|W_\mu|^2 - \tfrac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)^2 + \tfrac{1}{2}M_Z^2 Z_\mu^2 - \tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \sum_l [\bar{l}(i\not\partial - m_l)l + \bar{\nu}_l i\not\partial \nu_l] + \tfrac{1}{2}(\partial_\mu\phi)^2 - \tfrac{1}{2}m_\phi^2\phi^2$$

and $\mathcal{L}_I$ [Eq. (A4)]$\to\tilde{\mathcal{L}}_I$ by replacing $g_0 \to g_\mu$, $e_0 \to e$, $W_0 \to W$ etc. The mass terms in (A6) are the physical masses. As discussed in Sec. II, M_Z is to be calculated in terms of other renormalized parameters. [See Eq. (2.2).] The counterterms are in $\mathcal{L}_c$. We exhibit the part of $\mathcal{L}_c$ most important for calculations:

$$\begin{aligned}\mathcal{L}_c = &-\tfrac{1}{2}(Z_W - 1)|\partial_\mu W_\nu - \partial_\nu W_\mu|^2 + \left(\frac{Z_\lambda Z_g^2}{Z_{\nu_\mu} Z_{\mu L}} - 1\right)\frac{\lambda^2 g_\mu^2}{4}|W_\mu|^2 \\ &+ \sum_l \left[(Z_{lL}-1)\bar{l}_L(i\not\partial)l_L + (Z_{lR}-1)\bar{l}_R(i\not\partial)l_R - (Z_{Gl}-1)\frac{\lambda G_l}{\sqrt{2}}\bar{l}l + (Z_{\nu_l}-1)\bar{\nu}_l(i\not\partial)\nu_l\right] \\ &-\tfrac{1}{4}(Z_A - 1)(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \tfrac{1}{2}(Z_A Z_M)^{1/2}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu Z^\nu - \partial^\mu Z^\nu) - \tfrac{1}{4}(Z_Z + Z_M - 1)(\partial_\mu Z_\nu - \partial_\nu Z_\mu)^2 \\ &+ \tfrac{1}{2}\left\{\frac{\lambda^2 g_\mu^2}{4}\left(\frac{g_\mu^2}{g_\mu^2 - e^2}\right)\frac{Z_g^2 Z_\lambda Z_Z}{Z_W Z_{\nu_\mu} Z_{\mu L}}\left[1 - \frac{e^2}{g_\mu^2 - e^2}\left(\frac{Z_W Z_{\mu L} Z_{\nu_l}}{Z_g^2 Z_A} - 1\right)\right]^{-1} - M_Z^2\right\} Z_\mu^2 \\ &+ \frac{g_\mu}{\sqrt{2}}(Z_g - 1)(\bar{\nu}_\mu\gamma^\alpha\mu_L W_\alpha^+ + \bar{\mu}_L\gamma^\alpha\nu_\mu W_\alpha^-) + \frac{g_\mu}{\sqrt{2}}\left[Z_g\left(\frac{Z_{\nu_e}Z_{eL}}{Z_{\nu_\mu}Z_{\mu L}}\right)^{1/2} - 1\right][\bar{\nu}_e\gamma^\alpha e_L W_\alpha^+ + \bar{e}_L\gamma^\alpha\nu_e W_\alpha^-] \\ &+ e\sum_l [(Z_{lL}-1)\bar{l}_L\gamma_\alpha l_L + (Z_{lR}-1)\bar{l}_R\gamma_\mu l_R]A^\mu + \cdots .\end{aligned}$$

There are many more counterterms associated with $3W$ and $4W$ vertices, vertices involving the Higgs scalar, etc. Very few of these counterterms are independent and therefore many relations of the type verified in Appendix B must be satisfied.

APPENDIX B (Ref. 20)

Defining the Z's as in Table I and Appendix A we find

$$\frac{\delta M_W^2}{M_W^2} \equiv \frac{(M_W^2)_0 - M_W^2}{M_W^2} = \frac{Z_g^2 Z_\lambda}{Z_W Z_{\mu L} Z_{\nu_\mu}} - 1 = \frac{-\Pi_W(M_W^2)}{Z_W M_W^2}, \tag{B1}$$

$$(M_Z^2)_0 = \frac{M_W^2}{1 - e^2/g_\mu^2}\,\frac{Z_g^2 Z_\lambda}{Z_W Z_{\mu L} Z_{\nu_\mu}} \times\left[1 - \frac{e^2}{g_\mu^2 - e^2}\left(\frac{Z_W Z_{\mu L} Z_{\gamma\mu}}{Z_g^2 Z_A} - 1\right)\right]^{-1}. \tag{B2}$$

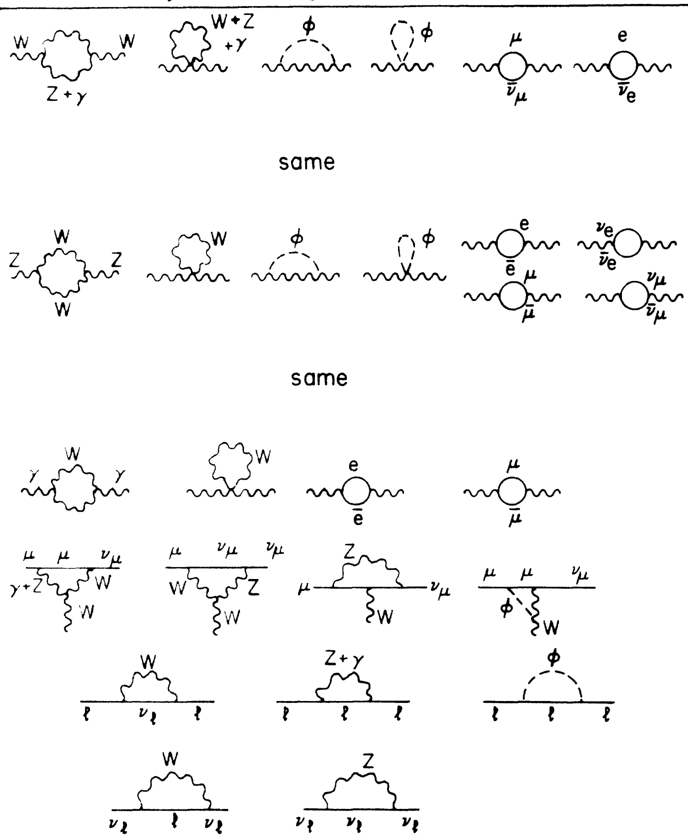
Thus if $M_Z^2 = M_W^2(1 - e^2/g_\mu^2)^{-1}$ + (ultraviolet-finite higher-order terms), then

$$\frac{\delta M_Z^2}{M_Z^2} = \frac{\delta M_W^2}{M_W^2} + \frac{e^2}{g_\mu^2 - e^2}\left[\frac{Z_W Z_{\mu L} Z_{\nu_\mu}}{Z_A Z_g^2} - 1\right] + \text{(ultraviolet-finite higher-order terms)}. \tag{B3}$$

The quantities involved in Eq. (B3) have been calculated using the 't Hooft-Veltman method[2] of continuing in the number of dimensions to obtain well-defined expressions. Each of these quantities can be written in terms of Γ functions with poles at $n = 2$ and $n = 4$ where n is the dimension. Letting X represent any one of these pieces, we have

$$X = 1 - \frac{g_\mu^2}{16\pi^2}\left[\frac{(-M_W^2\pi)^{n/2-2}}{4}\Gamma(1 - \tfrac{1}{2}n)A + \Gamma(2 - \tfrac{1}{2}n)B + \text{finite terms at } n = 4\right].$$

Figure 4 shows each of the X's along with the relevant Feynman diagrams and the corresponding co-



X	Feynman diagrams	A	B
$1+\frac{\delta M_W^2}{M_W^2}$		-3	$\frac{51R-9}{12R}+\frac{m_e^2+m_\mu^2}{2M_W^2}$
Z_W	same	$4R^2$	$R^2-R-\frac{8}{3}$
$1+\frac{\delta M_Z^2}{M_Z^2}$		-3	$\frac{52R^2+25R-35}{12R}+\frac{m_e^2+m_\mu^2}{2M_W^2}$
Z_Z	same	4	$\frac{7}{3R}-\frac{2}{3}-\frac{13R}{3}$
Z_A		0	$-(1-R)\frac{52}{12}$
Z_g		$4R^2-2R+3$	$R^2-\frac{3R}{2}+\frac{19}{6}+\frac{(1-2R)m_\mu^2}{8M_W^2}$
Z_{lL}		$4R^2-4R+3$	$R^2-2R+\frac{7}{4}+\frac{(4-2R)m_l^2}{4M_W^2}$
Z_{ν_l}		3	$\frac{3}{4}-\frac{3m_l^2}{4M_W^2}$

FIG. 4. Divergent parts of some subtraction constants and relevant Feynman diagrams. [Here, $R=(1-e^2/g^2)^{-1}$.]

efficients A and B. One can easily verify that Eq. (B3) is satisfied, thus showing that M_Z can be calculated in terms of M_W, e, and g_μ.

Figure 4 also contains all the information required to see that

$$g_e=g_\mu(1+\text{ultraviolet-finite piece of order }\alpha),$$

that is [see Eq. (A6)],

$$Z_{g_\mu}-\tfrac{1}{2}Z_{\mu L}-\tfrac{1}{2}Z_{\nu_\mu}$$
$$=(\text{ultraviolet-divergent terms independent of muon mass})+(\text{ultraviolet-finite terms of order }\alpha).$$

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†Junior Fellow, Society of Fellows.

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[4]T. W. Appelquist, J. R. Primack, and H. R. Quinn, Phys. Rev. D 6, 2998 (1972). Note the following corrections in this paper: The leading sign should be changed from − to + in Eqs. (20) and (26). The factors $(M_WM_Z)^{-2}$, M_Z^{-2}, and M_W^{-2} should appear in the left-hand side of Eqs. (25), (26), and (27), respectively.

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[7]The notation used in this paper differs slightly from

that used in I. The quantity g in I is identical to g_μ here, and the quantity g introduced in this paper [see Eq. (3.1)] is simply the infrared-finite part of g_μ. The notation g_e has exactly the same meaning here and in I.

[8]T. W. Appelquist and H. R. Quinn, Phys. Letters 39B, 229 (1972); T. Appelquist, J. Carrazone, T. Goldman, and H. Quinn, Harvard University report (unpublished).

[9]For the calculations discussed here, the quantities to be calculated can always be written in the form of spinor invariants multiplied by momentum-dependent quantities. By a regulator procedure, we mean a prescription for calculating these momentum-dependent factors; the spinor invariant is not included in the dimensional continuation. Anomalies do not arise in our calculation because we encounter no triangle diagrams in calculating order-α corrections to W decay and μ decay. We emphasize that the regulator techniques of Slavnov and Lee and Zinn-Justin (Ref. 3), which are sufficient to deal with the R formalism, are inadequate for the highly divergent U formalism.

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[11]One can introduce a photon mass while preserving the underlying gauge symmetry by adding a complex scalar field χ to the Lagrangian. If χ has vanishing SU(2) charge, and U(1) charge f, then the piece added to the Lagrangian has the form

$$\Delta\mathcal{L} = |(\partial_\mu - ifB_\mu)\chi|^2 - \mu_\chi{}^2\chi^\dagger\chi - h_\chi(\chi^\dagger\chi)^2 - a(\chi^\dagger\chi)(\varphi^\dagger\varphi).$$

Spontaneous development of a vacuum expectation value for χ will generate a term $\frac{1}{2}\tilde{\delta}^2 B_\mu B^\mu$, with $\tilde{\delta} = \sqrt{2}\, f\langle\chi\rangle_0$, and a few well-behaved corrections to the shifted theory. One can proceed back to the massless photon by taking $\langle\chi\rangle_0 \to 0$ or $f\to 0$. In the limit $f\to 0$ and $a\to 0$ the χ becomes a free particle.

[12]It is interesting to compare this result to that obtained for π decay assuming a universal pseudovector coupling of the pion to leptons by S. M. Berman [Phys. Rev. Letters 1, 468 (1958)] and T. Kinoshita [*ibid.* 2, 477 (1959)]. Writing their result as

$$\frac{\Gamma(\pi\to e\nu_e)}{\Gamma(\pi\to\mu\nu_\mu)} = R_0^\pi(1+\epsilon_\pi),$$

where $R_0^\pi = (m_e{}^2/m_\mu{}^2)[(m_\pi{}^2 - m_e{}^2)/(m_\pi{}^2 - m_\mu{}^2)]$, ϵ_π represents the corrections of order α. We note that Eq. (3.5) gives

$$\frac{\Gamma(W\to e\nu_e)}{\Gamma(W\to\mu\nu_\mu)} = R_0^W\left(1+\epsilon_W + \frac{2\alpha}{\pi}\ln\frac{m_\mu}{m_e}\right),$$

where $R_0^W = 1 + O(m_e/M_W)$ and $\epsilon_\pi \to \epsilon_W$ on replacing pion mass by W mass. The additional $\ln(m_\mu/m_e)$ term comes from a spin-dependent (nonconvective and noninfrared) part of the photonic vertex correction. The important new feature is that this ratio can now be calculated for a vector particle in the context of a spontaneously broken gauge theory. This was previously impossible since, unlike spin-zero electrodynamics, ordinary spin-one electrodynamics was plagued with nonrenormalizable divergences. See, for example, the ξ-limiting calculation of T. D. Lee [Phys. Rev. 128, 899 (1962)] where g_μ/g_e is infinite in the limit $\xi\to 0$. The formulas given here and in Eq. (3.5) are derived assuming photon mass $\ll$ lepton mass. The rate for W decay is divergent in the limit $m_l \to 0$, thus violating Kinoshita's theorem (see above reference).

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