can be freely chosen, only satisfying the condition that they be perpendicular to $\hat{v}$.

If S is a normal operator, so that $[S, S^\dagger]=0$, then S can be diagonalized by a unitary transformation

$$S = UDU^\dagger, \tag{B15}$$

where U is unitary and D is a diagonal matrix of the form $(\lambda, e^{id_2}, \ldots, e^{id_N})$, with λ the complex eigenvalue corresponding to the eigenvector $\hat{v}$. But since S is also symmetric, Eq. (B15) forces U to be a real orthogonal matrix, so that

$$S = ODO^T, \quad O \text{ orthogonal}, \tag{B16}$$

and thus, $A^* = a^*\hat{v}$, with $\hat{v}$ real. Equation (B16) is similar to the well-known decomposition of an arbitrary unitary symmetric matrix, except that D has entries of modulus one everywhere except for the first entry, where the magnitude of the eigenvalue λ is less than one.

We thus conclude that S has a unique real eigenvector $\hat{v}$ and a complex eigenvalue λ, which, using Eq. (B12b) is related to $\eta e^{2i\delta}$ by

$$\lambda a + \eta e^{-2i\delta} a^* = 0, \tag{B17}$$

so that $|\lambda| = \eta$ and $2\delta = \pi - 2\arg a - \arg\lambda$. Since neither δ nor $\arg a$ can be obtained independently, it is clear that one variable, which we choose to be the phase shift δ, must be given in order to determine $\arg a$.

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Multiparticle Partial-Wave Amplitudes and Inelastic Unitarity. III. Solution of the Three-Body Unitarity Equations Using Characteristic Operator Functions

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The most general solution to the unitarity equations involving $2\to2$, $2\to3$, and $3\to3$ processes is given when the total (including all disconnected processes) $3\to3$ partial-wave amplitude S is non-normal. The solution is given in terms of characteristic operator functions, using the theory of completely nonunitary operators. It is shown, once the characteristic operator function is given, how to compute the $2\to3$ partial-wave amplitude. An appendix shows that, if S can be exponentiated and all forces are two-body forces, no particle production is allowed, i.e., the $2\to3$ partial-wave amplitude is zero.

I. INTRODUCTION

In the preceding paper[1] (hereafter referred to as paper II) it was shown that the unitarity equations involving three reacting particles could be reformulated as eigenvalue-eigenvector equations, in that one could regard the $3\to3$ partial-wave amplitude S as an operator acting on a suitably defined Hilbert space, the $2\to3$ partial-wave amplitude being an element in this space of length less than or equal to one. This $2\to3$ partial-wave amplitude was then shown to be the unique eigenvector of $S^\dagger S$, with the eigenvalue being equal to the square of the inelasticity parameter of the corresponding $2 \to 2$ reaction.

These equations, derived in paper II and restated at the beginning of Sec. II for completeness, can be viewed in two ways. On the one hand, they can be viewed as providing a set of constraint equations on $3 \to 3$ and $2 \to 3$ partial-wave amplitudes, obtained from other theories in which it is necessary to make approximations in order to generate solutions and hence violate unitarity. For example, the partial-wave amplitudes generated from Feynman diagrams can be viewed as solutions to equations of quantum field theory which violate unitarity; one might hope to unitarize Feynman diagrams by inserting form factors of unspecified functional form and then forcing these diagrams to satisfy the eigenvalue-eigenvector equations.

On the other hand, one can view the unitarity equations as leading to a formulation of particle production without introducing Hamiltonians and other machinery of quantum field theory. In this view, similar to the viewpoint of S-matrix theory,[2] one would like to generate all the properties of partial-wave amplitudes from the unitarity equations supplemented by some additional hypotheses. The goal of this paper will be to show that it is possible to express three-body unitarity in a canonical form in which a so-called characteristic operator function carries almost all (up to the $2 \to 2$ phase shift) information about the interacting systems and automatically guarantees that three-body unitarity holds; in a succeeding paper, systems involving four- and higher-body unitarity will be discussed and the notion of crossing introduced as one of the possible additional hypotheses, used to constrain the possible characteristic operator functions. Section III will deal with the mathematics required to carry out this program and will involve the theory of completely nonunitary operators, while Sec. IV will discuss a simple (unphysical) example. Finally, Sec. V will show how one can calculate the $2 \to 3$ partial-wave amplitude from knowledge of the characteristic operator function.

It should be pointed out that a solution to the three-body unitarity equations was already given in paper II for the case when the total $3 \to 3$ partial-wave amplitude is a normal operator. But it will be shown in the Appendix that if one assumes that all forces involving the three interacting particles are two-body forces, and if S is normal, then there can be no particle production. Hence, while the case when S is normal generates solutions to the three-body unitarity equations, in the absence of three-body forces these solutions exclude the very phenomenon under investigation, namely particle production. For potential scattering, where the equations of motion are known and the existence of solutions for these equations guaranteed,[3] scattering involving only two-body forces means S must be non-normal.

II. FORMULATION OF THE THREE-BODY UNITARITY EQUATIONS AS EIGENVECTOR-EIGENVALUE EQUATIONS

We shall continue using the notation introduced in paper II, so that we are considering the following amplitudes that are to be connected by unitarity:

$2 \to 2$:

$$\langle 1''2''|\mathcal{T}|1'2'\rangle \to \text{generates a partial-wave amplitude } A_J(s)\,, \tag{2.1a}$$

$2 \to 3$:

$$\langle 1''2''3''|\mathcal{T}|1'2'\rangle \to \text{generates a partial-wave amplitude } A_{JM''}(s, s_q'')\,, \tag{2.1b}$$

$3 \to 2$:

$$\langle 1''2''|\mathcal{T}|1'2'3'\rangle \to \text{generates a partial-wave amplitude } A_{JM'}(s, s_q')\,, \tag{2.1c}$$

$3 \to 3$:

$$\langle 1''2''3''|\mathcal{T}|1'2'3'\rangle \to \text{generates a partial-wave amplitude } A_{JM''M'}(s, s_q'', s_q')\,. \tag{2.1d}$$

As before, we are labeling initial particles with primes, final particles with double primes, while the intermediate particles occurring in the unitarity equations are unprimed. $\sqrt{s}$ is the total invariant energy while J is the angular momentum. For simplicity we have assumed that all particles are spinless; the more complicated case involving spin will be dealt with in a succeeding paper. The labels M and s_q correspond to angular momentum projection and subenergies, respectively. For the case of three particles, two subenergies are needed, so that q runs over 1 and 2. At this point we will leave the specific choice for the subenergies unspecified.

It is to be noted that the partial-wave amplitude for the $3 \to 2$ process is separate from the $2 \to 3$ process. In paper II, time-reversal invariance was used to relate these two processes. However, in this paper we will drop the assumption of time-reversal invariance as it causes the unitarity equations to look unsymmetrical and complicates their solutions.

The unitarity equations relate the four partial-wave amplitudes given in (2.1) in a way that is made most clear by treating $A_{JM''M'}(s, s_q'', s_q')$ as an operator acting on functions that are elements of the Hilbert space

$$\mathcal{H}: \; \| f_J(s)\|^2 = \sum_{M=-J}^{+J} \int d^2 s_q |f_{JM}(s, s_q)|^2 < \infty \,. \quad (2.2)$$

The measure over the subenergies depends on the choice made for the subenergies; but whatever the choice the integration is over the Dalitz region. The partial-wave amplitudes, Eqs. (1.1b) and (1.1c), are elements of $\mathcal{H}$; elements, as will be shown, of length less than or equal to unity. It is useful to rewrite the $2\to 2$ partial-wave amplitude as

$$\eta_J(s) e^{2i\delta_J(s)} = 1 + iA_J(s) \,, \quad (2.3)$$

where η is the inelasticity parameter and δ the phase shift. Further, since the unitarity equations are diagonal in s and J, the dependence of the various partial-wave amplitudes on these two quantities will be suppressed. Also, rather than dealing with $A_{JM''M'}(s, s_q'', s_q')$, the connected $3\to 3$ partial-wave amplitude, it is useful to introduce the total $3\to 3$ partial-wave amplitude,

$$\begin{aligned} S_{JM''M'}(s, s_q'', s_q') = {} & J(s_q')\delta_{M''M'}\delta^2(s_q''-s_q') \\ & + i\sum_{j=1}^{3} {}^jA_{JM''M'}(s, s_q'', s_q') \\ & + iA_{JM''M'}(s, s_q'', s_q') \,, \end{aligned} \quad (2.4)$$

which is the sum of all disconnected and the connected $3\to 3$ partial-wave amplitudes. The one-line and totally disconnected $3\to 3$ partial-wave amplitudes are discussed in Appendix A of paper II, where, in particular, it is shown that the one-line disconnected partial-wave amplitudes ${}^jA_{JM''M'}(s, s_q'', s_q')$ (j stands for that particle not interacting with the other two particles) satisfy elastic unitarity. $J(s_q')$ is a Jacobian factor depending on the choice made of the subenergies s_q'.

Finally, regarding all quantities as depending on the basis $\{M, s_q\}$, it is possible to write the unitarity equations as

$$\mathcal{S} = \begin{pmatrix} \eta e^{2i\delta} & A^\dagger \\ B & S \end{pmatrix} \,,$$

$$\mathcal{S}^\dagger = \begin{pmatrix} \eta e^{-2i\delta} & B^\dagger \\ A & S^\dagger \end{pmatrix} ,$$

$\mathcal{S}^\dagger\mathcal{S} = \mathcal{S}\mathcal{S}^\dagger = \mathcal{I}$:

$$1 = \eta^2 + \|A\|^2 \,, \quad (2.5a)$$

$$1 = \eta^2 + \|B\|^2 \,, \quad (2.5b)$$

$$0 = SA + \eta e^{-2i\delta}B \,, \quad (2.5c)$$

$$0 = S^\dagger B + \eta e^{2i\delta}A \,, \quad (2.5d)$$

$$I = SS^\dagger + BB^\dagger \,, \quad (2.5e)$$

$$I = S^\dagger S + AA^\dagger \,, \quad (2.5f)$$

where A is the $2\to 3$ partial-wave amplitude and B the $3\to 2$ partial-wave amplitude.

The matrices in Eq. (2.5) are written in analogy with the actual infinite dimensional "matrices," as discussed in Appendix B of paper II. From Eq. (2.5a) and (2.5b) it is clear that

$$\|A\| = \|B\| \le 1 \,, \quad (2.6)$$

so that all $2\to 3$ partial-wave amplitudes are less than or equal to unity. Further, in the simple case being discussed in this paper, in which there is only one $2\to 2$ channel, $BB^\dagger$ and $AA^\dagger$ are rank-one operators and, as shown in paper II, by writing Eqs. (2.5e) and (2.5f) as

$$I - S^\dagger S = AA^\dagger \,, \quad (2.7a)$$

$$I - SS^\dagger = BB^\dagger \,, \quad (2.7b)$$

it is seen that A is the one unique eigenvector associated with the Hermitian operator $S^\dagger S$ while B is the one unique eigenvector associated with the Hermitian operator $SS^\dagger$. Further, the real eigenvalues x_A and x_B, associated with $S^\dagger S$ and $SS^\dagger$, respectively, can be obtained by multiplying Eqs. (2.7a) and (2.7b) by the appropriate unit-length eigenvectors:

$$(1 - x_A)\hat{A} = A(A, \hat{A}) \,, \quad (2.8a)$$

$$(1 - x_B)\hat{B} = B(B, \hat{B}) \,. \quad (2.8b)$$

Operating on Eqs. (2.8a) and (2.8b) from the left with A and B, respectively, then gives

$$1 - x_A = \|A\|^2 \,, \quad (2.9a)$$

$$1 - x_B = \|B\|^2 \,. \quad (2.9b)$$

But from Eqs. (2.5a) and (2.5b) we see that

$$x_A = x_B = \eta^2 \,. \quad (2.10)$$

Hence the unique eigenvalues of $S^\dagger S$ and $SS^\dagger$ are both η^2, the $2\to 2$ inelasticity parameter. As shown in paper II, however, the unitarity equations as expressed by Eq. (2.5) will not enable one to compute the $2\to 2$ phase shift δ.

It can also be seen that the two remaining unitarity equations, Eqs. (2.5c) and (2.5d), are compatible with Eqs. (2.5e) and (2.5f) by writing

$$\begin{aligned} SA &= -\eta e^{-2i\delta}B \,, \\ S^\dagger SA &= -\eta e^{-2i\delta}S^\dagger B \\ &= -\eta e^{-2i\delta}(-\eta e^{2i\delta})A \,, \\ S^\dagger SA &= \eta^2 A \,, \end{aligned} \quad (2.11)$$

in agreement with Eq. (2.8a) and (2.8b).

If Eq. (2.5f) is subtracted from (2.5e), there results

$$[S, S^\dagger] = BB^\dagger - AA^\dagger, \tag{2.12}$$

so that S is normal if and only if B equals A up to an over-all phase. This situation was discussed in paper II where use was also made of time-reversal invariance, so that $B = A^*$. The goal of this paper is to discuss the case when S is not normal.

III. THE CHARACTERISTIC OPERATOR FUNCTION

Since $\eta \leq 1$ and η^2 is the unique eigenvalue of $S^\dagger S$ and $SS^\dagger$, all other eigenvalues being unity, it is clear that S is a contraction, i.e., $\|Sf\| \leq \|f\|$ for all $f \in \mathcal{H}$. In such a situation it is possible to make use of the structure of non-normal operators given by Sz.-Nagy and Foias in Ref. 4, and in particular, to make use of their Theorem 3.2 (p. 9), which says that every contraction S can be uniquely decomposed into the direct sum of a unitary and completely nonunitary operator; more precisely, there exists a direct sum decomposition of $\mathcal{H}$ into $\mathcal{H}_U$ and $\mathcal{H}_{CNU}$, so that S acting on $\mathcal{H}_U$ is unitary and on $\mathcal{H}_{CNU}$ completely nonunitary. (A contraction S is completely nonunitary if there is no invariant subspace of $\mathcal{H}$ with the property that S restricted to it is unitary – Ref. 4, p. 8.) But it is clear that only the completely nonunitary part of S is of interest here, for only in the subspace $\mathcal{H}_{CNU}$ can there be nonzero $2 \to 3$ partial-wave amplitudes. If S were unitary, by Eqs. (2.5e) and (2.5f), $\|A\| = \|B\| = 0$ and $\eta = 1$. We will thus assume that S is completely nonunitary and $\mathcal{H} = \mathcal{H}_{CNU}$.

Now, since S is a contraction, it is possible to define defect operators (Ref. 4, p. 6),

$$D_S = (I - S^\dagger S)^{1/2}, \quad D_{S^\dagger} = (I - SS^\dagger)^{1/2}, \tag{3.1}$$

and see that, by virtue of Eqs. (2.5e) and (2.5f), D_S and $D_{S^\dagger}$ are both of rank one:

$$D_S{}^2 = AA^\dagger, \tag{3.2a}$$

$$D_{S^\dagger}{}^2 = BB^\dagger. \tag{3.2b}$$

In a succeeding paper it will be shown that the dimension of the defect spaces, $\dim \mathfrak{D}_S \equiv \dim D_S \mathcal{H}$ and $\dim \mathfrak{D}_{S^\dagger} \equiv \dim D_{S^\dagger} \mathcal{H}$, is determined by the number of $2 \to 2$ channels that are open at the energy $\sqrt{s}$ and by the intrinsic spin of the reacting particles. Since we are considering spinless particles and one $2 \to 2$ channel only, $\dim \mathfrak{D}_S = \dim \mathfrak{D}_{S^\dagger} = 1$.

What we now wish to show is that on an appropriately chosen subspace of a Hardy space, the operator S can always be realized as multiplication by a complex number z, i.e., it is a unilateral shift operator. To understand the meaning of these terms (discussed in Chaps. III and V of Ref. 4), consider the Hilbert space L^2 of square integrable functions on the circle

$$L^2: \quad \|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\varphi |f(\varphi)|^2 < \infty, \quad f = \sum_{n=-\infty}^{+\infty} a_n e^{in\varphi}. \tag{3.3}$$

Here a_n are the Fourier coefficients of f and satisfy

$$\|f\|^2 = \sum_n |a_n|^2 < \infty. \tag{3.4}$$

Now consider the (Hardy) space of functions $f \in L^2$ with the property that the corresponding Fourier coefficients vanish for negative index: $\{f \in L^2 | a_n = 0, \ n < 0\}$. Then it is possible to write

$$f = \sum_{n=0}^{\infty} a_n e^{in\varphi} \tag{3.5}$$

and think of such functions as being boundary values of functions analytic on the unit disk

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| \leq 1. \tag{3.6}$$

Such functions form the Hardy space H^2; and on this space it is possible to define the unilateral shift operator ("raising" operator) U_+ as multiplication by z:

$$(U_+ f)(z) = z f(z), \quad f \in H^2, \quad (U_+^\dagger f)(z) = z^* [f(z) - f(0)]. \tag{3.7}$$

Once the unitary map carrying $\mathcal{H}$ to a subspace $\mathfrak{M}^\perp$ of H^2 is given, it can be shown that the image of S in $\mathfrak{M}^\perp$ is just the unilateral shift operator U_+.

To see this, define a subspace $\mathfrak{M}$ of H^2 that is left invariant by U_+. Then it can be shown (Ref. 4, p. 198ff) that there exists a unique inner function $\Theta_S(z) \in H^2$ [inner means $|\Theta_S(z)| = 1$, $|z| = 1$ almost everywhere, and $\Theta_S \in H^\infty$ (Ref. 4, p. 101)], with the property that it generates $\mathfrak{M}$:

$$\mathfrak{M} = \Theta_S(z) H^2. \tag{3.8}$$

On the orthogonal complement

$$\mathfrak{M}^\perp = H^2 - \Theta_S(z) H^2, \tag{3.9}$$

and S is realized by U_+.

Before discussing the canonical form of $\Theta_S(z)$, we wish to show how it is connected with the operator S. Define the *operator* $\Theta_S(z)$ acting in $\mathcal{H}$ as (Ref. 4, p. 237ff)

$$\Theta_S(z) = -S + z D_{S^\dagger} (I - zS^\dagger)^{-1} D_S. \tag{3.10}$$

It carries $\mathfrak{D}_S$, the space spanned by A, into $\mathfrak{D}_{S^\dagger}$, the space spanned by B. This operator takes any element in $\mathfrak{D}_S$ and sends it into an element in $\mathfrak{D}_{S^\dagger}$, as can be seen by the following sequence of operations:

(1) D_S sends $\mathfrak{D}_S$ into $\mathfrak{D}_S$.

(2) $(I - zS^\dagger)^{-1}\mathfrak{D}_S$ is somewhere in $\mathcal{K}$, depending on S.

(3) $D_{S^\dagger}$ projects $(I - zS^\dagger)^{-1}\mathfrak{D}_S$ into the subspace $\mathfrak{D}_{S^\dagger}$.

Since $SA \propto B$, we see that indeed $\Theta_S(z)$ is an operator-valued function of z. Reference 4, p. 7 shows that

$$SD_S = D_{S^\dagger} S\,, \qquad D_S S^\dagger = S^\dagger D_{S^\dagger} \tag{3.11}$$

holds for all contractions S; from Eq. (2.11) it follows that

$$S(I - S^\dagger S) = (I - SS^\dagger)S\,, \qquad SAA^\dagger = BB^\dagger S \tag{3.12}$$

which, when multiplied to the right by A, gives

$$\| A \|^2 SA = B(B, SA) \tag{3.13}$$

or

$$SB \propto A\,. \tag{3.14}$$

Thus, in a sense, the unitarity equations (2.5c) and (2.5d) follow from (2.5e) and (2.5f). The main information contained in Eqs. (2.5c) and (2.5d) is that the proportionality factor in Eq. (3.14) is $-\eta e^{-2i\delta}$.

Now the operator $\Theta_S(z)$ defined by Eq. (3.10) can be regarded as a function in H^2 and in fact is the function defining the subspace $\mathfrak{M}$ (and hence $\mathfrak{M}^\perp$). $\Theta_S(z)$ is called the characteristic operator function, and knowledge of it completely specifies the operator S and the eigenvectors A and B.

That is, once $\Theta_S(z)$ is known, and the map from $\mathcal{K}$ to $\mathfrak{M}^\perp$ given, the unitarity Eqs. (2.5) are automatically satisfied, and in $\mathfrak{M}^\perp$, S is realized by U_+. Further, as can be seen from Eq. (3.10), $\Theta_S(0) = -S$, so that

$$|\Theta_S(0)| = \eta\,. \tag{3.15}$$

At this point one might well ask about the usefulness of $\Theta_S(z)$, for if S, A, and B are known, there is not much sense in computing $\Theta_S(z)$. But if dynamical equations for S, A, and B are not known, then $\Theta_S(z)$ can be a starting point for computing S, A, and B, for, as shown in Ref. 4, $\Theta_S(z)$ has a canonical form,

$$\Theta_S(z) = B(z)S(z)\,, \tag{3.16}$$

where $B(z)$ is a Blaschke product, of the form

$$B(z) = \prod_k \frac{a_k^*}{a_k}\,\frac{a_k - z}{1 - a_k^* z}\,, \tag{3.17}$$

each factor being the conformal map of the disk onto itself. The set $\{a_k\}$ generating the zeros of the Blaschke product are the discrete eigenvalues of S and satisfy $\sum_k (1 - |a_k|) < \infty$. The continuous eigenvalues are given by the singular function

$$S(z) = \exp\left(-\int_0^{2\pi} \frac{e^{it} + z}{e^{-it} - z}\, d\mu t\right), \tag{3.18}$$

where μ is a finite non-negative measure, singular with respect to Lebesgue measure (Ref. 4, p. 101).

In succeeding papers we will show several ways in which such a canonical decomposition can be used. First it will be shown that the zeros of the Blaschke product can be used to generate a complete biorthogonal set of functions which are superpositions of Breit-Wigner resonance functions.[5] Second, one can attempt to regard the breakup of $\Theta_S(z)$ into two factors as corresponding to the breakup of S into a part generating resonances and a part generating a background. And finally, by defining the notion of crossing for multiparticle processes, an attempt will be made to formulate equations for $\Theta_S(z)$, the solutions of which will predict the form of A and B.

The problem of finding the unitary map between $\mathfrak{M}^\perp$ and $\mathcal{K}$ can be obtained once $\Theta_S(z)$ is known. This subject is dealt with in the paper by Ahern and Clark.[6] What remains to be done in this paper is to compute the eigenvectors A and B as functions in $\mathfrak{M}^\perp$, for A and B correspond to quantities that can now be checked experimentally, in contrast to S, which involves three-particle scattering experiments. Before dealing with this subject, however, we give a simple (nonphysical) example.

IV. A SIMPLE EXAMPLE

When S is a finite-dimensional matrix, $\Theta_S(z)$ consists of a finite Blaschke product only. Consider as an example S, an $N\times N$ matrix, of the form

$$S = \begin{bmatrix} 0 & \\ \vdots & U \\ 0 & \\ -\eta e^{-2i\delta} & 0\cdots 0 \end{bmatrix}, \tag{4.1}$$

where U is an $(N-1)\times(N-1)$ unitary matrix. Then the square of the defect operator is

$$I - S^\dagger S = I_N - \begin{bmatrix} \eta^2 & 0 \cdots 0 \\ 0 & \\ \vdots & I_{N-1} \\ 0 & \end{bmatrix}$$

$$= \begin{bmatrix} 1-\eta^2 & 0 \cdots 0 \\ 0 & \\ \vdots & 0 \\ 0 & \end{bmatrix}$$

$$= AA^\dagger, \tag{4.2}$$

so that A is proportional to $\hat{e}_1$, of length $(1-\eta^2)^{1/2}$. Similarly,

$$I - SS^\dagger = I_N - \begin{bmatrix} I_{N-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \cdots 0 & 1-\eta^2 \end{bmatrix}$$

$$= BB^\dagger, \tag{4.3}$$

so that B is proportional to $\hat{e}_N$, the unit vector in direction N. Also, it is clear that S is non-normal and cannot be brought to diagonal form by a unitary transformation. To have A and B pointing in arbitrary directions, it is merely necessary to consider a "rotated" S of the form $U_B^\dagger S U_A$, where U_A is unitary, transforming $\hat{A} = \hat{e}_1$ into a new vector $U_A A$, and U_B is unitary, transforming B into $U_B B$.

In order to find the characteristic operator function $\Theta_S(z)$, it is necessary to compute $(I - zS^\dagger)^{-1}$ according to Eq. (3.10). Since this is in general difficult to compute, we will consider only the simple case when U is diagonal, consisting of $N-1$ phases, $\delta_1, \delta_2, \ldots, \delta_{N-1}$, with $\delta_N = \pi - 2\delta$. The eigenvalues of S are of the form

$$\begin{aligned} \lambda^N &= \eta \exp\left(i \sum_{j=1}^N \delta_j\right) \\ &= \eta e^{i\Delta}, \\ \Delta &= \sum_{j=1}^N \delta_j \\ \lambda_k &= \eta^{1/N} e^{(i/N)\Delta} e^{(2\pi i k)/N}, \quad k = 1, 2, \ldots, N \end{aligned} \tag{4.4}$$

and we wish to connect these eigenvalues with the Blaschke zeros of Eq. (3.17). Now, from Eq. (3.10), we get that

$$\begin{aligned} \Theta_S(z)A &= -SA + zD_{S^\dagger}(I - zS^\dagger)^{-1} D_S A \\ &= \eta e^{-2i\delta} B + z D_{S^\dagger}(I - zS^\dagger)^{-1} A \, \|A\| \\ &= \eta e^{-2i\delta} B + z(\hat{B}\hat{B}^\dagger)^{1/2} (I - zS^\dagger)\hat{A} \, \|A\|^3 \\ &= \eta e^{-2i\delta} B + \|A\|^2 z B(\hat{B}, (I - zS^\dagger)^{-1}\hat{A}) \\ &= [\eta e^{2i\delta} + z(1-\eta^2)(\hat{B}, (I - zS^\dagger)^{-1}\hat{A})]B, \end{aligned} \tag{4.5}$$

so that, as a function in H^2,

$$\Theta_S(z) = \eta e^{2i\delta} + z(1-\eta^2)(\hat{e}_N, (I - zS^\dagger)^{-1}\hat{e}_1). \tag{4.6}$$

Now

$$(I - zS^\dagger)^{-1} = \sum_{n=0}^\infty z^n (S^\dagger)^n,$$

so that we must compute $(S^\dagger)^n \hat{e}_1$; with the simple form chosen for U, this gives

$$(\hat{e}_N, (I - zS^\dagger)^{-1}\hat{e}_1) = \frac{z^{N-1}}{1 - z^N \eta e^{-i\Delta}} \times \exp\left(-i \sum_{j=1}^{N-1} \delta_j\right), \tag{4.7}$$

$$\Theta_S(z) = \frac{\eta e^{-2i\delta} + z^N e^{-i\Delta} e^{i\delta_N}}{1 - z^N \eta e^{-i\Delta}}.$$

Comparing this with the Blaschke product, Eq. (3.17), shows that

$$\prod_{k=1}^N (1 - a_k^* z) = 1 - z^N \eta e^{-i\Delta}, \tag{4.8}$$

so that indeed $a_N = \lambda_N = \eta^{1/N} e^{(+i\Delta)/N}$, with all the other zeros differing by the various roots of unity.

Had we started with a characteristic operator function of the form given in Eq. (4.7), one would be able to compute $\eta = |\Theta_S(0)|$ and get A and B, in a manner to be discussed in Sec. V. Notice that δ, the phase shift, is lost in the over-all phase Δ, as are all the other phases $\delta_1, \delta_2, \ldots, \delta_{N-1}$. This reflects the fact that the over-all phases multiplying the unit vectors $\hat{e}_1$ and $\hat{e}_N$ are not uniquely specified, and hence cannot be computed from $\Theta_S(z)$.

V. COMPUTATION OF A FROM $\Theta_S(z)$

We now wish to show how one can compute A, given $\Theta_S(z)$. It would, of course, be possible to realize S as U_+ on $\mathfrak{M}^\perp$, make use of the unitary map from $\mathfrak{M}^\perp$ to $\mathcal{H}$, and then solve the eigenvector problem for $S^\dagger S$ to obtain A; but it is clear that it will be much easier to carry out all the calculations in $\mathfrak{M}^\perp$, and then transform to $\mathcal{H}$.

Now, as shown in Ref. 6, the unitary map V between $\mathfrak{M}^\perp$ and H is known once $\Theta_S(z)$ is given. Since V is unitary, the unitarity equations in $\mathfrak{M}^\perp$

will be of the same form as in $\mathcal{H}$, given by Eq. (2.5). If all quantities in $\mathfrak{M}^\perp$ are denoted by tildes, Eq. (2.5f) in $\mathfrak{M}^\perp$ becomes

$$I_{\mathfrak{M}^\perp} - \tilde{S}^\dagger \tilde{S} = \tilde{A}\tilde{A}^\dagger , \tag{5.1}$$

when $I_{\mathfrak{M}^\perp}$ is the identity operator in $\mathfrak{M}^\perp$. Once $\tilde{A}$ is found from Eq. (5.1), A can be computed as $V\tilde{A}$, for its length $(1-\eta^2)^{1/2}$ is the same in both spaces.

But $\tilde{S}$ is realized as the unilateral shift in $\mathfrak{M}^\perp$; if an element of H^2 is not in $\mathfrak{M}^\perp$, it must be projected into $\mathfrak{M}^\perp$ so that

$$\begin{aligned}\tilde{S} &= P_{\mathfrak{M}^\perp} U_+ P_{\mathfrak{M}^\perp} \\ &= P_{\mathfrak{M}^\perp} z P_{\mathfrak{M}^\perp} ,\end{aligned} \tag{5.2}$$

where $P_{\mathfrak{M}^\perp}$ is the projection operator from H^2 to $\mathfrak{M}^\perp$.

Since $\{z^k\}$ forms a complete orthogonal set in H^2 and $\mathfrak{M}^\perp$ is contained in H^2, Eq. (5.1) can be written

$$\begin{aligned}&(I_{\mathfrak{M}^\perp} - \tilde{S}^\dagger\tilde{S}) P_{\mathfrak{M}^\perp} z^k = \tilde{A}(\tilde{A}, P_{\mathfrak{M}^\perp} z^k) , \\ &(P_{\mathfrak{M}^\perp} - P_{\mathfrak{M}^\perp} U_+^\dagger P_{\mathfrak{M}^\perp} U_+ P_{\mathfrak{M}^\perp}) z^k = \tilde{A}(\tilde{A}, z^k) ,\end{aligned} \tag{5.3}$$

or

$$\begin{aligned}\tilde{A} &\propto (P_{\mathfrak{M}^\perp} - P_{\mathfrak{M}^\perp} U_+^\dagger P_{\mathfrak{M}^\perp} U_+ P_{\mathfrak{M}^\perp}) z^k \\ &\propto [P_{\mathfrak{M}^\perp} - P_{\mathfrak{M}^\perp} U_+^\dagger (P_{H^2} - P_{\mathfrak{M}}) U_+ P_{\mathfrak{M}^\perp}] z^k \\ &\propto (P_{\mathfrak{M}^\perp} - P_{\mathfrak{M}^\perp} U_+^\dagger U_+ P_{\mathfrak{M}^\perp} + P_{\mathfrak{M}^\perp} U_+^\dagger P_{\mathfrak{M}} U_+ P_{\mathfrak{M}^\perp}) z^k \\ &\propto (P_{\mathfrak{M}^\perp} U_+^\dagger P_{\mathfrak{M}} U_+ P_{\mathfrak{M}^\perp}) z^k \\ &\propto (P_{\mathfrak{M}^\perp} z^* P_{\mathfrak{M}} z P_{\mathfrak{M}^\perp}) z^k ,\end{aligned} \tag{5.4}$$

where k is chosen so that $(\tilde{A}, z^k) \neq 0$. It is thus seen that the form of $P_{\mathfrak{M}}$ (or $P_{\mathfrak{M}^\perp}$) is needed. Now $P_{\mathfrak{M}}$ can be written as

$$P_{\mathfrak{M}} = \Theta_S P_{H^2} \Theta_S^* , \tag{5.5}$$

where Θ_S^* is the adjoint of the characteristic operator function and P_{H^2} is the projection operator into H^2. It is needed because in general Θ_S^* takes an element $f \in H^2$ out of H^2 (i.e., into L^2). That $P_{\mathfrak{M}}$ is a projection can be seen by setting $f = \Theta_S g$, $f \in \mathfrak{M}$, $g \in H^2$, and writing

$$\begin{aligned}P_{\mathfrak{M}} f &= \Theta_S P_{H^2} \Theta_S^\dagger f \\ &= \Theta_S P_{H^2} \Theta_S^\dagger \Theta_S g \\ &= \Theta_S P_{H^2} g \\ &= \Theta_S g \\ &= f .\end{aligned} \tag{5.6}$$

Since $\Theta_S^\dagger$ takes elements of H^2 out of H^2, it is useful to have an expression for P_{H^2} that does not depend on a power-series expansion. This is obtained by a Hilbert transform[7]

$$\begin{aligned}(P_{H^2} f)(\theta) &= P\int_{-\pi}^{+\pi} d\theta' \cot(\tfrac{1}{2}(\theta-\theta')) f(\theta) \\ &= g(\theta) , \quad f \in L^2, \quad g \in H^2 .\end{aligned} \tag{5.7}$$

For a given characteristic operator function $\Theta_S(z)$, $P_{\mathfrak{M}}$ (and thus $P_{\mathfrak{M}^\perp}$) can be computed from Eqs. (5.5) and (5.7). Written out explicitly, $\tilde{A}$ becomes

$$\begin{aligned}\tilde{A} &\propto [(P_{H^2} - P_{\mathfrak{M}}) z^* P_{\mathfrak{M}} z (P_{H^2} - P_{\mathfrak{M}})] z^k \\ &\propto [(P_{H^2} - \Theta_S P_{H^2} \Theta_S^*) z^* \Theta_S P_{H^2} \Theta_S^* z \\ &\quad \times (P_{H^2} - \Theta_S P_{H^2} \Theta_S^*)] z^k ,\end{aligned} \tag{5.8}$$

and this gives the "direction" of $\tilde{A}$, while its magnitude is $(1-\eta^2)^{1/2}$.

VI. CONCLUSION

It has been shown that when S, the total $3 \to 3$ partial-wave amplitude, is non-normal it is completely characterized by its characteristic operator function; and knowledge of the characteristic operator function also fixes η, the inelasticity parameter and the $2 \to 3$ partial-wave amplitude. Thus the characteristic operator function may be seen as expressing the content of the unitarity equations.

The question then arises as to how one might calculate characteristic operator functions. In a well-defined theory such as nonrelativistic potential scattering it should be possible to compute the characteristic operator function using the Schrödinger equation. This would hopefully allow one to get a feel for the kinds of characteristic operator functions of physical interest. Work along these lines is in progress, but it should be clear that knowledge of the characteristic operator function involves less information than that contained in the Schrödinger equation, for it seems as though the $2 \to 2$ phase shift can never be obtained from the characteristic operator function. This result seems quite strange, for one usually thinks of the phase shift as locked in with the other relevant parameters, such as inelasticity parameters and production amplitudes.

On the other hand, if the characteristic operator function is seen as a starting point and not derived from more basic equations, then it is necessary to formulate equations for the characteristic operator functions. A natural choice here involves crossing, since one already has a great deal of analytic control on the functions, and this possibility will be discussed in a succeeding paper. But here too a difficulty arises, along with that associated with the inability to calculate phase shifts: Since the unitarity equations have their simplest form in terms of partial-wave amplitudes, they are necessarily diagonal in energy and angular momentum. To generate amplitudes, more is necessary than just equations generating characteristic operator functions. It is also necessary to provide relations between different energies and angular momenta.

Finally, it should be pointed out that if S cannot be exponentiated, its spectrum must cover the boundary of the disk. Now in potential scattering it certainly is possible to exclude three-body forces. It will be interesting to see if indeed S cannot be exponentiated and its spectrum covers the boundary of the disk.

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APPENDIX: PROOF THAT IF S IS NORMAL AND THERE EXIST TWO-BODY FORCES ONLY, THEN THE PRODUCTION AMPLITUDES ARE ZERO

In paper II use was made of the fact that if S is normal, it can be brought to diagonal form, and in that diagonal form one of the eigenvalues is complex, the magnitude of that eigenvalue being the inelasticity parameter of the $2 \to 2$ reaction. What we wish to show is that if the forces generating S are two-body forces only, then S must necessarily be unitary (i.e., the defect operator is of rank zero) meaning that A is zero so that there is no particle production. Actually, what will be proved is somewhat weaker in that we will assume that S can be exponentiated, so that

$$S = e^{iH}, \tag{A1}$$

where H is an operator to be determined. If S is normal, it can be exponentiated only if its inverse exists, which means that S can have no zero eigenvalues [since the spectrum of S is defined as the complement of those values of λ for which $(S - \lambda I)^{-1}$ exists]. But the only possible zero eigenvalue that is allowed by the unitarity equations occurs for $\eta = 0$, in which case the energy and angular momentum dependence of $A_J(s)$, the $2 \to 2$ partial-wave amplitude, is trivial. Only when $\eta = 0$ can the (normal) operator S not be exponentiated.

Now for $\eta \neq 0$ it is possible to write out the exponential in (A1) so that

$$S = I + iH + \tfrac{1}{2}(iH)^2 + \cdots, \tag{A2}$$

and identify the first term (I) with the totally disconnected partial-wave amplitude discussed in Appendix A, Eq. (A1) of paper II.

The crucial part of the argument centers about the form of H occurring as the next term in the expansion. If there are two-body forces only, then H must be of the form of one-line disconnected partial-wave amplitudes, where, however, the phase shifts corresponding to the $2 \to 2$ scattering below three-body threshold are replaced by unknown functions $f_j(s_k)$, corresponding to "primitive" forces, which, when suitably summed, will give the actual phase shifts. The index j will still correspond to the angular momentum of system k (where $k = 1$ means 2 and 3 are interacting, etc.) and s_k is the appropriate subenergy.

In particular, if H were to consist of only one such one-line disconnected term, it would be of the form

$$H_{JM''M'}(s, s_q'', s_q') = 4\pi \sum_j Y_{jM''}(\theta'', 0) Y_{jM'}(\theta', 0) \times f_j(s_k')\, \delta(s_k'' - s_k')\, \delta_{M''M'}, \tag{A3}$$

as discussed in Appendix A of paper II, Eq. (A2ff).

For H of the form given in Eq. (A3), S of (A2) then consists of disconnected terms only, and is of the form

$$S_{JM''M'}(s, s_q'', s_q') = I_{JM''M'}(s, s_q'', s_q') + i\, {}^kA_{JM''M'}(s, s_q'', s_q'), \tag{A4}$$

where ${}^kA_{JM''M'}(s, s_q'' \cdot s_q')$ is given by Eq. (A3) except that the unknown functions $f_j(s_k)$ are replaced by the physical $2 \to 2$ phase shifts. Since these phase shifts occur in a one-line disconnected graph, they are evaluated below the three-body threshold in which region the inelasticity parameters are unity. But since the phase shifts are truly phases and since iH is the exponential of S, the unknown $f_j(s_k)$ must be real, making the operator H Hermitian. But if H is Hermitian, its spectrum is real, meaning that S is unitary.

It can be seen from (A3) that H Hermitian is nearly in diagonal form and has "eigenvalues" (up to a Jacobian factor) given by $f_j(s_k)$. This again shows that if $f_j(s_k)$ is real, then H is Hermitian and S is unitary.

Now consider H to be a sum of three one-line disconnected terms; then for any power of H, say H^n, the various terms occurring can be broken up into disconnected terms, of which there will be three types, and the remaining connected terms, consisting of products of the different one-line disconnected graphs. When all the powers in H are considered in Eq. (A2), there results the usual breakup of S into

$$S=I+i\sum_{k=1}^{3}{}^{k}A+iA_c\,, \tag{A5}$$

that is the sum of totally disconnected, one-line disconnected, and totally connected contributions. Now, however, the various one-line disconnected partial-wave amplitudes will be of the more complicated form given by Eq. (A4) of paper II; again, however, since $A_j(s_k)$ is the partial-wave amplitude of a physical process occurring below the three-body threshold, two-body unitarity dictates that $A_j(s_k)$ contain a phase shift only (i.e., the inelasticity parameter is unity) which again means that all the functions $f_j(s_k)$ are real.

But if $f_j(s_k)$ replaces $A_j(s_k)$ in Eq. (A4) of paper II, then H, as the sum of one-line disconnected "primitive two-body forces" is still Hermitian and hence S is unitary. But if S is unitary, then the $2\to 3$ partial-wave amplitudes must be zero.

Only two ingredients have been used to arrive at this result. First, the assumption that $\ln S$ exist and, second, that the meaning of a two-body force can be expressed by the notion of primitive diagrams, in which only two of the three particles interact. When these primitive diagrams are summed to give H, there always will be the connection between the disconnected one-line partial-wave amplitudes satisfying two-particle unitarity and those terms occurring in e^{iH} which are one-line disconnected; this connection always will have as a consequence that H is Hermitian. Thus, if S can be exponentiated, then there must be genuine three-body forces in order to have particle production.

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