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Meson Field Interacting with a Heavy Source*

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The nonrelativistic theory of a meson field interacting with a heavy source is treated in a self-consistent field approximation. The energy is obtained and compared with the static model. The mesonic contribution to the mass is computed. Subtraction of the center-of-mass kinetic energy gives an improved value for the energy of the system. Excitations of the system and meson scattering are discussed briefly.

I. INTRODUCTION

A self-consistent approximation for treating meson-field interactions has been proposed and applied to the cases of a system of fermions interacting via virtual-meson exchange[1] and a meson field interacting with a static source.[2] This paper is devoted to the same approximation in the case of a meson field interacting with a massive fermion. The idea is to use as far as possible the results of static-source theory, where the approximate solution describes a static meson field around the source. If the source is allowed to move slightly, then its wave function must satisfy the Schrödinger equation in the potential due to the static meson field. The result is very similar to the solution of the polaron problem.[3] There is some resemblance to the work of Krass,[4] but his solution is limited by his use of an older version of strong-coupling theory to treat the meson field.

The Hamiltonian for the case of a scalar field interacting with a nonrelativistic fermion field is

$$\begin{aligned}
H &= H_f + H_m + H_{\text{int}}\,,\\
H_f &= \int \psi^\dagger(\vec{x})\left(E_0 - \frac{\nabla^2}{2M}\right)\psi(\vec{x})\,d^3x\,,\\
H_m &= \int \omega(\vec{k})a^\dagger(\vec{k})a(\vec{k})\,d^3k\,,\\
H_{\text{int}} &= -g\int \psi^\dagger(\vec{x})\psi(\vec{x})\varphi(\vec{x})\,d^3x\,,\\
\varphi(\vec{x}) &= \int e_{\vec{k}}(\vec{x})[\,a(\vec{k}) + a^\dagger(-\vec{k})]\,d^3k\,,\\
e_{\vec{k}}(\vec{x}) &= e^{i\vec{k}\cdot\vec{x}}/[16\pi^3\omega(\vec{k})]^{1/2}\,,\\
\omega(\vec{k}) &= (\vec{k}^2 + m^2)^{1/2}\,,
\end{aligned} \tag{1}$$

with the usual field-operator commutation relations for $\psi, \psi^\dagger, a, a^\dagger$. Section II gives the details of the self-consistent field approximation, which consists in essence of minimizing the expectation value of H over the subspace of states that are products of a factor for the meson variables and a factor for the fermion variables.

As in the nuclear shell model, the vectors in the variational space do not have simple properties under translation. Each vector is a superposition of components with momenta spanning the whole range $-\infty$ to $+\infty$. Since only the expectation value of the total momentum is capable of being given a designated value, it is clear that a Lagrange-multiplier technique is applicable, and Sec. III shows how this technique can be used to determine the mass of the self-consistent ground state. The result is that the mass is equal to the fermion mass M plus a positive mesonic contribution; the formula has also been given by Gross.[5]

The derivation of the equation for the mass utilizes the fact that in nonrelativistic field theory the mass has nothing to do with the rest energy, but is only related to the kinetic energy; it is the constant relating T, the kinetic energy, and the square of the momentum $\vec{P}$. In relativistic theory this "kinetic" mass that relates T and $\vec{P}^2$ is required to be equal to the rest energy of the state. From the results of the present paper, it may be that the kinetic mass can be calculated in a relativistic theory by the techniques used here, at least for theories with strong coupling. Since the rest mass contains an adjustable renormalization, it can be set equal to the kinetic mass. In this sense, the kinetic mass is a more physical quantity to compute.

Section IV contains a description of the excitation spectrum of the one-fermion system, both for the case of a neutral scalar meson field and for the case of an isovector meson field. The scattering of mesons by fermions is considered briefly in Sec. V.

II. DETAILS

As in I and II, let the state vector be of the form $|g\rangle|F\rangle$, where $|g\rangle$ depends on the meson field, $|F\rangle$ on the fermion field. Then

$$\langle F|H|F\rangle = \langle F|H_f|F\rangle + \int \omega(\vec{k}) b^\dagger(\vec{k}) b(\vec{k})\, d^3k$$
$$- \int \omega(\vec{k})\,|d(\vec{k})|^2\, d^3k\,,$$
$$b(\vec{k}) = a(\vec{k}) + d(\vec{k})\,,$$
$$d(\vec{k}) = \frac{g}{[16\pi^3\omega^3(\vec{k})]^{1/2}}\tilde{\rho}(\vec{k})\,, \tag{2}$$
$$\tilde{\rho}(\vec{k}) = \int e^{-i\vec{k}\cdot\vec{x}}\rho(\vec{x})\, d^3x\,,$$
$$\rho(\vec{x}) = \langle F|\psi^\dagger(\vec{x})\psi(\vec{x})|F\rangle\,.$$

It follows that $|g\rangle$ is the vacuum of the $b(\vec{k})$ mesons,

$$b(\vec{k})|g\rangle = 0 \quad \text{for all } \vec{k}\,. \tag{3}$$

As in I, the state $|F\rangle$ is given by

$$|F\rangle = c^\dagger|0\rangle\,, \tag{4}$$

where $|0\rangle$ is the fermion vacuum and

$$c = \int f^*(\vec{x})\psi(\vec{x})\, d^3x\,. \tag{5}$$

The fermion wave function $f(\vec{x})$ is the normalized function that minimizes

$$\langle H\rangle = \int f^*(\vec{x})\left(E_0 - \frac{\nabla^2}{2M}\right) f(\vec{x})\, d^3x$$
$$- \frac{\gamma}{2}\int \rho(\vec{x})\,\frac{e^{-m|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}\rho(\vec{y})\, d^3y\, d^3x\,, \tag{6}$$
$$\gamma = g^2/4\pi\,,$$
$$\rho(\vec{x}) = |f(\vec{x})|^2\,.$$

A dimensionless form is obtained by the substitution

$$\vec{x} = \vec{\xi}/\gamma M\,,$$
$$f(\vec{x}) = (\gamma M)^{3/2} h(\vec{\xi})\,, \tag{7}$$
$$\rho(\vec{x}) = (\gamma M)^3 \sigma(\vec{\xi})\,,$$

which gives

$$\langle H\rangle = E_0 + \tfrac{1}{2}\gamma^2 M\left(-\int h^*(\vec{\xi})\nabla_\xi^2 h(\vec{\xi})\, d^3\xi \right.$$
$$\left. - \int \sigma(\vec{\xi})\,\frac{e^{-m|\vec{\xi}-\vec{\eta}|/\gamma M}}{|\vec{\xi}-\vec{\eta}|}\sigma(\vec{\eta})\, d^3\xi\, d^3\eta\right). \tag{8}$$

Clearly, the minimum of $\langle H\rangle$ satisfies

$$H_m \equiv \min_h \langle H\rangle = E_0 + \gamma^2 M F(m/\gamma M)\,. \tag{9}$$

The case $m/\gamma M = 0$ is known from the study of the polaron; it was shown by Pekar[3] that

$$F(0) = -0.054\,. \tag{10}$$

More generally, $f(\vec{x})$ satisfies the equations

$$\left[-\frac{\nabla^2}{2M} + V(\vec{x})\right] f(\vec{x}) = \epsilon f(\vec{x})\,,$$
$$V(\vec{x}) = -\gamma\int \frac{e^{-m|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}\rho(\vec{y})\, d^3y\,, \tag{11}$$

which can be solved self-consistently. If ϵ is the lowest eigenvalue of $-\nabla^2/2M + V(\vec{x})$, then $|F\rangle|g\rangle$ is the lowest state of the system in this approximation, but other solutions of (11) are possible and correspond to excited states. The approximate energy of the state i is given by

$$E_0 + \gamma^2 M F_i(m/\gamma M) = E_0 + \epsilon_i + \tfrac{1}{2}\int \rho_i(\vec{x}) V_i(\vec{x})\, d^3x$$
$$= E_0 + \epsilon_i - \int \omega(\vec{k})\,|d_i(\vec{k})|^2\, d^3k\,. \tag{12}$$

Figure 1 shows $F_0(m/\gamma M)$ as calculated by solving Eq. (11) numerically for the lowest state.

It is worth noting that the state $|g\rangle$ corresponds to the presence of a static meson field around the source $\rho(\vec{x})$. The static field $\chi(\vec{x})$ is given by

$$\chi(\vec{x}) = \langle g|\varphi(\vec{x})|g\rangle\,. \tag{13}$$

It follows easily that

$$\chi(\vec{x}) = g\int \frac{e^{-m|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}\rho(\vec{y})\, d^3y\,. \tag{14}$$

The correspondence with the results of the static model starts with the static-model formula

$$\langle H\rangle_{\rm SM} = E_0 - \frac{\gamma}{(2\pi)^2}\int \frac{|\tilde{\rho}(\vec{k})|^2}{\omega^2(\vec{k})}\, d^3k\,. \tag{15}$$

If $\tilde{\rho}(\vec{k})$ is taken to be the $\tilde{\rho}(\vec{k})$ determined in the self-consistent approximation, it is easy to see that

$$-\frac{\gamma}{(2\pi)^2}\int \frac{[\tilde{\rho}(\vec{k})]^2}{\omega^2(\vec{k})}\, d^3k = \tfrac{1}{2}\int \rho(\vec{x}) V(\vec{x})\, d^3x\,, \tag{16}$$

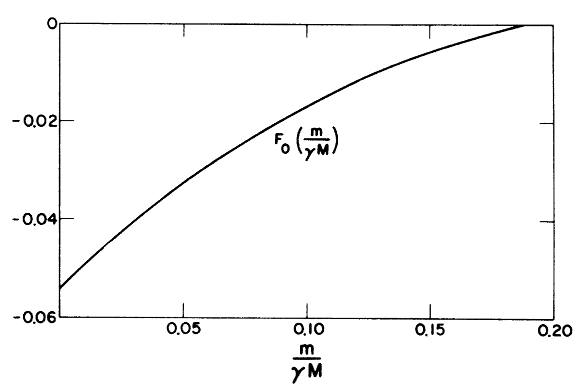

FIG. 1. The function $F_0(m/\gamma M)$ defined in Eq. (12).

so that the energy in the static model is just the potential energy of the self-consistent approximation, assuming that the self-consistent density function is used. In the limit $m/\gamma M \to 0$ it is possible to prove a virial theorem[6] by using standard techniques. The result is

$$\lim_{m/\gamma M\to 0}\left[\int f(\vec{x})\left(-\frac{\nabla^2}{2M}\right)f(\vec{x})d^3x+\tfrac{1}{4}\int\rho(\vec{x})V(\vec{x})d^3x\right]=0, \tag{17}$$

so that in this case the potential-energy contribution to $\langle H\rangle - E_0$ is $-0.109\gamma^2 M$ for both the static model and the self-consistent approximation, while the kinetic energy in the self-consistent approximation is $+0.054\gamma^2 M$. Clearly, as $m/\gamma M$ increases from zero, the magnitude of the ratio of kinetic to potential energy increases.

III. TRANSLATION

The momentum operator for the system is

$$\begin{aligned}\vec{P}^{op}&=\vec{P}_f^{op}+\vec{P}_m^{op}\\&=\frac{1}{2i}\int\psi^\dagger(\vec{x})\overleftrightarrow{\nabla}\psi(\vec{x})\,d^3x+\int\vec{k}a^\dagger(\vec{k})a(\vec{k})\,d^3k\,.\end{aligned}\tag{18}$$

The approximate state vector considered in Sec. II is not an eigenvector of $\vec{P}^{op}$, but it does give zero expectation value for $\vec{P}^{op}$. This situation is similar to that encountered in nuclear physics, where the shell-model wave function has the same properties with respect to translation. In both cases, it is a superposition of states belonging to a spectrum of eigenvalues of $\vec{P}^{op}$ that has relatively simple properties and can be written in a simple form.

It is now necessary to determine the ground-state energy E as a function of the expectation value of the momentum. Then the mass M^* of the system can be determined from

$$E(\vec{P})=E_0'+\frac{\vec{P}^2}{2M^*}+\cdots\,, \tag{19}$$

where $\vec{P}$ is written for $\langle\vec{P}^{op}\rangle$. The lack of eigenstates of $\vec{P}^{op}$ makes it clear that a Lagrange-multiplier method is called for here, with the following mode of operation. If it is desired to select from the set of states $|E_i,\vec{K}\rangle$, with E_i fixed and $\vec{K}$ taking all values, the particular state $|E_i,\vec{P}\rangle$ variationally, where $\vec{P}$ is a particular value of the momentum, this can be done by minimizing the expectation value not of H, but rather of $H-\vec{\mu}\cdot\vec{P}^{op}$, where $\vec{\mu}$ is a vector Lagrange multiplier. Over the set $|E_i,\vec{K}\rangle$ it follows that

$$\langle H-\vec{\mu}\cdot\vec{K}\rangle=E_i+\frac{\vec{K}^2}{2M^*}-\vec{\mu}\cdot\vec{K}, \tag{20}$$

so that the condition for the minimum is

$$\vec{K}=M^*\vec{\mu}\,. \tag{21}$$

The value of $\vec{\mu}$ must be chosen to make the expectation of $\vec{P}^{op}$ equal to the chosen value $\vec{P}$, namely,

$$\vec{P}=\langle\vec{P}^{op}\rangle=\vec{K}, \tag{22}$$

where the last part refers to the set $|E_i,\vec{K}\rangle$. For $\vec{\mu}=0$, it is clear that the variational principle of the preceding section is obtained. For $\vec{\mu}\neq 0$, it follows that $\vec{\mu}(\vec{P})$ is to be determined by solving

$$\langle\vec{P}^{op}\rangle=\vec{P} \tag{23}$$

for $\vec{\mu}$, where the left-hand side is a function of $\vec{\mu}$. Then M^* is given by (21), and hence

$$M^*=\lim_{\vec{\mu}\to 0}\frac{\vec{\mu}\cdot\vec{P}(\vec{\mu})}{\vec{\mu}^2}\,, \tag{24}$$

where the $\mu\to 0$ limit avoids $\vec{P}^4$, etc., terms.

There is an additional benefit here in that, if $|E_i\rangle$ is not an exact eigenstate of H and $\vec{P}^{op}$, it will have a nonzero expectation of $(P^{op})^2$; let

$$\langle(P^{op})^2\rangle\equiv\langle E_i,0|(\vec{P}^{op})^2|E_i,0\rangle\,. \tag{25}$$

Then

$$\begin{aligned}\langle E_i,\vec{P}|(P^{op})^2|E_i,\vec{P}\rangle&=\langle E_i,\vec{P}|(P^{op}-\vec{P})^2|E_i,\vec{P}\rangle+\vec{P}^2\\&=\vec{P}^2+\langle(P^{op})^2\rangle\,.\end{aligned}\tag{26}$$

The Lagrange-multiplier equations are unchanged, but the approximate energy E_0' can be improved by subtracting $\langle(P^{op})^2\rangle/2M^*$, the expectation value of the kinetic energy of the center of mass in the approximate state.

The self-consistent approximation can be applied to $H-\vec{\mu}\cdot\vec{P}^{op}$,

$$\begin{aligned}H-\vec{\mu}\cdot\vec{P}^{op}=&\int\psi^\dagger(\vec{x})\left(E_0-\frac{\nabla^2}{2M}+i\vec{\mu}\cdot\nabla\right)\psi(\vec{x})\,d^3x\\&+\int[\omega(\vec{k})-\vec{\mu}\cdot\vec{k}]a^\dagger(\vec{k})a(\vec{k})\,d^3k\\&-g\int\psi^\dagger(\vec{x})\psi(\vec{x})\varphi(\vec{x})\,d^3x\,.\end{aligned}\tag{27}$$

The equations to be solved are

$$\begin{aligned}&\left[-\frac{\nabla^2}{2M}+i\vec{\mu}\cdot\vec{\nabla}+V_{\vec{\mu}}(\vec{x})\right]f_{\vec{\mu}}(\vec{x})=\epsilon_{\vec{\mu}}f_{\vec{\mu}}(\vec{x}),\\&V_{\vec{\mu}}(\vec{x})=-g\chi_{\vec{\mu}}(\vec{x})\\&\qquad=-\gamma\int\frac{e^{-m|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}\rho_{\vec{\mu}}(\vec{y})\,d^3y\,,\\&\rho_{\vec{\mu}}(\vec{y})=|f_{\vec{\mu}}(\vec{y})|^2\,,\end{aligned}\tag{28}$$

where again

$$b_{\vec\mu}(\vec k) = a(\vec k) + d_{\vec\mu}(\vec k),$$
$$d_{\vec\mu}(\vec k) = \frac{g}{[16\pi^3\omega(\vec k)]^{1/2}[\omega(\vec k) - \vec\mu\cdot\vec k]}\,\tilde\rho_{\vec\mu}(\vec k). \tag{29}$$

Now let

$$f_{\vec\mu}(\vec x) = e^{iM\vec\mu\cdot\vec x} f'_{\vec\mu}(\vec x),$$
$$\left(-\frac{\nabla^2}{2M} + V_{\vec\mu}(\vec x)\right) f'_{\vec\mu}(\vec x) = [\epsilon_{\vec\mu}(\vec x) + \tfrac12 M_\mu{}^2] f'_{\vec\mu}(\vec x). \tag{30}$$

Since there is no direction-dependent interaction of the meson field with its source, $\rho_{\vec\mu}$ must be of the form

$$\rho_{\vec\mu}(\vec x) = \rho_0(\vec x) + \mu^2\rho_2(\vec x) + \cdots, \tag{31}$$

that is, there are no linear terms in $\vec\mu$. Similarly, $V_{\vec\mu}$ and $g_{\vec\mu}$ do not have terms linear in $\vec\mu$.

Now

$$\langle \vec P^{\,\mathrm{op}}\rangle_{\vec\mu} = \int f^*_{\vec\mu}(x)\,\frac1i\,\vec\nabla f(\vec x)\,d^3x + \int \vec k\,|d_{\vec\mu}(\vec k)|^2\,d^3k. \tag{32}$$

For small values of $\vec\mu$ it follows from (29) that

$$d_{\vec\mu}(\vec k) = d_0(\vec k)\left(1 + \frac{\vec\mu\cdot\vec k}{\omega(\vec k)}\right), \tag{33}$$

so that with (30), as $\vec\mu\to 0$,

$$\langle \vec P\rangle_{\vec\mu} \to \int f'_{\vec\mu} M\vec\mu f'_{\vec\mu}\,d^3x + \int \vec k\,|d_0(\vec k)|^2\left(1 + \frac{2\vec\mu\cdot\vec k}{\omega(\vec k)}\right)d^3k, \tag{34}$$

and hence

$$M^* = M + \tfrac23\int \frac{k^2|d_0(\vec k)|^2}{\omega(\vec k)}\,d^3k. \tag{35}$$

This formula has been derived by Gross[5] in a different way. In the present case, it follows that

$$M^* = M + \frac{\gamma}{6\pi^2}\int\frac{k^2|\tilde\rho_0(\vec k)|^2}{\omega^4(\vec k)}\,d^3k, \tag{36}$$

and since

$$(-\nabla^2 + m^2)V(\vec x) = -g^2\rho(\vec x),$$
$$g^2\,\frac{\tilde\rho_0(\vec k)}{\omega^2(\vec k)} = -\tilde V(\vec k), \tag{37}$$

it is easily seen that

$$\begin{aligned} M^* - M &= -\frac{1}{3g^2}\int V(\vec x)\nabla^2 V(\vec x)\,d^3x \\ &= -\tfrac13\int[\vec\nabla\chi(\vec x)]^2\,d^3x \\ &= -\tfrac13\int V(\vec x)\left(\rho(\vec x) + \frac{m^2}{g^2}V(\vec x)\right)d^3x. \end{aligned} \tag{38}$$

This last integral in configuration space is most convenient for numerical evaluation. The substitutions

$$\vec k = \gamma M\vec K,$$
$$\omega(\vec k) = \gamma M\nu(\vec K), \tag{39}$$
$$\nu(\vec K) = [K^2 + (m/\gamma M)^2]^{1/2}$$

in (36) give

$$\frac{M^*}{M} = 1 + \gamma^2\mathfrak{M}(m/\gamma M),$$
$$\mathfrak{M}(m/\gamma M) = \frac{1}{6\pi^2}\int\frac{K^2|\tilde\rho(\vec K)|^2}{\nu^4(\vec K)}\,d^3K. \tag{40}$$

The function $\mathfrak{M}(m/\gamma M)$ has been computed by using (38) and is shown in Fig. 2.

It is clear that the mesonic contribution to $M^* - M$ is positive. The form of (36) also indicates that in a theory with charged mesons the mesonic mass will be positive since, as was shown in I, the results for charged and neutral mesons differ mainly in the presence of a Lagrange multiplier in the charged case.

Now that M^* is available from (40), the center-of-mass kinetic energy can be evaluated. For the approximate state vector $|g\rangle|F\rangle$, it is clear that $\langle\vec P^{\,\mathrm{op}}_m\rangle = \langle\vec P^{\,\mathrm{op}}_f\rangle = 0$, so that

$$\langle(\vec P^{\,\mathrm{op}})^2\rangle = \langle(\vec P^{\,\mathrm{op}}_f)^2\rangle + \langle(\vec P^{\,\mathrm{op}}_m)^2\rangle. \tag{41}$$

In the usual way it follows that

$$\epsilon = \left\langle F\left|\frac{(\vec P^{\,\mathrm{op}}_f)^2}{2M}\right|F\right\rangle + \langle F|V(\vec x)|F\rangle,$$
$$\langle H\rangle - E_0 = \left\langle F\left|\frac{(\vec P^{\,\mathrm{op}}_f)^2}{2M}\right|F\right\rangle + \tfrac12\langle F|V(\vec x)|F\rangle, \tag{42}$$
$$\langle F|(\vec P^{\,\mathrm{op}}_f)^2|F\rangle = 2M[2\langle H\rangle - 2E_0 - \epsilon].$$

The expectation value of $(\vec P^{\,\mathrm{op}}_m)^2$ is easily calculated:

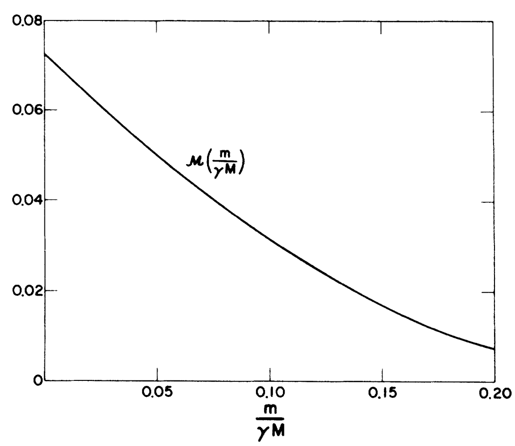

FIG. 2. The function $\mathfrak{M}(m/\gamma M)$ defined in Eq. (40).

$$\begin{aligned}\langle g|(\vec{P}_m^{op})^2|g\rangle &= \int k^2 |d(\vec{k})|^2 d^3k \\ &= \frac{\gamma}{4\pi^2}\int \frac{k^2|\tilde{\rho}(\vec{k})|^2}{\omega^3(\vec{k})} d^3k \\ &= \frac{\gamma^3 M^2}{4\pi^2}\int \frac{k^2|\tilde{\rho}(\vec{K})|^2}{\nu^3(\vec{K})} d^3K\,. \end{aligned} \tag{43}$$

Clearly

$$\begin{aligned}\langle(\vec{P}_f^{op})^2\rangle &= \gamma^2 M^2 F_{P,f}(m/\gamma M)\,,\\ \langle(\vec{P}_m^{op})^2\rangle &= \gamma^3 M^2 F_{P,m}(m/\gamma M)\,.\end{aligned} \tag{44}$$

The center-of-mass energy is not reducible to a function of $m/\gamma M$. Figure 3 shows the center-of-mass energy divided by $\gamma^2 M$ as a function of $m/\gamma M$ for various values of γ. The total energy obtained by subtracting $E_{c.m.}$ from $\langle H\rangle$ is shown in Fig. 4, with E_0 taken to be zero.

IV. EXCITATIONS

For the case of a scalar-meson field, the excited states of the system occur when the function $f(\vec{x})$ in Eqs. (11) is taken to belong to an eigenvalue ϵ higher than the lowest one. These fermion excitations can be calculated from the self-consistent field Eqs. (11); if it seems desirable, additional Lagrange multipliers can be used to ensure orthogonality of an excited state to all lower states. The static meson field in one of these states with fermion excitation is not very different from that in the lowest state, so that orthogonality of the states comes essentially from orthogonality of the fermion wave functions $f(\vec{x})$.

When the meson field is charged, it has long been known that in the static-source limit with strong coupling there are excited states of the meson cloud; the excitation energies of certain of these states have been calculated. The self-consistent-field method has also been used to treat these mesonic excitations in the static-source limit, and there is no difficulty in principle in extending the results of II to the case of a nonrelativistic recoiling source. It seems likely that the spectrum of mesonic excitations for a recoiling source is very similar to that for a static source. Moreover, it is to be expected that each state with only fermion excitation forms the basis for a band of states, each having the same fermion excitation but with varying degrees of mesonic excitation.

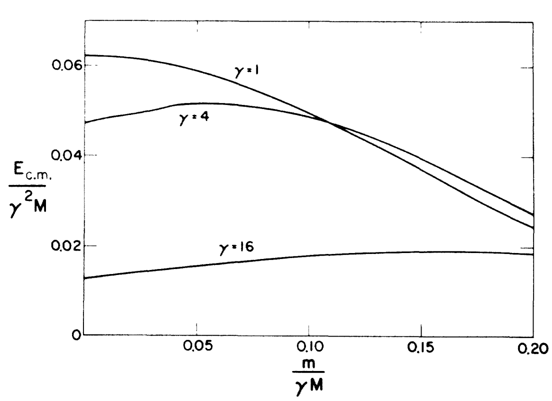

FIG. 3. The center-of-mass energy $\langle(\vec{P}^{op})^2\rangle/2M^*$ in units of $\gamma^2 M$ plotted versus $m/\gamma M$ for various values of γ.

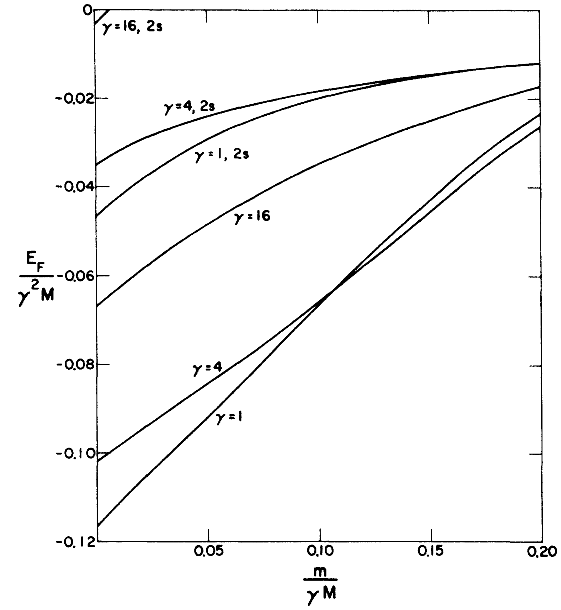

FIG. 4. The total energy E_F in units of $\gamma^2 M$ for various values of γ. The curves labeled 2s are for the excited 2s state, the others for the ground state.

Since, as was noted earlier, the ground-state energy is of the order of $-0.05\gamma^2 M$, the energy of a fermion excitation may be expected to be of order $0.02\gamma^2 M$. The energy of a mesonic excitation is of order m/γ for $\gamma > 2$. The ratio of spacing within a band to spacing between band bases is of order

$$\frac{50m}{\gamma^3 M} = \frac{50m}{\gamma^3 M^*}\times\frac{M^*}{M}\,. \tag{45}$$

Because the fermion field is spread out and is not a point source, the orthogonality difficulties pointed out in II do not arise for a recoiling source. The problems that do arise come from nonorthogonality of approximate states whose exact counterparts must be orthogonal. For example, the approximate state that has a 2s fermion with its self-consistent meson field is not orthogonal to the approximate ground state; the meson parts $|g_{1s}\rangle$ and $|g_{2s}\rangle$ are quite similar, but they are different, so that $V_{1s}(\vec{x})$ and $V_{2s}(\vec{x})$ are different and $f_{1s}(\vec{x})$ and $f_{2s}(\vec{x})$ are not orthogonal. As noted above, extra Lagrange multipliers can be used to ensure orthogonality, but this is a cumbersome procedure. It seems better to accept the fact that the exact eigenstates of H are orthogonal, but that the self-consistent approximations to the same states are not. The approximate states cannot give good results for quantities that are sensitive to the orthogonality properties of the eigenstates.

V. MESON SCATTERING

In the case of the scalar meson field, where there are no mesonic excitations, the inelastic scattering of mesons leading to final states with fermion excitation is easily calculated in the Born approximation. Consider the inelastic scattering of a meson with momentum $\vec{p}$ off the ground state, so with the final state consisting of a meson with momentum $\vec{q}$ and fermion state s_f,

$$S_{fi} = \langle \vec{q}, s_f^{\text{out}} | \vec{p}, s_0^{\text{in}} \rangle . \tag{46}$$

It is relatively simple to construct H_0 and H_{int} so that

$$\begin{aligned} &H = H_0^0 + H_{\text{int}}^0 , \\ &H_0^0 |\vec{p}, s_0^{\text{in}}\rangle = E |\vec{p}, s_0^{\text{in}}\rangle , \end{aligned} \tag{47}$$

with E given by (12). This is achieved, as in I, by expanding the fermion field operator $\psi(\vec{x})$ in terms of the eigenfunctions $f_i^0(x)$ of the operator $-\nabla^2/2M + V_0(\vec{x})$ of Eq. (11) and the residual meson field $\varphi(\vec{x}) - \chi_0(\vec{x})$ in terms of plane waves. Here the subscript (superscript) nought is used to indicate that V_0 is calculated with $\rho_0(\vec{x})$, the fermion density in the ground state s_0, and $f_i^0(\vec{x})$ is an eigenfunction of an operator that depends on $V_0(\vec{x})$; similarly, $\chi_0(\vec{x})$ is the static-meson field in the ground state s_0. It is clear that $b(\vec{k})$, $d(\vec{k})$, etc., all require such subscripts. When the expansions are substituted into H, the result is

$$\begin{aligned} H_0^0 &= E - \epsilon_0^0 + \sum_i \epsilon_i^0 c_0^\dagger(i) c_0(i) \\ &\quad + \int \omega(\vec{k}) b_0^\dagger(\vec{k}) b_0(\vec{k}) \, d^3k , \\ H_{\text{int}}^0 &= -g \sum_{ij} \int d^3k \, C_{ij}^0(\vec{k}) : c_0^\dagger(j) c_0(i) : b_0(\vec{k}) + \text{H.c.} , \\ C_{ij}^0(\vec{k}) &= \int e_{\vec{k}}^*(\vec{x}) f_i^{0*}(\vec{x}) f_j^0(\vec{x}) \, d^3x , \end{aligned} \tag{48}$$

with $\sum_i$ representing summation over discrete and integration over continuum states. The operator $c_0(i)$ annihilates a fermion in the state $f_i^0(\vec{x})$.

In terms of the b and c operators

$$\begin{aligned} &|\vec{p}, s_0\rangle_a = b_0^\dagger(\vec{p}) c_0^\dagger(0) |g_0\rangle , \\ &|\vec{q}, s_f\rangle_a = b_f^\dagger(\vec{q}) c_f^\dagger(f) |g_f\rangle , \end{aligned} \tag{49}$$

where the subscript a denotes the self-consistent-field approximation to the state in question. Now the Born approximation to T is

$$\begin{aligned} T_{fi}^B &= {}_a\langle \vec{q}, s_f | H_{\text{int}}^0 | \vec{p}, s_0 \rangle_a \\ &= -g \sum_{i,j} \int \langle g_f | b_f(\vec{q}) b_0(\vec{k}) b_0^\dagger(\vec{p}) | g_0 \rangle C_{ij}^0(\vec{k}) \langle 0 | c_f(f) : c_0^\dagger(j) c_0(i) : c_0^\dagger(0) | 0 \rangle \, d^3k \\ &\quad - g \sum_{i,j} \int \langle g_f | b_f(\vec{q}) b_0^\dagger(\vec{k}) b_0^\dagger(\vec{p}) | g_0 \rangle C_{ij}^{0*}(\vec{k}) \langle 0 | c_f(f) : c_0^\dagger(i) c_0(j) : c_0^\dagger(0) | 0 \rangle \, d^3k . \end{aligned} \tag{50}$$

For the fermion matrix elements, it is easy to see that

$$\begin{aligned} \langle 0 | c_f(f) : c_0^\dagger(j) c_0(i) : c_0^\dagger(0) | 0 \rangle &= \langle 0 | c_f(f) c_0^\dagger(j) | 0 \rangle \delta_{i,0} \\ &= \delta_{i,0} \int f_f^{f*}(\vec{x}) f_j^0(\vec{x}) \, d^3x . \end{aligned} \tag{51}$$

The meson matrix elements are

$$\begin{aligned} \langle g_f | b_f(\vec{q}) b_0(\vec{k}) b_0^\dagger(\vec{p}) | g_0 \rangle &= \langle g_f | b_f(\vec{q}) | g_0 \rangle \, \delta(\vec{k} - \vec{p}) \\ &= [d_f(\vec{q}) - d_0(\vec{q})] \langle g_f | g_0 \rangle \, \delta(\vec{k} - \vec{p}) , \\ \langle g_f | b_f(\vec{q}) b_0^\dagger(\vec{k}) b_0^\dagger(\vec{p}) | g_0 \rangle &= \delta(\vec{k} - \vec{q}) [d_0^*(\vec{p}) - d_f^*(\vec{p})] \langle g_f | g_0 \rangle + \delta(\vec{q} - \vec{p}) [d_0^*(\vec{k}) - d_f^*(\vec{k})] \langle g_f | g_0 \rangle \\ &\quad + [d_f(\vec{q}) - d_0(\vec{q})] [d_0^*(\vec{p}) - d_f^*(\vec{p})] [d_0^*(\vec{k}) - d_f^*(\vec{k})] \langle g_f | g_0 \rangle , \\ \langle g_f | g_0 \rangle &= \exp\left(-\tfrac{1}{2} \int | d_f(\vec{k}) - d_0(\vec{k}) |^2 \, d^3k \right) . \end{aligned}$$

Elastic scattering is a second-order process. It is best calculated in this case by using unitarity and the inelastic T matrix to give a dispersion relation for the real part of the elastic T matrix.

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[6]This was pointed out to me by D. Liberman.

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Massive Quarks and Deep-Inelastic Phenomena*

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A possible theory of hadrons in terms of a massive-quark field theory is proposed. Some consequences for deep-inelastic phenomena are worked out.

I. INTRODUCTION

The problematic aspects indicated by SLAC experiments[1] and their scaling behavior have insistently hinted at the relevance of a picture of hadrons in terms of constituents, the most popular ones being Feynman's partons.[2] However, these objects seem to have puzzling and disturbing characteristics, such as their small mass and their invisibility.[3]

Attempts to construct a quark field theory of the hadrons have recently appeared in the literature, most notably by the Cambridge school[4]; unfortunately these approaches do not provide any answer to the aforementioned problems. In fact they look all the more unsatisfactory if one tries to think about the alleged constituents of the hadrons in realistic terms, and to use them to build up a coherent picture which explains the basic facts of scaling without running into serious contradictions with experiments.

A possible theory of hadrons in terms of a massive-quark field theory was proposed in 1969.[5] The preliminary results of that study exhibited some attractive features, but the details of the particular model considered were somehow unrealistic.

The motivation for looking at a field theory of quarks is the standard set of arguments in favor of the quark model (e.g., SU_3, current algebra, etc.), which suggest that the quark is a relevant degree of freedom of hadronic matter. On the other hand, the impressive list of negative results in quark searches[6] requires the quarks, if they exist at all as particles, to have a very large mass [>25 GeV according to CERN Intersecting Storage Rings (ISR) experiments[6]], or else the forces have to be so singular as to prevent the quarks from coming out of hadrons even though they can be kinematically produced.[7] We do not think that the latter possibility is particularly appealing in view of the linearity of Regge trajectories.[8]

If we then subscribe to a massive-quark field theory (one can eventually take the limit $M_q \to \infty$ and get rid of "real" quarks) we are led to consider dynamical field-theoretical equations, such as the Bethe-Salpeter (BS) equation, to describe the binding of quarks to form physical particles.[9] In order to solve a BS equation we need as an input a BS kernel, and it is here that dynamics creeps in in a crucial way. We feel, however, that the dynamical situation in the case of superstrong (in the limit infinite) binding may well be considerably simpler than in the intermediate-binding case. This fact appears to be suggested by the regularities of the spectrum of hadrons, the smallness of their widths as compared with their masses, exchange degeneracy, linearity of Regge trajectories, the nonexistence of exotic states, etc.

Thus one can try to guess a simple form for the kernel in the BS equation for the $q\bar{q}$ scattering amplitude which embodies these basic features, and this was indeed taken up in Ref. 5.

From these attempts we are now going to abstract some properties which, in our opinion, should be at the basis of any realistic quark field theory of hadronic matter.

(i) Hadronic wave functions. We can associate