[10]This property distinguishes crucially our approach from the one advocated by the Cambridge group (Ref. 4). In fact, owing to the singularity at infinity of our amplitudes, it is not possible to perform the rotation of contours in the complex plane, by which these authors show the vanishing of the diffractive contribution in several cases.

[11]Any amplitude is so defined that every quark leg carries only the square root of the quark propagator. In this way we need not take explicitly into account quark propagators in both formulas and figures. No claim is made as to the validity of the unsubtracted Lehmann-Källén representation for the quark two-point function.

[12]A. H. Mueller, Phys. Rev. D 2, 2963 (1970).

[13]See, for example, R. Brandt and G. Preparata, lectures given at the 1971 Hamburg Summer School (unpublished) and references therein.

[14]We thank Frank Paige for help in this estimate.

[15]V. N. Gribov and L. N. Lipatov, Phys. Letters 37B, 78 (1971), and report (unpublished); P. M. Fishbane and J. D. Sullivan, Phys. Rev. D 6, 645 (1972).

[16]This has been also noticed, with the mentioned differences, in parton models by P. Landshoff and J. C. Polkinghorne [Nucl. Phys. B33, 221 (1971)] and in an interesting series of papers by R. Blankenbecler, S. J. Brodsky, and J. F. Gunion [Phys. Letters 39B, 649 (1972) and recent SLAC reports (unpublished)].

[17]Recent work has been carried out along similar lines by M. Bohm, H. Joos, and M. Krammer, Nucl. Phys. B51, 397 (1973).

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Multiparticle Partial-Wave Amplitudes and Inelastic Unitarity. II. Analysis of Three-Body Unitarity

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The three-body unitarity equations are analyzed and it is shown that the partial-wave amplitude for three reacting particles almost fixes the form of the partial-wave amplitude for two reacting particles and the production partial-wave amplitude. Further, it is shown that unitarity imposes constraints on the form taken by the three-body partial-wave amplitude. A program is suggested for utilizing these ideas.

I. INTRODUCTION

Of the various model-independent constraints imposed on relativistically invariant scattering amplitudes, unitarity is, in general, one of the most difficult to satisfy. Only at low energies, where two-body or quasi-two-body reactions predominate, is it reasonably clear how to satisfy unitarity constraints. But in those energy regions where (nonresonant) particle production reactions are important the constraints imposed by unitarity become very complicated.

In this paper we will analyze three-body unitarity involving $2\to 2$, $2\to 3$, and $3\to 3$ reactions. (The numbers designate the numbers of reacting particles.) The motivation for such an analysis arises from the pion-nucleon system, where unitarity relates the reactions $NN\to NN$, $NN\to NN\pi$, and $NN\pi \to NN\pi$; but at this stage the three particles under consideration will be spinless and distinguishable in order to focus on the unitarity equations alone. In a later paper, dealing specifically with the pion-nucleon system, spin and other complications will be taken into account.

Thus, we wish to look at a system consisting of three spinless, distinguishable particles, interacting at energies between the three-body and four-body thresholds. The starting point for the analysis is the observation that in this region the unitarity equations form a closed set of nonlinear equations relating the amplitudes $2\to 2$, $2\to 3$, and $3\to 3$. The main problem is how to deal with the $3\to 3$ amplitude, for the corresponding reaction is not experimentally accessible (unless one of the initial particles is viewed as a bound state of two particles) and very little is known about it theoretically. Further, it will be shown that the $3\to 3$ amplitude dominates the unitarity equation in the sense that knowledge of it almost fixes the $2\to 2$ and $2\to 3$ amplitudes.

A related problem has to do with the analysis of experimental data concerning production processes. While a phase-shift analysis is feasible for two-body scattering processes, not only because the scattering amplitude depends on very few (namely two) parameters, but also because some of the analytic structure of the amplitude is known, the same is not true for production processes. Thus with the production of just one particle the number of parameters in the scattering amplitude in general increases to five. The exhibition of data for production processes in a reasonably model-independent manner is a very difficult task. Further, recent papers have brought into question whether it even makes sense to carry out a $2\to 2$ phase-shift analysis in those regions where inelasticity is significant.[1]

The point of this paper is to take these two problems and deal with them together. Since it is difficult to experimentally analyze a production process in order to learn what the underlying scattering amplitude is, we will pretend that the $3\to 3$ amplitude is known and use the unitarity equations to predict the form of the $2\to 2$ and $2\to 3$ amplitudes, which are experimentally accessible. Now, getting any realistic $3\to 3$ amplitude may seem to be a hopeless task in light of the limited theoretical knowledge of such amplitudes, but it will be shown in Sec. III that the unitarity equations put constraints on the type of $3\to 3$ amplitudes that will satisfy three-body unitarity.

A program then might be to start with simple $3\to 3$ amplitudes, generated, for example, from Feynman graphs, and view the unitarity constraints as imposing corrections on the amplitude. Once the unitarity constraint on the $3\to 3$ amplitude is satisfied, the other parts of the unitarity equations can be used to predict the form of the $2\to 2$ and $2\to 3$ amplitudes.

More precisely, in Sec. II the general form of the unitarity equations will be discussed and it will be shown how these equations relate partial-wave amplitudes of the relevant reactions (rather than the amplitudes themselves). Then in Sec. III the unitarity equations will be analyzed and it will be shown that the $3\to 3$ partial-wave amplitudes do not uniquely fix the $2\to 2$ and $2\to 3$ partial-wave amplitudes. Rather, what happens is that, in order to fix the $2\to 3$ partial-wave amplitude (up to an over-all phase) and the inelasticity parameter of the $2\to 2$ process, knowledge of the $2\to 2$ phase shift *below*[2] the three-body threshold is also required. In order to fix the over-all phase of the $2\to 3$ partial-wave amplitude, it is necessary to also know the $2\to 2$ phase shift *above* the three-body threshold.

The constraints imposed on the $3\to 3$ partial-wave amplitudes by unitarity will turn out to involve restrictions on the eigenvalues and eigenvectors that such partial-wave amplitudes can have. The detailed analysis of these restrictions will actually be carried out in Appendix B.

II. STRUCTURE OF THREE-BODY UNITARITY EQUATIONS

To analyze the three-body unitarity equations it is necessary to carefully specify the variables that occur in $2\to 2$, $2\to 3$, and $3\to 3$ amplitudes, since it is the corresponding partial-wave amplitudes that appear in the equations. For clarity of exposition, only spinless distinguishable particles are being considered, so that the amplitudes can be written as follows:

$2\to 2$:

$$\langle 1''2''|\mathcal{T}|1'2'\rangle \propto A(s,x) = \sum_J (2J+1)^{1/2} P_J(x) A_J(s), \tag{2.1a}$$

$2\to 3$:

$$\langle 1''2''3''|\mathcal{T}|1'2'\rangle \propto A(s,\hat{n},s_q'') = \sum_{JM''} (2J+1)^{1/2} Y_{JM''}(\hat{n}) \times A_{JM''}(s,s_q''), \tag{2.1b}$$

$3\to 3$:

$$\langle 1''2''3''|\mathcal{T}|1'2'3'\rangle \propto A(s,R,s_q'',s_q') = \sum_{JM''M'} (2J+1)^{1/2} D^J_{M''M'}(R) \times A_{JM''M'}(s,s_q'',s_q'). \tag{2.1c}$$

$\sqrt{s}$ is the total invariant energy while J is the angular momentum; since both quantities are conserved they appear as diagonal labels in the unitarity equations and are considered arbitrary, but fixed. x, $\hat{n}$, and R involve angles between incoming and outgoing systems in the over-all c.m. system. The most general rotation, R, involves three Euler angles and is to be understood as relating the body-fixed frame of the outgoing particles to the body-fixed frame of the incoming particles in the $3\to 3$ amplitude. In the $2\to 3$ amplitude the two incoming particles form a line rather than a plane so that only two angles, $\hat{n}$, are required to specify this line relative to the outgoing plane. Finally, in the $2\to 2$ reaction only one angle is needed to relate the incoming direction to the outgoing final direction.

What remains are the subenergies, specifying the relationship between the initial particles (primed quantities) and the relationship between the final particles (double-primed quantities). Quite generally in this paper (as in I)[3] the initial particles will be labeled with primes, the final particles with "double primes," while the intermediate particles occurring in the unitarity equations will be unprimed. All the subenergies can of course be written explicitly as relativistically invariant quantities. For the case of three particles (e.g., the final particles) s_q'' can be chosen, e.g., as $(p_1''+p_2'')^2$ and $(p_1''+p_3'')^2$; but it is better to leave the actual choice of subenergies unspecified, to be chosen after the unitarity equations are developed.

It is also possible to write all of the rotations appearing in the amplitudes (2.1) as relativistic scalars. For example, in the $2\to 3$ reaction, $\hat{n}$ can be associated with the momentum transfers $(p_1'-p_1'')^2$, $(p_1'-p_2'')^2$, since

$$\begin{aligned}(p_1'-p_1'')^2 &= M_1^2+M_1^2 - 2(E_1'E_1''-|\vec{p}_1'||\vec{p}_1''|\cos\theta_{1'1''}),\\ (p_1'-p_2'')^2 &= M_1^2+M_2^2 - 2(E_1'E_2''-|\vec{p}_1'||\vec{p}_2''|\cos\theta_{1'2''}).\end{aligned} \tag{2.2}$$

E_1', E_1'', E_2'', etc., are all functions of subenergies, while the angle between 1′ and 2″, $\theta_{1'2''}$, can be related to the angles between 1″ and 2″, which is a function of subenergies, and φ, the azimuthal angle. Although for some purposes it is useful to have explicit expressions for the angles in terms of relativistic invariants for the unitarity equations, this is not the case here since only relations between partial-wave amplitudes are involved, in which the rotations are eliminated. The partial-wave amplitudes are denoted by A's in Eqs. (2.1) and the variables for the various reactions explicitly indicated.

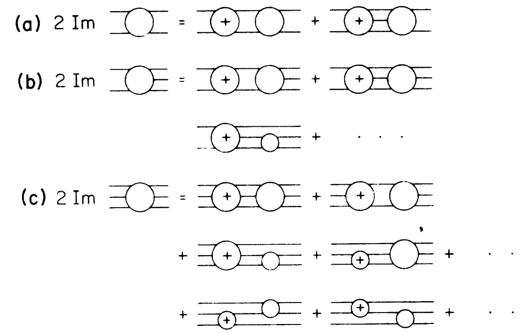

FIG. 1. Bubble diagrams for three-body unitarity equations.

That the unitarity equations involve relations between the partial-wave amplitudes can be seen by starting with the operator equation

$$i(\mathcal{T}^\dagger-\mathcal{T})=\mathcal{T}^\dagger\mathcal{T} \tag{2.3}$$

and considering various possible matrix elements. Remembering that we are considering reactions in a world consisting only of particles 1, 2, and 3 means that a complete set of states inserted between $\mathcal{T}^\dagger$ and $\mathcal{T}$ gives the following equations:

$$2\,\mathrm{Im}\langle 1''2''|\mathcal{T}|1'2'\rangle=\int_{\substack{\text{intermediate}\\ \text{momenta}}}[\langle 1''2''|\mathcal{T}^\dagger|12\rangle\langle 12|\mathcal{T}|1'2'\rangle+\langle 1''2''|\mathcal{T}^\dagger|123\rangle\langle 123|\mathcal{T}|1'2'\rangle], \tag{2.4a}$$

$$\begin{aligned}2\,\mathrm{Im}\langle 1''2''3''|\mathcal{T}|1'2'\rangle=\int_{\substack{\text{intermediate}\\ \text{momenta}}}&[\langle 1''2''3''|\mathcal{T}^\dagger|12\rangle\langle 12|\mathcal{T}|1'2'\rangle+\langle 1''2''3''|\mathcal{T}^\dagger|123\rangle\langle 123|\mathcal{T}|1'2'\rangle\\ &+\langle 1''2''3''|\mathcal{T}^\dagger|123\rangle\langle 12|\mathcal{T}|1'2'\rangle E_3\delta^3(\vec{p}_3-\vec{p}_3')]+\cdots,\end{aligned} \tag{2.4b}$$

$$\begin{aligned}2\,\mathrm{Im}\langle 1''2''3''|\mathcal{T}|1'2'3'\rangle=\int_{\substack{\text{intermediate}\\ \text{momenta}}}&[\langle 1''2''3''|\mathcal{T}^\dagger|123\rangle\langle 123|\mathcal{T}|1'2'3'\rangle+\langle 1''2''3''|\mathcal{T}^\dagger|12\rangle\langle 12|\mathcal{T}|1'2'3'\rangle\\ &+\langle 1''2''3''|\mathcal{T}^\dagger|123\rangle\langle 12|\mathcal{T}|1'2'\rangle E_3\delta^3(\vec{p}_3-\vec{p}_3')+\cdots].\end{aligned} \tag{2.4c}$$

Time-reversal invariance has been used in writing $i\langle\,|\mathcal{T}^\dagger-\mathcal{T}|\rangle=2\,\mathrm{Im}\langle\,|\mathcal{T}|\rangle$. If the partial-wave amplitudes introduced in Eqs. (2.1) are substituted into the unitarity Eqs. (2.4) and the integrals over intermediate momenta used to extract the relevant rotations, there results a set of three equations for the partial-wave amplitudes which can be written most expressively in graphical form, as shown in Fig. 1. The number of legs on the bubble(s) designates the relevant reaction and partial-wave amplitude. It is to be noted that not all reactions have been included in Figs. (1b) and (1c); the missing reactions are those occurring when the particle labels are permuted and are indicated by the dots.

Written out in terms of the partial-wave amplitudes the unitarity equations become

$$2\,\mathrm{Im}A_J(s)=|A_J(s)|^2+\sum_M\int ds_q|A_{JM}(s,s_q)|^2, \tag{2.5a}$$

$$2\,\mathrm{Im}A_{JM''}(s,s_q'')=A^*_{JM''}(s,s_q'')A_J(s)+\sum_M\int ds_q[A^*_{JMM''}(s,s_q,s_q'')+{}^3A^*_{JMM''}(s,s_q,s_q'')]A_{JM}(s,s_q)\,, \tag{2.5b}$$

$$\begin{aligned}2\,\mathrm{Im}A_{JM''M'}(s,s_q'',s_q')=&\sum_M\int ds_q A^*_{JMM''}(s,s_q,s_q'')A_{JMM'}(s,s_q,s_q')+A^*_{JM''}(s,s_q'')A_{JM'}(s,s_q')\\&+\sum_M\int ds_q\,{}^3A^*_{JMM''}(s,s_q,s_q'')A_{JMM'}(s,s_q,s_q')+\sum_M\int ds_q A^*_{JMM'}(s,s_q,s_q'')\,{}^3A_{JMM'}(s,s_q,s_q')\\&+\sum_M\int ds_q\,{}^3A^*_{JMM''}(s,s_q,s_q'')\,{}^1A_{JMM'}(s,s_q,s_q')\,.\end{aligned} \tag{2.5c}$$

The $3\to 3$ partial-wave amplitudes with superscripts correspond to those reactions in which one particle does not interact with the other two; thus ${}^2A_{JM''M'}(s,s_q'',s_q')$ is the partial-wave amplitude corresponding to the reaction in which particles 1 and 3 interact, and particle 2 does not. The form of such "disconnected" amplitudes is given in Appendix A, where it is shown that they depend only on the phase shift *below* the three-body threshold. Only those terms in Eqs. (2.5) have been written out which correspond to the bubbles in Fig. 1. Thus, in Eqs. (2.5) there are also missing terms, corresponding to suitable permutations of the "disconnected" partial-wave amplitudes.

From Eqs. (2.5) it is clear that the quantities s and J are diagonal in the unitarity equations. To simplify the notation, they will be suppressed with the understanding that they are arbitrary but fixed. (s of course lies between the three-body and four-body thresholds.) Further, it is to be noticed that the discrete index M, running between $-J$ and $+J$, and the integration over the subenergies ds_q, always constitute the "measure" over the intermediate states; if we replace $\{M, s_q\}$ by the single symbol x_i, then dx_i "=" $\sum_M\int ds_q$ and the unitarity equations become

$$2\,\mathrm{Im}A=|A|^2+\sum\int dx_i\,|A(x_i)|^2\,, \tag{2.6a}$$

$$2\,\mathrm{Im}A(x_i'')=A^*(x_i'')A+\sum\int dx_i[A^*(x_i,x_i'')+{}^3A^*(x_i,x_i'')]A(x_i)\,, \tag{2.6b}$$

$$\begin{aligned}2\,\mathrm{Im}A(x_i'',x_i')=&\sum\int dx_i\,A^*(x_i,x_i'')A(x_i,x_i')+A^*(x_i'')A(x_i')\\&+\sum\int dx_i[{}^3A^*(x_i,x_i'')A(x_i,x_i')+A^*(x_i,x_i'')\,{}^3A(x_i,x_i')+{}^3A^*(x_i,x_i'')\,{}^1A(x_i,x_i')]\,.\end{aligned} \tag{2.6c}$$

Several general observations can be made about these equations. First, Eq. (2.6a) is usually rewritten in a form which puts a bound of 1 on the magnitude of the $2\to 2$ partial-wave amplitude by introducing

$$1+iA=\eta e^{2i\delta}\,, \tag{2.7}$$

where η is the inelasticity parameter and δ the phase shift. Equation (2.6a) then becomes

$$1=\eta^2+\sum\int dx_i\,|A(x_i)|^2\,, \tag{2.8}$$

indicating that $\eta\le 1$ and any singularity that $|A(x_i)|^2$ might have is integrable. But in contrast to inelastic $2\to 2$ reactions [see Appendix B, Eq. (B8)] no bound can be put on the $2\to 3$ partial-wave amplitude, using only unitarity arguments. However, from an experimental point of view, with counters and detectors always involving finite widths, Eq. (2.8) can be used to put a bound on $|A(x_i)|^2$, of the form

$$\int_{\Delta s_q} ds_q\,|A_{JM}(s,s_q)|^2\le 1\,, \tag{2.9}$$

where Δs_q is the window size of the detector in terms of the subenergy s_q. Equation (2.8) relates experimentally observable quantities, namely η, obtained in a phase-shift analysis and $|A(x_i)|$, which occurs in the differential cross section of the $2\to 3$ reaction holding s and s_q fixed, and integrating over all $\hat{n}$:

$$\frac{d^2\sigma(2\to 3)}{ds_q}\propto\sum_{JM}|A_{JM}(s,s_q)|^2\,. \tag{2.10}$$

In contrast, Eqs. (2.6b) and (2.6c) involve the $3\to 3$ reaction, which has not been experimentally observable; for this reason these two equations have been less useful than Eq. (2.8), since very little is known about $3\to 3$ reactions. One might hope to use the second and third unitarity equations to learn something about $3\to 3$ partial-wave amplitudes, assuming that experimental information is available for the $2\to 2$ and $2\to 3$ process. But it is to be noted that in Eq. (2.6b), half of the variables in the $3\to 3$ partial-wave amplitude are integrated over, so at best one might extract some information relating to suitable averages over $3\to 3$ partial-wave amplitudes. And in Eq. (2.6c), the $3\to 3$ partial-wave amplitude occurs in almost every term.

It thus seems hopeless to learn anything about the $3\to 3$ amplitude from knowledge of the $2\to 2$ and $2\to 3$ amplitudes, for the $3\to 3$ amplitude quite clearly "dominates" the other amplitudes. Rather the point of view to be taken here is that one should start with some guessed form of the $3\to 3$ amplitude and use the unitarity equations to test the guess by seeing what sorts of results are predicted for the physically accessible $2\to 2$ and $2\to 3$ reactions. Such a point of view will be discussed more fully in the conclusion. Section III will show that, assuming $A(x_i'', x_i')$ and the phase shifts below the three-body threshold are known, η and $A(x_i)$ – up to an over-all phase – can be computed. If the phase shift δ is also known between the three-body and four-body thresholds, then the over-all phase of the $2\to 3$ partial-wave amplitude can be computed.

III. SOLUTION OF THE THREE-BODY UNITARITY EQUATIONS IN TERMS OF THE $3\to 3$ PARTIAL-WAVE AMPLITUDE

To analyze the unitarity equations, we first rewrite Eq. (2.6c) by defining

$$T(x_i'', x_i') = A(x_i'', x_i') + \sum_{k=1}^{3} {}^kA(x_i'', x_i'), \tag{3.1}$$

i.e., T is the sum of the connected plus one-line disconnected graphs (see Appendix A). Then, making use of the fact that the one-line disconnected partial-wave amplitudes satisfy two-particle elastic unitary [see Eq. (A6)],

$$2\,\mathrm{Im}\,{}^kA(x_i'', x_i') = |{}^kA(x_i'', x_i')|^2, \tag{3.2}$$

it is possible to write Eq. (2.6c) as

$$2\,\mathrm{Im}\left[A(x_i'', x_i') + \sum_{k=1}^{3} {}^kA(x_i'', x_i')\right] = \sum\int dx_i\left[A^*(x_i, x_i'') + \sum_k {}^kA^*(x_i, x_i'')\right]\left[A(x_i, x_i') + \sum_{k'} {}^{k'}A(x_i, x_i')\right] + A^*(x_i'')A(x_i'). \tag{3.3}$$

Here all the disconnected terms have been included, in contrast to Eq. (2.6c), where the permuted terms were left out for convenience. Equation (3.3) becomes

$$2\,\mathrm{Im}\,T(x_i'', x_i') = \sum\int dx_i\, T^*(x_i, x_i'')\, T(x_i, x_i') + A^*(x_i'')A(x_i'). \tag{3.4}$$

To analyze the nature of the constraints imposed by unitarity on the reaction matrix $T(x_i'', x_i')$, we introduce the S matrix by including the totally disconnected $3\to 3$ partial-wave amplitude [Eq. A1)], so that

$$S(x_i'', x_i') = J(x_i)\delta^3(x_i'' - x_i') + iT(x_i'', x_i'). \tag{3.5}$$

Also the "measure"

$$dx_i = \sum_{M=-J}^{+J}\int d^2 s_q$$

can be used to generate a Hilbert space H with inner product

$$(f, g) = \sum\int dx_i f^*(x_i)g(x_i),$$

and the operators S and T can be thought of as operating on this space, while the $2\to 3$ partial-wave amplitudes are to be seen as elements of this space with norm less than one.

The unitarity equations become

$$1 = \eta^2 + \|A\|^2 \quad \text{[rewritten form of Eq. (2.8)]}, \tag{3.6a}$$

$$0 = \eta e^{-2i\delta}A^*(x_i'') + \sum\int dx_i\, S(x_i'', x_i)A(x_i) \quad \text{[rewritten form of Eq. (2.6b)]}, \tag{3.6b}$$

$$A(x_i'')A^*(x_i') = J(x_i')\delta(x_i'' - x_i') - \sum\int dx_i\, S^*(x_i, x_i'')S(x_i, x_i') \quad \text{[rewritten form of Eq. (3.4)]}. \tag{3.6c}$$

Now since $\sum\int dx_i\, S^*(x_i, x_i'')S(x_i, x_i')$ is self-adjoint, it has a real spectrum and can be diagonalized by some unitary transformation. Further, written in operator form as $I - S^\dagger S$, it is seen that the right-hand side of Eq. (3.6c) is an (defect)[4] operator of rank one. Thus, the eigenvalues of $S^\dagger S$ are of the form $(x, 1, 1, 1, \ldots, 1, \ldots)$. Let $v \in H$, $\|v\| = 1$ be the eigenvector associated with the eigenvalue x; operating with v to the right of Eq. (3.6c) gives

$$A(A, v) = v(1 - x), \tag{3.7}$$

and taking the inner product with A to the left gives

$$\|A\|^2(A, v) = (A, v)(1 - x). \tag{3.8}$$

Combining Eq. (3.8) with (3.6a) shows that $x=\eta^2$; thus, the one unique eigenvalue of $S^\dagger S$ is η^2.

The argument now proceeds in the same manner as in Appendix B, where the operators are finite-dimensional matrices. If S is normal, there is a unitary operator that diagonalizes S (the case when S is non-normal will be dealt with in the following paper), and hence diagonalizes $S^\dagger S$; using Eq. (3.6b) it then follows that the vector v of Eq. (3.7) is an eigenvector of S associated with a complex eigenvalue λ of magnitude η. Conversely, the same unitary transformation that diagonalizes $S^\dagger S$ also diagonalizes S, but since S is symmetric, the unitary transformation is real orthogonal; collecting all these results together gives

$$\begin{aligned} &S=ODO^T\,,\\ &Sv=\lambda v\,,\quad v \text{ real, of length one}\\ &A=av\,,\quad |a|=\|A\|\\ &S^\dagger S=O|D|^2O^T\,, \end{aligned}\tag{3.9}$$

where D is a diagonal operator whose first entry is the eigenvalue λ and all other entries are of modulus one; O is real orthogonal while a is a complex number relating the $2\to 3$ partial-wave amplitude A to the eigenvector v. If the relations of Eq. (3.9) are substituted into Eq. (3.6), the following scalar equations are obtained:

$$\begin{aligned} &1=\eta^2+|a|^2\,,\\ &|\lambda|=\eta\,,\\ &0=\eta e^{-2i\delta}a^*+\lambda a\,. \end{aligned}\tag{3.10}$$

From the last of Eqs. (3.10) it can be seen that

$$\begin{aligned} &0=e^{-2i\delta}\,e^{-i\arg a}+e^{i\arg\lambda}\,e^{+i\arg a}\,,\\ &e^{i(\pi+\arg\lambda)}=e^{2i(\arg a-\delta)}\,,\\ &\arg\lambda=\pi-2(\arg a+\delta)\,. \end{aligned}\tag{3.11}$$

For a given operator S, which is properly unitary, the eigenvalue λ can be computed. But there is no way to independently compute either δ or $\arg a$. Therefore one of these quantities must be given – either from experiment or from some theory. We will regard the $2\to 2$ phase shift δ as given and use Eq. (3.11) to compute $\arg a$ and thus compute A, the $2\to 3$ partial-wave amplitude. It is interesting to note that being able to write A as av means that A is real except for an over-all phase, which of course can depend on s and J.

To conclude this section we translate the basic results of Eqs. (3.9) and (3.11) into results on the T matrix, remembering that T includes all the one-line disconnected partial-wave amplitudes as well as the connected $3\to 3$ partial-wave amplitude. In order for any such T matrix to satisfy unitarity it is necessary and sufficient that it have one unique eigenvalue $(\eta e^{i\arg\lambda}-1)/i$ and associated real eigenvector v. All the other eigenvalues are of the form $(e^{i\Delta}-1)/i$, where Δ is a phase. The $2\to 3$ partial-wave amplitude is related to the eigenvector v by $A=iav$. More explicitly

$$\begin{aligned} &\sum\int dx_i'\;T_{sJ}(x_i'',x_i')v_{sJ}(x_i')=\frac{\eta(s,J)e^{i\arg\lambda(sJ)}-1}{i}\,v_{sJ}(x_i'')\,,\\ &A_{sJ}(x_i)=ia(s,J)v_{sJ}(x_i)\,,\\ &2\arg a(s,J)=2\delta(s,J)+\arg\lambda(s,J)\,; \end{aligned}\tag{3.12}$$

the dependence on $\sqrt{s}$ and J, the total invariant energy and angular momentum, is to be noted.

Thus, we have shown that η and $|A|$, the inelasticity parameter and the magnitude of the $2\to 3$ partial-wave amplitude, can be computed as appropriate eigenvalues and eigenvectors of the T matrix, which involves knowing the connected $3\to 3$ partial-wave amplitude and the phase shifts of the $2\to 2$ reactions below the three-body threshold. Finally, if the phase shift of the $2\to 2$ reaction between the three-body and four-body thresholds is known, the over-all phase of the $2\to 3$ partial-wave amplitude can be computed.

IV. CONCLUSION

It is now necessary to explain in somewhat more detail how we hope to make use of the unitarity equations developed in Sec. III. It is well known that it is difficult to simultaneously satisfy unitarity and crossing. Thus, in generalized Veneziano models, crossing is explicitly built into the models and a problem is then how to incorporate unitarity.[5] Conversely, when potentials can be used, for example, with the Faddeev equations,[6] unitarity is automatically satisfied, but then the meaning of crossing is not clear.

The point of view to be taken here is that any (relativistically invariant) model amplitudes should first be forced to satisfy three-body unitarity and then their crossing properties analyzed. The most obvious way to obtain (partial-wave) amplitudes is to start with simple Feynman graphs – leaving the functional form of certain (form) factors undetermined – and force these amplitudes to satisfy three-body unitarity. It can be argued that such a starting point is not entirely arbitrary since Feynman graphs with appropriate form factors often give reasonable results.[7]

Concretely, what this means is to choose a Feynman graph for a $3\to 3$ process, compute its partial-wave amplitude, make use of some of the undetermined form factors to unitarize the graph via the Eqs. (3.12), and then see how the unitarized graph agrees with experiment by computing the magnitude of the $2\to 3$ partial-wave amplitude and the inelasticity parameter for the $2\to 2$ process. Thus the laboratory for the $3\to 3$ amplitude is to be found in the predictions that are made for the experimentally accessible $2\to 2$ and $2\to 3$ reactions.

Now it must be conceded that such a program will probably be difficult to carry out – but if it can be, then since the forms of the $3\to 3$ partial-wave amplitudes are explicitly given, it should be possible to cross them into a domain involving the $2\to 4$ reaction which is also experimentally accessible. For example, the amplitude for the reaction $\pi NN\to \pi NN$ can be crossed to give $NN\to \pi\pi NN$. The degree to which this amplitude violates four-body unitarity can be determined via the four-body constraint equations and then crossed over to the original channel. In this way one might hope to use crossing in an iterative manner in conjunction with unitarity in the various channels.

It should be pointed out that such a program is much less ambitious than a full S-matrix program, in that in each energy region only a finite number of particles are involved, and some experimental information is fed into the equations.

The goal of the next papers then is both to proceed with an analysis of the case when the $3\to 3$ partial-wave amplitude is non-normal, using the theory of characteristic operator functions,[4] and to find some simple models that satisfy the three-body unitarity equations. There are several ways of doing this, including the expansion of $3\to 3$ partial-wave amplitudes in a superposition of resonances using anharmonic Fourier analysis and seeing under what conditions a $3\to 3$ partial-wave amplitude will generate a resonance. If it is possible to find amplitudes that satisfy three-body unitarity and also agree reasonably well with experiment, then it will be necessary to extend the unitarity equations to include four particles in order to implement crossing. But such an extension should not be difficult to carry out. Finally, in order to be able to deal with the πNN system it is necessary to include complications arising from spin, and also to include in the unitarity equations the fact that there is another two-body open channel, namely $NN\to \pi d$.

APPENDIX A: PARTIAL-WAVE AMPLITUDES FOR DISCONNECTED DIAGRAMS

In many of the calculations dealing with unitarity, it proves useful to have expressions for partial-wave amplitudes of disconnected processes. Only two such disconnected amplitudes can occur in three-body unitarity equations, one involving three particles which do not interact, the other allowing two particles to interact while the third does not. Consider first the totally disconnected partial-wave amplitude (PWA); we have

$$[\text{Totally disconnected PWA}] = J(s_q')\delta^2(s_q'' - s_q')\delta_{M''M'}, \tag{A1}$$

where $J(s_q')$ is a phase-space Jacobian. That the partial-wave amplitude for the totally disconnected term is of this form can be seen by noticing either that a general $3\to 3$ partial-wave amplitude is written $A_{JM''M'}(s, s_q'', s_q')$ and the δ functions indicate no change in subenergies or reference frame, or by writing an "amplitude" for three noninteracting particles with prescribed subenergy relations between them and computing the partial-wave amplitude. The form of the Jacobian will of course depend on the variables chosen for the subenergies.

The partial-wave amplitude for the disconnected process denoted ${}^3A(x_i)$ in Eq. (2.5a), in which particle 3 does not interact with particles 1 and 2, can be written

$$ {}^3A_{JM''M'}(s, s_q'', s_q') = \langle [sJ]\vec{p} = \vec{0}\,\sigma''; M''s_q'' | T_{12} | [sJ]\vec{p} = \vec{0}\,\sigma'; M's_q'\rangle , \tag{A2}$$

where $|[sJ]\vec{p} = \vec{0}\,\sigma; Ms_q\rangle$ is a three-particle state, labeled by the total invariant energy $\sqrt{s}$, total angular momentum J, spin projection σ, and taken in the over-all c.m. system of the three particles ($\vec{p} = \vec{0}$). M is a spin projection along a body-fixed axis of the three particles, while s_q, as before, represents some choice of two subenergies. Finally, T_{12} is the reaction operator, so labeled to indicate that only particles 1 and 2 interact.

To write (A2) in a more convenient form, we use the Racah coefficients of the Poincaré group[3] to go to a coupling scheme where particles 1 and 2 are coupled first and then the resultant coupled to particle 3. We have[8] (deleting all primes)

$$|[sJ]\vec{p} = \vec{0}\,\sigma; Ms_q\rangle = \sum_{jm} (2j+1)^{1/2} D^J_{Mm}[R(\hat{p}_3)] D^j_{m0}[R^{-1}(\hat{p}_3)R(\hat{p}_1(12\,\text{c.m.}))]\,|[sJ]\vec{p} = \vec{0}\,\sigma; jms_{12}\rangle , \tag{A3}$$

where $s_{12} = (p_1 + p_2)^2$ and $R(\hat{p}_1(12\,\text{c.m.}))$ is the rotation specifying the direction of particle 1 in the 12 c.m. frame. Substituting Eq. (A3) (and its adjoint) into (A2) gives

$$^3A_{JM''M'}(s, s_q'', s_q') = \sum_{jm} (2j+1) D^{J*}_{M''m}[R(\hat{p}_3'')] D^J_{M'm}[R(\hat{p}_3')] D^{j*}_{m0}[R^{-1}(\hat{p}_3'') R(\hat{p}_1''(1''2''\,\text{c.m.}))] \times D^j_{m0}[R^{-1}(\hat{p}_3') R(\hat{p}_1'(1'2'\,\text{c.m.}))] A_j(s_{12}') \delta(s_{12}'' - s_{12}') . \quad \text{(A4)}$$

A further simplification occurs if the "third" particle of both initial and final states defines the z axis of the initial and final body-fixed frames, respectively. Then $D^J_{Mm}[R(\hat{p}_3)] = \delta_{Mm}$ and (A4) becomes

$$^3A_{JM''M'}(s, s_q'', s_q') = 4\pi \sum_j Y_{jM''}(\theta'', 0) Y_{jM'}(\theta', 0) A_j(s_{12}') \delta(s_{12}'' - s_{12}') \delta_{M''M'} , \quad \text{(A5)}$$

where θ is the angle between particle 3 and particle 1 in the 12 c.m. system. Equation (A5) indicates that the natural subenergy variables to use in disconnected three-body partial-wave amplitudes are the energy of the noninteracting particle, which is proportional to s_{12} and obviously conserved in the reaction, and the angle between the noninteracting particle and one of the interacting particles in the c.m. system of the two interacting particles. In a succeeding paper it will be shown that such a choice of subenergy variables generates a rectangular Dalitz plot and is a natural set to use for processes involving resonances.

It is also to be noted that $^3A(x_i)$ (and of course $k = 1, 2$ also) satisfies two-particle unitarity, of the form

$$2\,\text{Im}\,^3A(x_i'', x_i') = \sum \int dx_i \, ^3A^*(x_i, x_i'') \, ^3A(x_i, x_i') .$$

To see this, use Eq. (A4) and write

$$\begin{aligned} 2\,\text{Im}\,^3A(x_i'', x_i') &= \sum \int dx_i \, ^3A^*(x_i, x_i'') \, ^3A(x_i, x_i') \\ &= \sum_M \int ds_q \sum_{jm} (2j+1) D^J_{Mm}[R(\hat{p}_3)] D^{J*}_{M''m}[R(\hat{p}_3'')] D^j_{m0}(\theta) D^{j*}_{m0}(\theta'') A_j^*(s_{12}) \delta(s_{12} - s_{12}'') \\ &\quad \times \sum_{j'm'} (2j'+1) D^J_{Mm'}[R(\hat{p}_3)] D^J_{M'm'}[R(\hat{p}_3')] D^{j'*}_{m'0}(\theta) D^{j'}_{m'0}(\theta') A_{j'}(s_{12}) \delta(s_{12} - s_{12}') \\ &= \sum_{j,m} (2j+1) D^{J*}_{M''m}[R(\hat{p}_3'')] D^J_{M'm}[R(\hat{p}_3')] D^{j*}_{m0}(\theta'') D^j_{m0}(\theta') \, |A_j(s_{12}')|^2 \delta(s_{12}'' - s_{12}') \\ &= \sum_{j,m} (2j+1) D^{J*}_{M''m}[R(\hat{p}_3'')] D^J_{M'm}[R(\hat{p}_3')] D^{j*}_{m0}(\theta'') D^j_{m0}(\theta') 2\,\text{Im} A_j(s_{12}') \delta(s_{12}'' - s_{12}') . \quad \text{(A6)} \end{aligned}$$

To get to the last line of Eq. (A6) involves using the orthonormality of the D^j functions and choosing as a set of variables for the subenergies the quantities $\sqrt{s_{12}}$ and the angle θ between $\hat{p}_3$ and $\hat{p}_1$ in the 12 c.m. system. The δ function in the subenergy s_{12}, the orthonormality of the $D^j(\theta)$ functions, and the group properties of the rotation $R(\hat{p}_3)$ eliminate the "integration" over the measure $dx_i = \sum_M \int ds_q$.

More generally, it can be seen that the natural set of subenergy variables to use in the disconnected partial-wave amplitude ^kA is the invariant energy of the kth particle and the angle between the kth particle and one other particle in the c.m. system of the other two particles. The "rectangular" Dalitz plots are bounded by $\cos\theta$ going from -1 to $+1$, and the invariant subenergy going from its minimum value, the sum of the masses of the two other particles to its maximum value, the difference of the total invariant energy and the mass of the kth particle. It is to be noted that $A_j(s_{12}')$ involves only the phase shift below the three-body threshold, and in this region η, the inelasticity parameter is always one. Thus the partial-wave amplitude $T(x_i'', x_i') \equiv A(x_i'', x_i') + \sum_{k=1}^{3} {}^kA(x_i'', x_i')$ is known if the connected $3 \to 3$ partial-wave amplitude and the phase shifts in the disconnected terms are known.

APPENDIX B: SOLUTION OF N-DIMENSIONAL UNITARITY EQUATIONS

In this appendix we wish to look at unitarity equations arising from finite-dimensional unitary matrices.[9] Such equations arise when it is possible to approximate three-body equations as quasi-two-body equations, and should be a good approximation to the more complicated three-body equations when resonance effects dominate the three-body amplitudes. Since in this approximation we have only two particles in the initial and final states, it is possible to label each possible set of two particles by a number j; we then have the following possible reactions:

$$\begin{matrix} 1 \to 1, & 1 \to 2, & \ldots & 1 \to N, \\ 2 \to 1, & 2 \to 2, & \ldots & 2 \to N, \\ \ldots & & & \ldots \\ N \to 1, & & & N \to N . \end{matrix} \quad \text{(B1)}$$

Each diagonal entry represents an elastic reaction of the form $j\to j$, so there are N elastic reactions. Because of time-reversal invariance, $j\to k = k\to j$; further, if all the particles are spinless, a given amplitude can be written as

$$A^{j\to k}(E,x)=\sum_J P_J(x)A_J^{j\to k}(E). \quad \text{(B2)}$$

Now unitarity can be written in operator form as

$$i(\mathcal{T}^\dagger-\mathcal{T})=\mathcal{T}^\dagger\mathcal{T}, \quad \text{(B3)}$$

which, in terms of allowed matrix elements, becomes

$$i\langle k|\mathcal{T}^\dagger-\mathcal{T}|j\rangle=\sum_n\int\langle k|\mathcal{T}^\dagger|n\rangle\langle n|\mathcal{T}|j\rangle, \quad \text{(B4)}$$

where a complete set of states $|n\rangle$ has been inserted between $\mathcal{T}^\dagger$ and $\mathcal{T}$. Expanding the amplitudes into their partial waves and making use of time-reversal invariance then gives

$$2\,\mathrm{Im}A_J^{j\to k}(E)=\sum_n\rho_n A_J^{k\to n*}(E)A_J^{j\to n}(E), \quad \text{(B5)}$$

where ρ_n is the phase-space factor arising from the integration over the momentum variables in channel n. Equation (B5) is often written as a matrix equation by setting $A_J^{j\to k}(E)=A_{kj}$:

$$2\,\mathrm{Im}A=A^\dagger\rho A. \quad \text{(B6)}$$

By defining a new partial-wave amplitude which includes a square root of the phase-space factor, it is possible to eliminate the matrix ρ completely; thus, in the following equations, we will set $\rho=I$.

Now set $1+iA^{j\to j}=\eta^j e^{2i\delta_j}$; then the diagonal elements of Eq. (B5) give

$$1=(\eta^j)^2+\sum_{n\neq j}|A^{j\to n}|^2, \quad \text{(B7)}$$

which shows that all partial-wave amplitudes are bounded by 1, i.e.,

$$|A_J^{j\to k}(E)|\leq 1. \quad \text{(B8)}$$

The argument leading to Eq. (B8) is substantially unchanged even if the reacting particles have spin. But if the variables that occur in the unitarity equations include continuous variables such as subenergies, then it is no longer possible to define a bound such as (B8); rather a bound of the type given in Eq. (2.9) seems to be the best that can be done.

The preceding argument suggests that it is useful to have a discrete matrix analog of the "continuous matrices" that occur in the actual three-body unitarity equations.[10] Such matrices can be obtained either by considering the subenergy variables to be discrete, or suppressing them completely and dealing only with the quantum number M, the projection of angular momentum running between $-J$ and J. Thus, consider the reduced "S matrix"

$$\begin{aligned}\mathcal{S}&=I+i\mathcal{T}\\&=I+i\begin{pmatrix}t&\vec{\mathrm{T}}\\\vec{\mathrm{T}}&T\end{pmatrix},\end{aligned} \quad \text{(B9)}$$

where $\mathcal{S}$ is a symmetric, unitary, $(N+1)\times(N+1)$ matrix, and the "reaction matrix" has three parts: t, corresponding to a $2\to 2$ partial-wave amplitude; $\vec{\mathrm{T}}$, an N component vector, corresponding to a $2\to 3$ partial-wave amplitude; and T, symmetric, corresponding to a $3\to 3$ partial-wave amplitude. We will pretend that T is known and see what constraints T has to satisfy, and what can be learned about $\vec{\mathrm{T}}$ and t from the unitarity equations, (B3).

It turns out to be simpler to work directly with the $\mathcal{S}$ matrix, so we write

$$\mathcal{S}=\begin{pmatrix}\eta e^{2i\delta}&A^\dagger\\A^*&S\end{pmatrix}, \quad \text{(B10)}$$

where A is an N-component column vector; there is a natural inner product on vectors which can be written

$$\begin{aligned}(A,A')&=\sum_{i=1}^N A_i^*A_i'\\&=A^\dagger A',\end{aligned} \quad \text{(B11)}$$

and $\mathcal{S}$ is to be regarded as an operator acting on the space of vectors defined relative to the inner product (B11).

With $\mathcal{S}$ satisfying $\mathcal{S}^\dagger\mathcal{S}=I$, the unitarity equations become

$$\eta^2+\|A\|^2=1, \quad \text{(B12a)}$$

$$SA+\eta e^{-2i\delta}A^*=0, \quad \text{(B12b)}$$

$$AA^\dagger=I-S^\dagger S. \quad \text{(B12c)}$$

Now $S^\dagger S$ is Hermitian, therefore it can be diagonalized; since $AA^\dagger$ is of rank 1, it is clear that the eigenvalues of $S^\dagger S$ are of the form $(x,1,1,\ldots,1)$. Denote the normalized eigenvector associated with the real eigenvalue x as $\hat{v}$ and apply Eq. (B12c) to $\hat{v}$:

$$A(A,\hat{v})=\hat{v}(1-x). \quad \text{(B13)}$$

Operating on (B13) with A from the left then gives

$$\begin{aligned}&\|A\|^2(A,\hat{v})=(A,\hat{v})(1-x),\\&\|A\|^2=1-x,\end{aligned} \quad \text{(B14)}$$

which shows, using Eq. (B12a), that $x=\eta^2$. The possible solution $(A,\hat{v})=0$ of Eq. (B14) will not be considered since it corresponds to $\|A\|=0$, as can be seen from Eq. (B13). Thus, the nondegenerate eigenvalue of $S^\dagger S$ equals the square of the inelasticity parameter. Since all the remaining eigenvalues are one, the orthonormal eigenvectors can be freely chosen, only satisfying the condition that they be perpendicular to $\hat{v}$.

If S is a normal operator, so that $[S, S^\dagger]=0$, then S can be diagonalized by a unitary transformation

$$S=UDU^\dagger\,, \tag{B15}$$

where U is unitary and D is a diagonal matrix of the form $(\lambda, e^{id_2}, \ldots, e^{id_N})$, with λ the complex eigenvalue corresponding to the eigenvector $\hat{v}$. But since S is also symmetric, Eq. (B15) forces U to be a real orthogonal matrix, so that

$$S=ODO^T\,,\quad O \text{ orthogonal}\,, \tag{B16}$$

and thus, $A^*=a^*\hat{v}$, with $\hat{v}$ real. Equation (B16) is similar to the well-known decomposition of an arbitrary unitary symmetric matrix, except that D has entries of modulus one everywhere except for the first entry, where the magnitude of the eigenvalue λ is less than one.

We thus conclude that S has a unique real eigenvector $\hat{v}$ and a complex eigenvalue λ, which, using Eq. (B12b) is related to $\eta e^{2i\delta}$ by

$$\lambda a+\eta e^{-2i\delta}a^*=0\,, \tag{B17}$$

so that $|\lambda|=\eta$ and $2\delta=\pi-2\arg a-\arg\lambda$. Since neither δ nor $\arg a$ can be obtained independently, it is clear that one variable, which we choose to be the phase shift δ, must be given in order to determine $\arg a$.

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PHYSICAL REVIEW D VOLUME 7, NUMBER 10 15 MAY 1973

Multiparticle Partial-Wave Amplitudes and Inelastic Unitarity. III. Solution of the Three-Body Unitarity Equations Using Characteristic Operator Functions

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The most general solution to the unitarity equations involving $2\to2$, $2\to3$, and $3\to3$ processes is given when the total (including all disconnected processes) $3\to3$ partial-wave amplitude S is non-normal. The solution is given in terms of characteristic operator functions, using the theory of completely nonunitary operators. It is shown, once the characteristic operator function is given, how to compute the $2\to3$ partial-wave amplitude. An appendix shows that, if S can be exponentiated and all forces are two-body forces, no particle production is allowed, i.e., the $2\to3$ partial-wave amplitude is zero.

I. INTRODUCTION

In the preceding paper[1] (hereafter referred to as paper II) it was shown that the unitarity equations involving three reacting particles could be reformulated as eigenvalue-eigenvector equations, in that one could regard the $3\to3$ partial-wave amplitude S as an operator acting on a suitably defined Hilbert space, the $2\to3$ partial-wave amplitude being an element in this space of length less than or equal to one. This $2\to3$ partial-wave amplitude was then shown to be the unique eigenvector of $S^\dagger S$, with the eigenvalue being equal to the square of the inelasticity parameter of the corres-