# Dispersion Calculation of the Anomalous Magnetic Moment of the Electron

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Using a self-consistent electron-positron elastic scattering amplitude developed previously, we present a cutoff-independent dispersion calculation of the two-body-intermediate-state correction to the electron magnetic moment. The complete second-order (in $\alpha$, the fine-structure constant) contribution to the anomalous part of the magnetic moment, $\kappa = \frac{1}{2}(g-2)$, due to two-body exchange is $\kappa^{(2)} - \kappa^{(1)} = (-0.940)\alpha^2/\pi^2$. The coefficient of the second-order term has the proper sign but is considerably larger than the experimental value (−0.328). The large difference between our result and the correct value indicates that many-body effects may be as important as two-body exchange in dispersion calculations for electrodynamics. We conclude that the published cutoff dispersion-theoretic estimates of the higher-order corrections to the electron magnetic moment based only on a consideration of the two-body intermediate state are not likely to be correct.

Using a self-consistent electron-positron elastic scattering amplitude developed previously,[1] we present a dispersion-theoretic calculation of the two-body-intermediate-state contribution to the electron anomalous magnetic moment. The result is cutoff-independent and includes all second-order (in $\alpha$, the fine-structure constant) corrections due to two-body exchange. This note, then, represents a first step toward an exact dispersion calculation of the anomalous magnetic moment of the electron which does not refer to the usual perturbation formalism. We note that there have been other attempts toward a dispersion calculation of the electron magnetic moment. However, the most recent of these have either relied on cutoffs to regulate the behavior of the dispersion integrals[2,3] or they have been dependent on an analysis of the discontinuities of Feynman graphs to determine the necessary absorptive parts.[4] Also, the intent of Refs. 2–3 is not to attempt an exact calculation but to achieve an estimate of the higher-order corrections without the complexity of a full calculation. In any case, the work presented here is completely independent of the usual formulation of perturbation theory and is free of arbitrary cutoffs. Instead, we employ the general formalism of analytic $S$-matrix theory. For example, in Ref. 1, using a low-energy theorem, analyticity, and a requirement of self-consistency, we were able to construct a satisfactory second-order electron-positron elastic scattering amplitude. In addition to analyticity, cutoff independence, and self-consistency, the amplitude exhibits Regge asymptotic behavior and a finite Jacob-Wick expansion. The purpose of this note is to apply the results of Ref. 1 to a calculation of the electron magnetic moment. Thus, in a sense, the calculation presented here represents a continuation of our previous work.

At this point, one might question the necessity to present a partial evaluation of the second-order term, even though the approach is based on analytic $S$-matrix theory rather than the usual Feynman-diagram techniques, since the perturbation calculation of the third-order correction to the electron magnetic moment is essentially complete.[5] There are, in fact, several reasons why the analysis presented here should be of value. In the first place, one of the principal reasons for doing dispersion calculations in electrodynamics is to test different methods on familiar problems for which solutions are available and to determine precisely what are the predictions of the new approach. We also note that since an individual Feynman graph may contribute to several unitarity diagrams, the results of this calculation could not have been anticipated by any argument based on a study of perturbation theory. There is no simple way to infer the value of a particular unitarity diagram to a given order from the sum of Feynman diagrams. Thus, in a sense our results are new. Since our calculation is free from cutoff ambiguities and, presumably, exact, we feel that it may aid in uncovering the foundations of an analytic $S$-matrix theory of electrodynamics. A final consideration is that the results presented here strongly indicate that previous attempts[2,3] to estimate the higher-order corrections to the electron magnetic moment using dispersion theory are not likely to be correct. Our work indicates that the many-body intermediate states neglected in these calculations may be fully as important as the two-body state.[6] We feel it is appropriate at this time to indicate that a failure to give a correct estimate of the third-order term does not imply a general failure of dispersion theory but only of the particular approximations which have been employed previously.

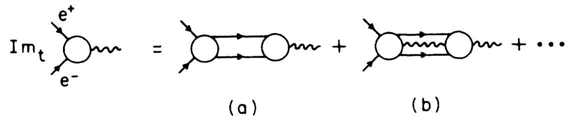

FIG. 1. Unitarity diagrams for the electron vertex amplitude: (a) Two-body (electron-positron) intermediate state, (b) three-body contribution.

Our approach to the dispersion calculation of the electron anomalous magnetic moment is essentially that of Drell and Zachariasen.[7] Their paper contains most of the necessary background to our work although their aim is somewhat different. Thus, we will consider unitarity contributions to the imaginary parts of the electron form factors as given in Fig. 1. We note that each successive diagram is of order $\alpha$ with respect to the previous one due to the two additional vertices, but that each diagram may contribute to all higher orders. In this note we will consider only the two-body contribution, diagram 1(a). We will, however, go beyond the lowest-order contribution of this diagram and evaluate exactly its second-order corrections. We can do this relatively easily since an analytic second-order electron-positron elastic scattering amplitude is available from our previous work. Since, presumably, the complete second-order contribution to the magnetic moment is known exactly from perturbation theory, we thus obtain the magnitude of the three-body contribution to the second-order magnetic moment without the necessity of a laborious calculation. Eventually, of course, it will be necessary to evaluate the second-order contribution of diagram 1(b) explicitly to show completely that our dispersion approach will give the correct experimental value. However, in order to evaluate this term we would have to extend our formalism for the construction of self-consistent electromagnetic scattering amplitudes to include production processes. Although work is in progress toward that end, it will be some time before it is completed. Meanwhile, we feel that the results of this work are significant enough to warrant publication at this time.

In order to evaluate the unitarity diagrams of Fig. 1, we need an explicit expression for the vertex amplitude. It is convenient to work in the helicity representation so that the vertex can be written

$$\langle\epsilon|A(e^+ + e^- \to \gamma)|\lambda_1\lambda_2\rangle = 3\langle\epsilon|a(t)|\lambda_1\lambda_2\rangle D^{J=1}_{\lambda\epsilon}(\theta,\phi) \,, \tag{1}$$

where the $\langle\epsilon|a(t)|\lambda_1\lambda_2\rangle$ are linear combinations of the electron form factors and the $D^J_{\lambda\epsilon}(\theta,\phi)$ are elements of an irreducible representation of the rotation group. Equation (1) is just the Jacob-Wick expansion[8] of the vertex amplitude in which $\theta$ and $\phi$ specify the angles of incidence of the incoming electron (positron). The photon, without loss of generality, is assumed to be emitted along the positive $z$ axis. The explicit form of the $\langle\epsilon|a(t)|\lambda_1\lambda_2\rangle$ in terms of the form factors can be obtained relatively simply from the expression for the vertex given in Ref. 7. Also, in order to evaluate diagram 1(a), we need the Jacob-Wick expansion of the elastic electron-positron scattering amplitude, which we write in the form

$$\langle\lambda_3\lambda_4|H(s,t,u)|\lambda_1\lambda_2\rangle = \sum_{J,M}(2J+1)\langle\lambda_3\lambda_4|h^J(t)|\lambda_1\lambda_2\rangle \times D^{J*}_{M\mu}(\theta',\phi')D^J_{M\lambda}(\theta,\phi) \,. \tag{2}$$

Using (1) and (2) and the orthogonality properties of the $D$ matrices, the unitarity diagram 1(a) can be evaluated to yield the equation

$$\operatorname{Im}\langle\epsilon|a(t)|\lambda_1\lambda_2\rangle = 2\pi\rho\operatorname{Re}\sum_{\lambda_3,\lambda_4}\langle\epsilon|a^*(t)|\lambda_3\lambda_4\rangle\langle\lambda_3\lambda_4|h^{J=1}(t)|\lambda_1\lambda_2\rangle \,. \tag{3}$$

TABLE I. $P(t)$, $R(t)$, and $G(t)$ are defined in Eqs. (7), (8) of the text. The functions of which $G(t)$ is a linear combination are displayed in Eq. (9).

| $\langle\lambda_3\lambda_4\|\lambda_1\lambda_2\rangle$ | $P(t)$ | $R(t)$ | $G(t)$ |
|---|---|---|---|
| $\langle\frac{1}{2}\frac{1}{2}\|\frac{1}{2}\frac{1}{2}\rangle$ | $(\gamma-1)(t-2)$ | $4(\gamma-\frac{23}{18})(t-2)-\frac{4}{3}q^{-2}$ | $-\frac{1}{3}[\Gamma^{(1)}(t)+2q^2\kappa^{(1)}F_2^{(1)}(t)]$ |
| $\langle\frac{1}{2}\frac{1}{2}\|-\frac{1}{2}-\frac{1}{2}\rangle$ | 0 | $-\frac{4}{3}q^{-2}$ | $-\frac{1}{3}[\Gamma^{(1)}(t)+2q^2\kappa^{(1)}F_2^{(1)}(t)]$ |
| $\langle\frac{1}{2}\frac{1}{2}\|-\frac{1}{2}\frac{1}{2}\rangle$ | $-\frac{W}{2\sqrt{2}}$ | $\frac{-W}{2\sqrt{2}}[-4(\gamma-\frac{23}{18})+\frac{4}{3}q^{-2}]$ | $\frac{W}{2\sqrt{2}}(\frac{2}{3})[\Gamma^{(1)}(t)+q^2\kappa^{(1)}F_2^{(1)}(t)]$ |
| $\langle-\frac{1}{2}\frac{1}{2}\|-\frac{1}{2}\frac{1}{2}\rangle$ | $\frac{1}{4}(4\gamma-3)(t-2)$ | $-\frac{1}{2}(t-2)[-6(\gamma-\frac{23}{18})+\frac{2}{3}q^{-2}]$ | $-\frac{2}{3}(\frac{1}{4}t)\Gamma^{(1)}(t)$ |
| $\langle-\frac{1}{2}\frac{1}{2}\|\frac{1}{2}-\frac{1}{2}\rangle$ | $\frac{1}{2}$ | $-2(\gamma-\frac{23}{18})-\frac{2}{3}q^{-2}$ | $-\frac{2}{3}(\frac{1}{4}t)\Gamma^{(1)}(t)$ |

In Eq. (3), $\rho(t)=[q/(2\pi)^2W]\,\theta(t-4)$ is the two-body phase space. $\theta(z)$ is the Heaviside function and $q=\frac{1}{2}(t-4)^{1/2}$ is the c.m. momentum of the incoming electron (positron), $W=\sqrt{t}$ is the total energy. We choose units such that $m_e=\hbar=c=1$. Note that in order to ensure reality of the imaginary part of the form factors we have chosen one-half the sum of incoming plus outgoing boundary conditions for the sum over intermediate states. This is necessary in any approximation which neglects some intermediate states or uses approximate amplitudes. Finally, Eq. (3) can be solved to give the following expressions for the form factors

$$\begin{aligned}
q^2\,\mathrm{Im}F_1(t) &= 2\pi\rho\,\mathrm{Re}[A_{11}(t)F_1^*(t)+A_{12}(t)\kappa F_2^*(t)]\\
&= 2\pi\rho\,\mathrm{Re}\left(\left[\tfrac{1}{4}tf_{22}^1(t)-f_{11}^1(t)+\frac{W}{2\sqrt{2}}f_{12}^1(t)\right]F_1^*(t)+\tfrac{1}{4}t\left[f_{22}^1(t)-f_{11}^1(t)+\frac{8-t}{2\sqrt{2}\,W}f_{12}^1(t)\right]\kappa F_2^*(t)\right)\,,\\
-q^2\kappa\,\mathrm{Im}F_2(t) &= 2\pi\rho\,\mathrm{Re}[A_{21}(t)F_1^*(t)+A_{22}(t)\kappa F_2^*(t)]\\
&= 2\pi\rho\,\mathrm{Re}\left(\left[f_{22}^1(t)-f_{11}^1(t)+\frac{t-2}{\sqrt{2}\,W}f_{12}^1(t)\right]F_1^*(t)+\left[f_{22}^1(t)-\tfrac{1}{4}tf_{11}^1(t)+\frac{W}{2\sqrt{2}}f_{12}^1(t)\right]\kappa F_2^*(t)\right)\,.
\end{aligned}\tag{4}$$

In Eq. (4)

$$\begin{aligned}
f_{11}^J(t) &= \langle\tfrac{1}{2}\tfrac{1}{2}|h^J(t)|\tfrac{1}{2}\tfrac{1}{2}\rangle+\langle\tfrac{1}{2}\tfrac{1}{2}|h^J(t)|-\tfrac{1}{2}-\tfrac{1}{2}\rangle\,,\\
f_{12}^J(t) &= 2\langle\tfrac{1}{2}\tfrac{1}{2}|h^J(t)|-\tfrac{1}{2}\tfrac{1}{2}\rangle\,,\\
f_{22}^J(t) &= \langle-\tfrac{1}{2}\tfrac{1}{2}|h^J(t)|-\tfrac{1}{2}\tfrac{1}{2}\rangle+\langle-\tfrac{1}{2}\tfrac{1}{2}|h^J(t)|\tfrac{1}{2}-\tfrac{1}{2}\rangle
\end{aligned}\tag{5}$$

are parity-conserving partial-wave amplitudes[9] (parity and total spin diagonal), $F_1(t)$ and $F_2(t)$ are the usual charge and magnetic moment form factors, respectively. $\kappa=\frac{1}{2}(g-2)$ is the anomalous part of the magnetic moment. We note that only the $J=1$ amplitudes contribute to the form factors. Moreover, the singlet state and the triplet $J=L$,

$$\begin{aligned}
f_0^J(t) &= \langle\tfrac{1}{2}\tfrac{1}{2}|h^J(t)|\tfrac{1}{2}\tfrac{1}{2}\rangle-\langle\tfrac{1}{2}\tfrac{1}{2}|h^J(t)|-\tfrac{1}{2}-\tfrac{1}{2}\rangle\,,\\
f_1^J(t) &= \langle-\tfrac{1}{2}\tfrac{1}{2}|h^J(t)|-\tfrac{1}{2}\tfrac{1}{2}\rangle-\langle-\tfrac{1}{2}\tfrac{1}{2}|h^J(t)|\tfrac{1}{2}-\tfrac{1}{2}\rangle\,,
\end{aligned}\tag{6}$$

do not appear in Eq. (4). We see that our expressions (4)–(5) are essentially the same as Eqs. (18)–(20) of Drell and Zachariasen. However, we do not write the amplitudes $f_{ik}^J(t)$ in terms of triplet phase shifts and a mixing angle since the electron-positron elastic scattering amplitude does not exactly satisfy a simple elastic unitarity condition at any energy. This is due not only to vacuum-polarization effects but also to the fact that elastic and inelastic thresholds coincide. In view of its simplicity, one might question why Eq. (4) has not been used more extensively in dispersion calculations of the electron magnetic moment. (We note that an analytic solution can be simply obtained in terms of the helicity amplitudes.[10]) The answer is simple; the usual perturbation expansion of the electron-positron elastic scattering amplitude does not have a finite Jacob-Wick expansion. Thus, in using (essentially) this equation Drell and Zachariasen were forced to employ the Coulomb amplitude obtained from the nonrelativistic Schrödinger equation to do the integration over the forward angles. The results of this integration were then matched to the results of an integration over the nonforward angles using the relativistic Born term. Even for the lowest-order corrections to the form factors this procedure is rather cumbersome and, in any case, it is not possible to obtain the higher-order contributions in this fashion. Thus, it has not heretofore been possible to do a pure dispersion calculation of the higher-order corrections to the electron magnetic moment without introducing cutoffs. (Note that while the procedure of using a "sidewise" dispersion relation introduced by Drell and Pagels obviates the difficulty in doing the angular integrations, the high-energy behavior which results necessitates a cutoff.) However, in a recent paper[1] we introduced a self-consistent, analytic electron-positron elastic scattering amplitude which is accurate through second order. Moreover, this amplitude is cutoff-independent and has a well-defined Jacob-Wick expansion. We also note that while the perturbation expansion of the electron-positron elastic scattering amplitude does not have a finite Jacob-Wick expansion, the partial-wave projections of our self-consistent amplitude can be expanded in a regular power series in $\alpha$. Thus, it provides a suitable basis for the calculation of the contribution of diagram 1(a) to the electron magnetic moment. It is only necessary to project out the $J=1$ helicity amplitudes; the form factors can then be evaluated using Eq. (4).

In considering the partial-wave projections of the scattering amplitude given in Ref. 1, we note first that only the real parts are necessary for the evaluation of diagram 1(a) through second order. The argument is elementary. If we remember that $F_1(0)=F_2(0)=1$, then we see that in Eq. (3) the real part of a $\langle\epsilon|a(t)|\lambda_3\lambda_4\rangle$ is on the order of unity. According to Fig. 1, however, the imaginary part is of first order in $\alpha$. Since the lowest nonvanishing contribution to the imaginary part of the electron-positron elastic scattering amplitude is of second order, we find that, in taking the real part of the product and neglecting higher-order terms, only the real part of the helicity amplitude contributes. This result will simplify our procedure somewhat. In the following we will not present the details of the calculation of the partial-wave projections of the spin-$\frac{1}{2}$–spin-$\frac{1}{2}$ elastic scattering amplitude given in Ref. 1. The process is exceedingly tedious and the general problem has been treated in detail in the paper by Goldberger, Grisaru, MacDowell, and Wong.[9] In our case, most of the integrals encountered are familiar to dispersion theorists or are given in Appendix A of Ref. 1. In any case, the integrations are straightforward. We find that, neglecting terms of higher order,[11] the second-order expression for the real part of the $J=1$ helicity amplitudes can be written

$$\mathrm{Re}\langle\lambda_3\,\lambda_4\,|\,h^1(t)\,|\,\lambda_1\,\lambda_2\rangle=\frac{4\pi\alpha}{4q^2}\,\mathrm{Re}\langle\lambda_3\,\lambda_4\,|\,P(t)-\kappa^{(1)}Q(t)\,|\,\lambda_1\,\lambda_2\rangle+\frac{4\pi\alpha}{t}\,\Gamma_1^{(1)}(0)\,\mathrm{Re}\langle\lambda_3\,\lambda_4\,|\,G(t)\,|\,\lambda_1\,\lambda_2\rangle\ . \tag{7}$$

$\Gamma_1^{(1)}(0)=1+\kappa^{(1)}$, where $\kappa^{(1)}=\alpha/2\pi$ is the Schwinger correction to the electron magnetic moment. $P(t)$ is a polynomial in $t$; $G(t)$ is essentially a linear combination of $\Gamma^{(1)}(t)$, the first-order vertex function and $F_1^{(1)}(t)$ and $F_2^{(1)}(t)$, the first-order electron form factors. $Q(t)$ has the form

$$Q(t)=R(t)+\sum_{k=0}^{3}\beta_k I^k(t)\ , \tag{8}$$

where $\beta_0=2(\gamma-\frac{11}{12})$, $\beta_1=-2(2\gamma-\frac{10}{3})$, $\beta_2=\frac{8}{3}$, $\beta_3=-2$. $\gamma=0.577\ldots$ is the Euler constant. The $I^k(t)$ are quadratic functions of Legendre functions of the second kind. The argument of each $Q_l(z)$ is $1/v=E/q$, the reciprocal of the relativistic velocity of the incoming (outgoing) electron (positron). $R(t)$ is a ratio of polynomials. $P(t)$, $R(t)$, $G(t)$, and the $I^k(t)$ are given in Tables I and II. For completeness, we also give the first-order expressions for $\Gamma^{(1)}(t)$, $F_1^{(1)}(t)$, and $F_2^{(1)}(t)$, although these are also given in Ref. 1. We have

TABLE II. The $I^k(t)$ are defined in Eq. (8). Each $I^k(t)$ is a quadratic function of the $Q_l$, where $Q_l=Q_l(1/v)$ is a Legendre function of the second kind. The argument of the $Q_l$ is the reciprocal of the relativistic velocity of the incoming (outgoing) electron (positron).

| $\langle\lambda_3\lambda_4\|\lambda_1\lambda_2\rangle$ | $I^0(t)$ | $I^1(t)$ |
|---|---|---|
| $\langle\frac{1}{2}\frac{1}{2}\|\frac{1}{2}\frac{1}{2}\rangle$ | $\frac{2}{3}Q_0^2+2(t-3)Q_1^2+\frac{4}{3}Q_2^2$ | $(t-2)Q_1+\frac{t-2}{q^2}Q_0^2+\frac{1}{q^2}Q_1^2$ |
| $\langle\frac{1}{2}\frac{1}{2}\|-\frac{1}{2}-\frac{1}{2}\rangle$ | $\frac{2}{3}Q_0^2-2Q_1^2+\frac{4}{3}Q_2^2$ | $\frac{1}{q^2}Q_1^2$ |
| $\langle\frac{1}{2}\frac{1}{2}\|-\frac{1}{2}\frac{1}{2}\rangle$ | $\frac{W}{2\sqrt{2}}(-\frac{4}{3})(Q_0^2-Q_2^2)$ | $\frac{W}{2\sqrt{2}}\left(\frac{1}{q^2}\right)(Q_0^2+Q_1^2)$ |
| $\langle-\frac{1}{2}\frac{1}{2}\|-\frac{1}{2}\frac{1}{2}\rangle$ | $(t-2)[\frac{2}{3}Q_0^2+Q_1^2+\frac{1}{3}Q_2^2]$ | $(t-2)\left[Q_1+\frac{1}{4q^2}(3Q_0^2+Q_1^2)\right]$ |
| $\langle-\frac{1}{2}\frac{1}{2}\|\frac{1}{2}-\frac{1}{2}\rangle$ | $\frac{4}{3}Q_0^2-2Q_1^2+\frac{2}{3}Q_2^2$ | $-\frac{1}{2q^2}(Q_0^2-Q_1^2)$ |
| $\langle\lambda_3\lambda_4\|\lambda_1\lambda_2\rangle$ | $I^2(t)$ | $I^3(t)$ |
| $\langle\frac{1}{2}\frac{1}{2}\|\frac{1}{2}\frac{1}{2}\rangle$ | $\frac{1}{60}(t-2)[6Q_1-Q_3]+\frac{t-1}{2q^2}Q_1+\frac{1}{2q^4}Q_0^2$ | $(t-2)Q_1+\frac{t-2}{q^2}Q_0^2+\frac{t-2}{2q^2}Q_1^2$ |
| $\langle\frac{1}{2}\frac{1}{2}\|-\frac{1}{2}-\frac{1}{2}\rangle$ | $\frac{1}{2q^2}Q_1+\frac{1}{2q^4}Q_0^2$ | $\frac{t}{4q^2}Q_1^2$ |
| $\langle\frac{1}{2}\frac{1}{2}\|-\frac{1}{2}\frac{1}{2}\rangle$ | $\frac{W}{2\sqrt{2}}\left(\frac{1}{2q^4}\right)(2q^2Q_1+Q_0^2)$ | $\frac{W}{2\sqrt{2}}\left(\frac{t}{4q^2}\right)(Q_0^2+Q_1^2)$ |
| $\langle-\frac{1}{2}\frac{1}{2}\|-\frac{1}{2}\frac{1}{2}\rangle$ | $(t-2)\left[\frac{1}{60}(6Q_1-Q_3)+\frac{1}{2q^2}Q_1+\frac{1}{8q^4}Q_0^2\right]$ | $(t-2)\left[Q_1+\frac{1}{4q^2}(3Q_0^2+Q_1^2)\right]$ |
| $\langle-\frac{1}{2}\frac{1}{2}\|\frac{1}{2}-\frac{1}{2}\rangle$ | $\frac{1}{4q^4}Q_0^2$ | $\frac{-t}{8q^2}(Q_0^2-Q_1^2)$ |

TABLE III. $\mathcal{P}(t)$, $\mathcal{R}(t)$, and $\mathcal{S}(t)$ are defined in Eqs. (10), (11).

| $q^{-2}A_{ij}$ | $\mathcal{P}(t)$ | $\mathcal{R}(t)$ | $\mathcal{S}(t)$ |
|---|---|---|---|
| $q^{-2}A_{11}$ | $(\gamma-1)(t-2)+\frac{1}{4}t$ | $(\gamma-\frac{23}{18})(3t-8)-\frac{1}{3}(t+8)q^{-2}$ | $\frac{1}{3}[-(t+2)\Gamma^{(1)}(t)+\frac{1}{2}(3t-8)\kappa^{(1)}F_2^{(1)}(t)]$ |
| $q^{-2}A_{12}$ | $\frac{1}{2}t$ | 0 | $-2(\frac{1}{4}t)\Gamma^{(1)}(t)$ |
| $q^{-2}A_{21}$ | $-1$ | $4(\gamma-\frac{23}{18})-4q^{-2}$ | $\frac{1}{3}(t+2)\kappa^{(1)}F_2^{(1)}(t)$ |
| $q^{-2}A_{22}$ | $(1-\gamma)(t-2)$ | 0 | 0 |

$$F_2^{(1)}(t)=F(1,1;\tfrac{3}{2};\tfrac{1}{4}t)\ ,$$
$$F_1^{(1)}(t)=1+\kappa^{(1)}F_2^{(1)}(t)\sum_{k=0}^{2}\gamma_k t^{1-k}-\kappa^{(1)}\sum_{k=1}^{2}\gamma_k t^{1-k}-\tfrac{4}{9}\kappa^{(1)}\ , \tag{9}$$
$$\Gamma^{(1)}(t)=\Gamma_1^{(1)}(0)+\kappa^{(1)}F_2^{(1)}(t)\sum_{k=0}^{2}\beta_k t^{1-k}-\kappa^{(1)}\sum_{k=1}^{2}\beta_k t^{1-k}-\tfrac{4}{9}\kappa^{(1)}\ ,$$

where $F(\alpha,\beta;\gamma;z)$ is a Gauss hypergeometric function, and $\gamma_0=\frac{1}{2}\beta_0-\frac{1}{6}$, $\gamma_1=\frac{1}{2}\beta_1-\frac{2}{3}$, $\gamma_2=\frac{1}{2}\beta_2+\frac{4}{3}$. The $\beta_k$ are introduced in Eq. (8).

Finally, the particular linear combinations of helicity amplitudes which appear as coefficients of the form factors in Eq. (4) can be written

$$\frac{1}{q^2}\mathrm{Re}A_{ij}(t)=\frac{4\pi\alpha}{4q^2}\mathrm{Re}[\mathcal{P}_{ij}(t)-\kappa^{(1)}\mathcal{Q}_{ij}(t)]+\frac{4\pi\alpha}{t}\Gamma_1^{(1)}(0)\mathrm{Re}\mathcal{S}_{ij}(t)\ , \tag{10}$$

where

$$\mathcal{Q}_{ij}(t)=\mathcal{R}_{ij}(t)+\sum_{k=0}^{3}\beta_k\mathcal{J}_{ij}^{k}(t)\ . \tag{11}$$

The explicit forms of the $\mathcal{P}_{ij}(t)$, $\mathcal{R}_{ij}(t)$, $\mathcal{S}_{ij}(t)$, and $\mathcal{J}_{ij}^k(t)$ are given in Tables III and IV. We note that $\mathcal{P}_{ij}(t)$, $\mathcal{Q}_{ij}(t)$, and $\mathcal{S}_{ij}(t)$ are finite[12] at $t=4$, so that the coefficients $\rho(t)q^{-2}A_{ij}(t)$ are well behaved at threshold. Thus, there is no reason to introduce a photon mass or infrared cutoff. The asymptotic behavior of these terms, neglecting logarithmic contributions, is such that $q^{-2}A_{21}(t)\to t^{-1}$ for large $t$, while the three remaining coefficients approach constant values. This implies that $F_1(t)$ requires one subtraction (which we have already noted in Ref. 1) while $F_2(t)$ satisfies an unsubtracted dispersion relation. This corresponds to the usual situation in which the electron charge is

TABLE IV. The $\mathcal{J}^k(t)$ are defined in Eq. (11). Again, each $\mathcal{J}^k(t)$ is a quadratic function of the $Q_l$. Contributions to $A_{12}$ and $A_{22}$ are omitted since they would only contribute to third order.

| $q^{-2}A_{ij}$ | $\mathcal{J}^0(t)$ | $\mathcal{J}^1(t)$ |
|---|---|---|
| $q^{-2}A_{11}$ | $\frac{2}{3}(t+2)Q_0^2+(t-8)Q_1^2+\frac{1}{3}(t+8)Q_2^2$ | $(t+2)Q_1+\left(\frac{3t-8}{4q^2}\right)Q_0^2+\left(\frac{t+8}{4q^2}\right)Q_1^2$ |
| $q^{-2}A_{21}$ | $-4Q_1^2+4Q_2^2$ | $\frac{1}{q^2}Q_0^2+\frac{3}{q^2}Q_1^2$ |

| $q^{-2}A_{ij}$ | $\mathcal{J}^2(t)$ | $\mathcal{J}^3(t)$ |
|---|---|---|
| $q^{-2}A_{11}$ | $\frac{1}{60}(t-2)(6Q_1-Q_3)+\frac{t}{2q^2}Q_1+\left(\frac{t+8}{8q^4}\right)Q_0^2$ | $(t-2)Q_1+\left(\frac{7t-16}{8q^2}\right)Q_0^2+\left(\frac{5t-8}{8q^2}\right)Q_1^2$ |
| $q^{-2}A_{21}$ | $\frac{2}{q^2}Q_1+\frac{3}{2q^4}Q_0^2$ | $\left(\frac{t-1}{2q^2}\right)(Q_0^2+Q_1^2)$ |

a parameter of the theory while the anomalous magnetic moment is completely determined.

We are now in a position to evaluate the second-order corrections to the form factors which result from diagram 1(a). If we insert the explicit values for the $A_{ij}(t)$ and the lowest-order form factors into the right-hand side of Eq. (4), we obtain the second-order expressions for the absorbtive parts. The complete form factors are obtained as usual by a simple dispersion integral. In particular, the second-order contribution to the anomalous magnetic moment is just the coefficient of $F_2(0)$, where we impose the condition

$$F_2^{(2)}(0) = \frac{1}{\pi}\int_4^\infty \frac{dt}{t}\,\mathrm{Im}F_2^{(2)}(t) = 1. \tag{12}$$

This integral (12) can be evaluated analytically. We find that the contribution of diagram 1(a) to $\kappa^{(2)}$ is given by

$$\kappa^{(2)} - \kappa^{(1)} = \left[\tfrac{1}{4}(\tfrac{9}{4} - 7\gamma)\zeta(2) + \tfrac{1}{2}(3\gamma - \tfrac{77}{36})\right]\frac{\alpha^2}{\pi^2}$$

$$\simeq (-0.940)\frac{\alpha^2}{\pi^2}\;, \tag{13}$$

where $\zeta(2) = \frac{1}{6}\pi^2$ is the Riemann zeta function of argument two.

Our result for the coefficient of the second-order term is nearly three times the experimental value (−0.328). Excluding the possibility of computational error,[13] we see that retaining only the two-body intermediate state [diagram 1(a)] does not give a satisfactory value for the electron magnetic moment. We conclude that the contributions of multiparticle intermediate states may be as important in electrodynamics as that of the two-body state. Of course, this result should not be particularly surprising. A basic concept of dispersion theory is that "distant" (i.e., multiparticle) singularities may be neglected for suitably restricted values of the energy. However, in electrodynamics the physical threshold is common to both elastic and inelastic intermediate states. Thus, there is no *a priori* reason to suppose that many-body effects will be small. All cuts may contribute significantly to the dispersion integrals. One aspect of the calculation presented here is that these considerations have now been placed on a quantitative basis. There is no ambiguity due to the presence of arbitrary cutoffs. Finally, we conclude that the published cutoff-dispersion-theory estimates of the higher-order corrections to the electron magnetic moment based only on two-body exchange are probably not reliable. There is some uncertainty due to the difference in approach, but the relative importance of many-body intermediate states should not be drastically affected by the choice of dispersion variable. At least, there is no compelling physical argument that the reverse is true. In any case, our results indicate that a pure dispersion calculation of the electron magnetic moment may be more difficult than anticipated. On the other hand, we have introduced the possibility that the higher-order corrections to the anomalous magnetic moment of the electron can be evaluated without the introduction of cutoffs or other arbitrary parameters and without reference to the usual forms of perturbation theory.

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[6]There is some ambiguity due to the fact that Refs. 2 and 3 employ a "sidewise" dispersion relation while our approach is through the direct channel. However, the conclusion that many-body effects may be important should not be drastically affected by the particular choice of dispersion variable.

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[11]Some apparent third-order terms are retained to simplify the notation. These are to be ignored in any calculation.

[12]The singularities of $\mathcal{R}(t)$ and $\mathcal{I}^2(t)$ exactly cancel.

[13]Every step in the calculation was tested by at least one independent computation. In addition, checks for internal self-consistency were utilized whenever possible. The essential simplicity and straightforwardness of this calculation add some assurance that the results will be accurate.