Relativistic Quantum Many-Body Theory in Riemannian Space-Time*

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A relativistic quantum many-body theory, which includes the strong interactions between elementary particles in curved space-time, is constructed. Using a generalized statistical-density operator, which incorporates the effects of gravitation as given by Einstein's field equations, as well as observables constructed from a generally covariant Lagrangian for matter fields, the definition of an N-point function of second-quantized matter fields is presented. The Yukawa coupling of a spinor field is then introduced. Coupled integral equations for the fermion and boson two-point functions in terms of the vertex function are given, which contain density and temperature effects in curved space-time; they are coupled to Einstein's equations through the expectation value of the energy-momentum density operator. Renormalization to effective masses and charge, as well as regularization, are discussed. The curved-space-time statistical-density operator is examined in the flat-space-time limit, and also in the nonrelativistic limit. The former agrees with previous work in relativistic statistical mechanics. The introduction of temperature and density as boundary conditions on flat-space-time N-point functions is carried out yielding a relativistic formalism, which may be applied to the calculation of such quantities as the equation of state of a superdense system of strongly interacting baryons. The nonrelativistic limit suggests a new approach to the statistical mechanics of Newtonian gravitation, in which such parameters as temperature become functions of coordinates. The relativistic flat-space-time limit is applicable to neutron stars at densities $\rho > 10^{15}$ g/cm^3 consisting of strongly interacting matter.

I. INTRODUCTION

Second-quantized many-body theory, high-energy theory, and general relativity are three areas of physics with little overlap. Their apparent mutual exclusiveness results from the lack of measurable phenomena in our solar system where more than one of these effects is of major significance at a given time. This is not the case in astronomy and cosmology. Astrophysicists have suggested that superdense matter is formed in the final stages of stellar evolution for sufficiently massive stars,[1] and possibly in the initial stages of stellar and galactic evolution as envisioned by Ambartsumyan.[2,3] It is also used to describe the initial stages of some cosmological models.[4] The discussion of such phenomena as these requires a knowledge of the macroscopic properties of superdense matter. For densities in excess of about 10^{15} g/cm^3, the dynamics will be determined by strong interactions. In special cases, not necessarily at super-nuclear densities,[5] curved space-time will make significant contributions to the description of the system. In general a correct study of the phenomena associated with superdense matter must therefore combine elementary-particle, many-body, and space-time theories in a fundamental manner. It is the purpose of this paper to develop such a formalism – a relativistic quantum many-body theory in Riemannian space-time – and to discuss its consequences for the theory of superdense matter.

That there might exist stars with densities greater than the density of a white dwarf was suggested as early as 1932 by Landau.[6] Two years later Baade and Zwicky[7] observed that the remnants of supernovae might contain such objects as their core. Until this time, it had been believed that all stars evolve, through sufficient mass loss, to the white dwarf stage, which was assumed to represent the final stage of stellar evolution. However, it was shown by Oppenheimer and Volkoff[8] in 1939 that another stable configuration of higher density existed for masses below about $0.7M_\odot$. The central densities which they found were in the range $10^{13} < \rho < 10^{16}$ g/cm^3. As a consequence of such high densities, the principal constituent in the stellar core is found to be neutrons. Thus the concept of a neutron star was born. Subsequent investigations have indicated that beyond the neutron-star stage there exist no configurations of matter in its ground state which are stable against gravitational collapse to a singularity.

Although the idea of a neutron star was theoretically attractive, it received little attention before

1959, except as a source of interesting fundamental problems associated with such an anomalous object. With few exceptions it was commonly believed that white dwarf stars represented the final stage of stellar evolution after the exhaustion of nuclear fuel and the ejection of sufficient mass to bring the total below the Chandrasekhar mass limit. In 1959 a renewed interest in neutron stars as being the end product of supernovae was taken by Cameron.[9] The discovery of pulsars in 1968 stimulated interest in attempts to understand the detailed properties of these objects.[10] Finally the observation of a pulsar among the remnants of the Crab Nebula has virtually placed the neutron star in the list of physically accepted entities.

With the renewed interest in neutron stars, there has developed a similar interest in the states of matter beyond the Landau-Oppenheimer-Volkoff mass limit M_{LOV}. As seen by a distant observer, a star of mass greater than M_{LOV} that has exhaust-

TABLE I. The table illustrates various effects which become important at successively higher densities for matter in curved space-time. The entries do not necessarily refer to a single system, although they may represent different stages in the evolution of a star of mass greater than the mass of 10^{57} baryons, as it undergoes gravitational collapse to a singularity. The five columns contain (1) the density range in g/cm^3; (2) the corresponding interparticle separation $l \equiv (m/\rho)^{1/3}$ in terms of the baryon mass $m = 2\times 10^{-24}$ g; (3) the nature of particle interactions; (4) the field-theoretic nature of gravitation; (5) and the macroscopic nature of matter as described by the equation of state.

Density (g/cm^3)	$l = (m/\rho)^{1/2}$ (cm)	Interaction	Gravitational field	Equation of state
$\rho < 10^{15}$	$l > 10^{-13}$	Nonrelativistic; description in terms of potentials valid.	c numbers	Determined by atomic and nuclear physics in flat space-time. No gravitational contributions. Macroscopic nature (such as mass-radius relation, etc.) determined predominantly by Newtonian gravity.
$10^{15} < \rho < 10^{17}$	$l \gtrsim 10^{-13}$	Relativistic; description in terms of potentials invalid. Strong interactions and decay important.	c numbers	Determined by strong interactions and weak decay of baryons in flat space-time. No gravitational contributions on microscopic level. Macroscopic nature determined by Newtonian gravity.
$10^{17} < \rho < 10^{20}$	$l > 10^{-15}$	Strong interactions and decay important. Hadronic matter.	c numbers	Determined by strong interactions and weak decay of baryons in flat space-time. No gravitational contributions generally on local level, but macroscopic nature determined by generally relativistic equations, for example, generally relativistic equation of hydrostatic equilibrium, etc. Nonlinear gravitational effects become important at the center.
$10^{20} < \rho < 10^{49}$	$l > 10^{-24}$	Extreme relativistic interactions; hadronic matter; quarks, partons, (?).	c numbers	Extreme relativistic energies. The description of interactions is uncertain, but possibly including such exotic states as quarks, or parton models, etc. If collective effects exist whose range of correlation is greater than 10^4 cm, then curvature contributes locally to interactions and determines the equation of state. General relativity governs all.
$10^{49} < \rho < 10^{93}$	$l > 10^{-33}$	(?)	c numbers	Whatever interactions exist contain significant contributions due to the curvature of space-time, even in the absence of collective effects.
$\rho > 10^{93}$	$l < l^* = (\hbar G/c^3)^{1/2} \sim 10^{-33}$	(?)	q numbers	(?)

ed its nuclear energy sources will appear to asymptotically approach its Schwarzschild radius; as seen by a comoving observer, the star will collapse in a finite time to a singularity. That a physically reasonable equation of state might stop such an effect is unlikely, but cannot be ruled out.[11]

Interest in superdense matter has developed in other areas as well. Ambartsumyan has suggested that it might play a significant part in the initial stages of stellar and galactic formation. Contrary to the popular consensus of astrophysicists, he has hypothesized the formation of stars and galaxies from superdense protostellar bodies to states of lower density. This has led to the study of systems composed of hyperons with average densities in excess of 10^{16} g/cm^3.

Finally, there is the question of early cosmologies, which take as their initial state matter and radiation at temperatures of the order of 10^{12} °K, the subsequent formation of the elements, and the still undetermined question of the total baryonic change of the universe.

In spite of their differing origins, the problems mentioned above contain the common characteristic of being determined primarily by strong interactions in curved space-time (see Table I). The latter enter via the equation of state. In all calculations of the equation of state of superdense matter, the strong interactions as well as gravitation have been ignored. The interactions, when they are included, are treated in a nonrelativistic manner. Decay processes are taken into account phenomenologically, and no distinction is made between elementary particles and bound states. The asymptotic behavior of the equation of state is usually bounded or approximated by arguments based on causality.[12]

For densities at or somewhat below nuclear densities $10^{13} \lesssim \rho \lesssim 10^{15}$ g/cm^3, various nuclear potentials have been used to describe interactions, and the equation of state obtained through the use of nonrelativistic many-body theory.[13] Above $\rho \approx 10^{15}$ g/cm^3 relativistic effects and strong interactions must be included. However, in this region previous calculations have essentially been those of a noninteracting relativistic Fermi gas. In some instances, interactions have been simulated by excluding some of the volume, which acts as a relativistic hard core of the particle. In order to appreciate the inadequacies of these models, it need only be mentioned that the interparticle separation for densities $\rho \sim 10^{16}$ g/cm^3 is roughly one-third that in normal nuclear matter ($\rho \sim 2.5 \times 10^{14}$ g/cm^3). Furthermore, the average energy per particle at such densities is about 200 MeV. Consequently nucleons may be converted into hyperons via weak decay, with subsequent energy loss in the form of neutrinos.

A correct calculation of the equations of state at high densities must include strong interactions. At these densities it is not possible to make a distinction between strong interactions and strongly interacting particles. The latter are the manifestation of the strong interactions and if we neglect these forces, then we neglect the basis of the very existence of these particles.

It is usually argued that a clean separation exists between gravitational effects, as described by c-number fields, and the interactions between elementary particles, particularly if early cosmologies and the final stages of stellar evolution accompanied by gravitational collapse are ignored.[14] This assumption is based on the argument that gravitation will contribute significantly to elementary particle interactions only when the curvature of space-time is comparable to the particle's Compton wavelength, and this does not happen until densities of the order of 10^{49} g/cm^3 are reached. For higher densities curvature described by c-number fields contributes significantly to the interactions. This line of reasoning ignores the potential importance for superdense matter of collective effects. Superdense matter consisting of strongly interacting fermions and bosons may exhibit such effects as superconductivity, superfluidity, ferromagnetism, and Bose-Einstein condensation, to mention but a few. The occurrence of such phenomena would suggest that gravitation contributes significantly when the curvature of space-time is comparable to the range of correlations. The latter are usually intermediate between the dimensions characteristic of the system's microscopic and macroscopic size. It is reasonable then to expect the curvature of space-time to contribute to interactions at a density much lower than 10^{49} g/cm^3, as is usually assumed. As an example, consider the situation in which the range of correlations is comparable to the dimensions of a neutron star core, say $r_c \sim 10^4$ cm. The density at which curvature is expected to contribute significantly to the interactions between particles is then found to be $\rho \sim 10^{20}$ g/cm^3.

It is as yet uncertain as to whether such densities are reached inside neutron stars. Nevertheless, it is certain that they will be reached for stars which are unable to shed sufficient mass to lie below the Landau-Oppenheimer-Volkoff limit.[1] Curved space-time must be included in the calculation of the macroscopic properties in this central crush region.

Quantum space-time effects may well be the principal factors in determining the final issue of gravitational collapse.[15] The fundamental length at which such effects become important is usually assumed to be $L \sim (\hbar G/c^3)^{1/2} \sim 10^{-33}$ cm. This corresponds to a density of about $\rho \sim 10^{93}$ g/cm^3, above which the metric must be described by a q-number field, and Einstein's equations become quantum field equations. As long as we restrict ourselves to systems whose density is $\rho \ll 10^{93}$ g/cm^3, the problems of quantization of space-time do not enter, and gravitation may be described by c-number fields.

The most natural quantities that lend themselves to the union of general relativity, high-energy theory, and many-body theory are thermodynamic Green's functions. Green's functions have the advantage over other formalisms of being more transparent due to their physical interpretations. In addition, this formalism is common to all areas of physics, and has a graphical representation which simplifies an otherwise complicated subject. The development of elementary particle physics[16] and thermodynamic many-body theory[17] in terms of Green's functions has been investigated extensively in the literature. Their covariant generalization can be achieved in a straightforward manner. In this approach the strong interactions are taken into account by means of a relativistic Lagrangian. Curved space-time is included consistently by requiring that the equations be generally covariant in a Riemannian manifold. The relativistic Heisenberg equations of motion lead to an infinite set of coupled equations for the Green's functions. The boundary conditions on the Green's functions give the many-body effects.

Before proceeding with the development of the formalism, a review of existing methods of treating relativistic dense systems will be given. We shall not dwell here on the various nonrelativistic many-body techniques which have been highly successful in treating ordinary nuclear matter ($10^{12} < \rho < 10^{14}$ g/cm^3). In this density region the use of potentials to describe interactions is valid, and relativistic effects are small. The problems associated with this region are primarily of a calculational nature, or involve questions of constructing an appropriate empirical potential best suited to the descriptions of nuclear matter. They are not problems of principle, as is the case of densities above $\rho \sim 10^{15}$ g/cm^3, or as in the presence of strong gravitational fields.

The first fully relativistic many-body theory in flat space-time was developed by Fradkin[18] and coworkers. Relativistic thermodynamic Green's functions were defined as solutions of the relativistic equations of motion. The treatment is field-theoretic in nature and includes interactions through a general nonderivative coupling. The approach is essentially an extension of the method first developed by Matsubara, in which the time dependence of the Green's function is replaced by an imaginary time. As a result the time-development operator e^{-iHt} and the density operator $e^{-\beta H}$ are formally alike. The many-body effects are then introduced as boundary conditions on the imaginary time dependence of the Green's function. Although this approach is a very powerful one, especially for calculations, it breaks the covariance of the theory at the very beginning, and is therefore not a useful starting point for a many-body theory in curved space-time.

In a recent paper,[19] a flat space-time relativistic many-body theory is developed from a different viewpoint. Starting with time-dependent Green's functions, the method of Martin and Schwinger is generalized to include relativistic dynamics. The approach retains the Lorentz invariance of the fields, and therefore serves as a useful starting point for a generalization to include gravitation.

Various discussions of the equation of state of superdense matter have been given in the literature, although none of them is based upon a many-body theory which includes gravitation through the curvature of space-time.[20]

The formalism to be developed in the following sections represents the first work of its kind which

(i) includes relativistic interactions as given by a theory of elementary particles;

(ii) incorporates both the finite density and finite temperature of a system from the principles of statistical mechanics;

(iii) and includes the effects of gravitation through Einstein's general theory of relativity.

This approach not only unifies three fundamental branches of theoretical physics in a natural and appealing manner, but leads to new effects which can result only from such a formulation. Specifically it introduces into the calculation of the macroscopic properties of a many-body system effects due to the fundamental structure of particles, which are themselves directly coupled to macroscopic properties of the system under study. This is a point of central importance for the problem of superdense matter and gravitational collapse, since it becomes artificial to distinguish between interparticle and intraparticle interactions at densities such that particle separations are on the order of their Compton wave length. The fact that elementary particle interactions, which determine individual particle structure, are also of importance in discussing superdense matter has not been sufficiently emphasized in the past,[21] and our ability to incorporate them in a fundamental manner is considered to be one of the major achievements of this work.

Other features of interest which result from our ability to include elementary particles in a natural manner are

(i) the direct dependence of macroscopic properties on the mass, spin, parity, and other quantum numbers of the constituent particles, in addition to the coupling constants;

(ii) the possibility of including gravitational contributions to interactions between particles from first principles;

(iii) the possibility of including all density- and temperature-dependent collective effects in a manner analogous to the well-tested nonrelativistic many-body theory;

(iv) the unification of interparticle and intraparticle renormalization effects (density-dependent and self-interaction-dependent, respectively) at a fundamental level;

(v) and the adaptability to our formalism of existing approximation techniques developed in elementary particle theory and thermodynamic many-body theory.

We shall develop and discuss the relativistic many-body formalism of finite-temperature and finite-density systems in four sections (see Fig. 1). In Sec. II a generally covariant Lagrangian for spin-one-half fermions and spin-zero bosons will be written which includes a Yukawa coupling. The resulting equations of motion are coupled to Einstein's field equations yielding a self-consistent set of equations for quantized matter fields, and the c-numbers $g_{\mu\nu}$ which describes the curvature of space-time. Next, a generalized N-point function of the quantized matter fields is defined which includes curvature. A generalized statistical-density operator is also defined which includes the curvature of space-time through the metric; its nonrelativistic and special relativistic limits are discussed. Section III contains a discussion of the flat-space-time limit of the results presented in the second section. In particular the relativistic fermion and boson two-point functions are constructed, which contain finite temperature and density effects as boundary conditions on the noninteracting Green's functions. The results are discussed, and the role of particle and antiparticle states explored. The relativistic formalism in curved space-time is presented in Sec. IV. Coupled equations for the fermion and boson two-point functions in curved space-time are written in terms of the three-point function. These are then coupled to Einstein's equations through the pressure and density. The latter are expressed in terms of the two-point function. Finally, it is shown that the many-body effects may be introduced through boundary conditions on the homogeneous part of the Green's function. The latter is written in terms of geometrical factors containing the effects of curved-space-time and the flat-space-time noninteracting Green's function. The renormalization to effective masses and charge is then discussed in Sec. V, where it is em-

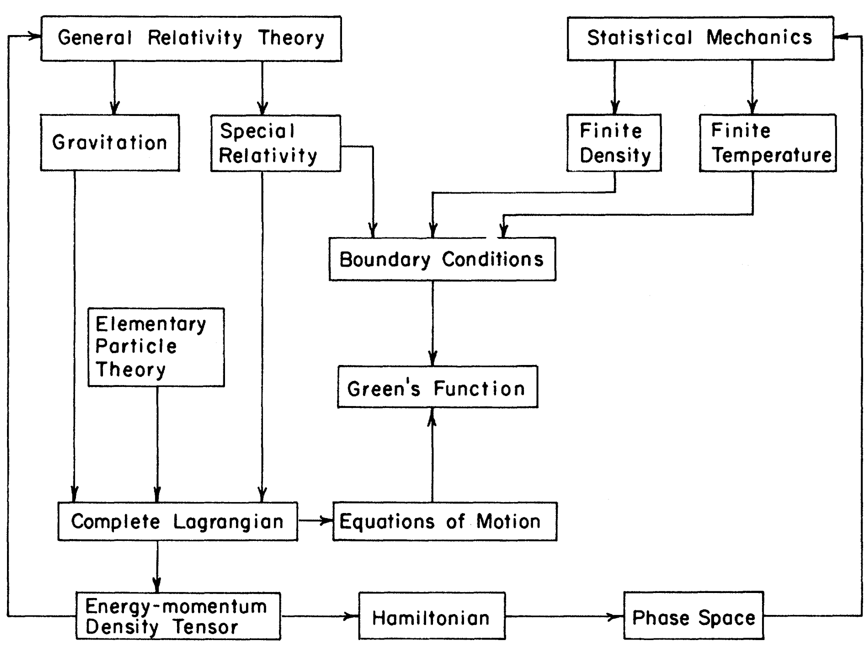

FIG. 1. Flow chart summarizing the stages at which relativity, elementary particle theory, and statistical mechanics enter into the specification of boundary conditions for the Green's function. It is assumed that a theory of each is initially given.

phasized that these quantities will in general depend upon the geometry of space-time.

Notation

The notation to be used throughout the following sections will be summarized below. The signature of $g_{\mu\nu}$ is -2. Tensor indices are denoted by lower-case Greek characters $\alpha, \beta, \ldots = 0, 1, 2, 3$, while spinor indices are denoted by lower-case Latin characters $a, b, \ldots = 0, 1, 2, 3$. The summation convention is assumed for both types of indices. Latin indices in parenthesis (a) will be used to denote different objects, and are not to be confused with tensor or spinor indices. Finally, we have set $\hbar = c = 1$, but have retained the gravitational coupling constant κ. Partial differentiation is denoted by a comma and covariant differentiation by a semicolon.

In some applications the inner product of two 4-vectors a^μ and b^μ will be denoted by $a \cdot b \equiv a^\mu b_\mu$. Frequently a function of the 4-momentum p^μ will be denoted $f(p)$. In flat space-time $\not{p} \equiv \gamma^\mu p_\mu$. Flat-space-time γ matrices will always be denoted by γ, whereas in curved space-time they will be written as $\gamma(x)$. Quantities referring to antiparticles will be denoted by the same symbol as their particle counterpart with a bar (e.g., antiparticle Fermi momentum $\bar{p}_F$).

II. ELEMENTARY PARTICLES IN CURVED SPACE-TIME

A. Covariant Equations of Motion

In order to discuss the behavior of a many-body system of elementary particles in curved space-time, the equations describing their dynamics will be written in a generally covariant form.[22] Since the discussion will be field-theoretic in nature, a covariant Lagrangian will serve as the most convenient method of introducing particle dynamics and gravitation. The equations of motion will be given, and the effects of curved space-time included by coupling the energy-momentum density tensor for matter fields to Einstein's field equations. Throughout the discussion continual reference will be made to the Yukawa coupling between spin-one-half particles.

The generally covariant Lagrangian density is

$$\mathfrak{L}(x) = \mathfrak{L}_E(x) + \mathfrak{L}_G(x)\,, \tag{2.1}$$

where the dynamics of elementary particles are introduced through $\mathfrak{L}_E$:

$$\begin{aligned}\mathfrak{L}_E = &(-g)^{1/2} Z_2 : \bar{\psi}(x)\{i\gamma^\mu(x)[\partial_\mu - \Gamma_\mu] - m\}\psi(x) : \\ &+ \tfrac{1}{2}(-g)^{1/2} Z_3 : \{\varphi(x)^{;\mu}\varphi(x)_{;\mu} - \mu^2\varphi(x)^2\} : \\ &- (-g)^{1/2} Z_1^{-1} Z_2 Z_3^{1/2} g_r : \bar{\psi}(x)\gamma_5(x)\psi(x)\varphi(x) : \\ &+ (-g)^{1/2} Z_2 \delta m : \bar{\psi}(x)\psi(x) : + (-g)^{1/2}\delta\mu^2 : \varphi(x)\varphi(x) :\end{aligned} \tag{2.2}$$

and the gravitational Lagrangian density $\mathfrak{L}_G$ is given by

$$\begin{aligned}\mathfrak{L}_G &= (-g)^{1/2} R \\ &= (-g)^{1/2} g^{\beta\delta}\{\Gamma^\alpha_{\beta\alpha,\,\delta} - \Gamma^\alpha_{\beta\delta,\,\alpha} + \Gamma^\alpha_{\tau\delta}\Gamma^\tau_{\beta\alpha} - \Gamma^\alpha_{\tau\alpha}\Gamma^\tau_{\beta\delta}\}\,.\end{aligned} \tag{2.3}$$

All spinor indices have been omitted above.

Quantization of the equations (with respect to matter fields only) will be accomplished by imposing generally covariant commutation relations on the fields. In particular, it is required that the real boson field operator $\varphi(x)$ and its conjugate momentum $\pi^\mu(x)$ satisfy the commutation relations

$$[\varphi(x), \varphi(x')] = [\pi^\mu(x), \pi^\nu(x')] = 0 \tag{2.4}$$

for all spacelike separations $(x_\mu - x'_\mu)^2 < 0$, and that

$$\int_{\Sigma(x)} d\sigma^\mu(x')[Z_3^{1/2}\varphi(x), Z_3^{1/2}\pi_\mu(x')] = 1\,, \tag{2.5}$$

where $\Sigma(x)$ is a spacelike hypersurface $d\sigma^\mu d\sigma_\mu < 0$ containing the space-time point x^μ. The differential tensor surface area is given by $d\sigma^\mu$ and is orthogonal to the timelike unit vector η^μ:

$$\eta_\mu d\sigma^\mu = 0\,. \tag{2.6}$$

The fermion field operators $\psi(x)$ and $\bar{\psi}(x)$ satisfy the anticommutation relations

$$\{\psi_a(x), \psi_b(x')\} = \{\bar{\psi}_a(x), \bar{\psi}_b(x')\} = 0 \tag{2.7}$$

for spacelike separations, and

$$\int_{\Sigma(x)} d\sigma^\mu(x')\{Z_2^{1/2}\psi_a(x), [Z_2^{1/2}\bar{\psi}(x')\gamma_\mu(x')]_b\} = \delta_{ab} \tag{2.8}$$

on the hypersurface $\Sigma(x)$. Under the restriction that the spacelike hypersurface $\Sigma(x)$ be given at constant time $x_0 = t$, the usual equal-time commutation relations result:

$$\begin{aligned}&[\varphi(x), \varphi(x')]_{t=t'} = [\dot{\varphi}(x), \dot{\varphi}(x')]_{t=t'} = 0\,, \\ &\left[\varphi(x), \frac{d\varphi(x')}{\partial t'}\right]_{t=t'} = Z_3^{-1}\delta(\vec{x} - \vec{x}')\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}&\{\psi_a(x), \psi_b(x')\}_{t=t'} = \{\psi_a^\dagger(x), \psi_b^\dagger(x')\}_{t=t'} = 0\,, \\ &\{\psi_a(x), \psi_b^\dagger(x)\}_{t=t'} = Z_2^{-1}\delta_{ab}\delta(\vec{x} - \vec{x}')\,.\end{aligned} \tag{2.10}$$

The commutation relations (2.4) and (2.5) and the anticommutation relations (2.7) and (2.8) contain field-operator renormalization coefficients Z_2 and Z_3. The term Z_1 is the coupling-constant renormalization factor. Mass renormalization counterterms δm and $\delta\mu^2$ for fermions and bosons, respectively, are included. The renormalized strong-interaction coupling constant is denoted by g_r. The spinor adjoint is defined in terms of the Hermitian adjoint

$\psi^\dagger(x)$ by

$$\psi_b^\dagger(x)\gamma_{ba}^0(x)=\bar{\psi}_a(x)\,. \tag{2.11}$$

The generally covariant Dirac gamma matrices $\gamma^\mu(x)$ are defined by the anticommutator

$$\{\gamma_a^\mu(x),\gamma_b^\nu(x)\}=2g^{\mu\nu}(x)\delta_{ab}\,. \tag{2.12}$$

The pseudoscalar $\gamma_5(x)$ is by definition

$$\gamma^5(x)\equiv\gamma_5(x)=i\gamma^0(x)\gamma^1(x)\gamma^2(x)\gamma^3(x)\,. \tag{2.13}$$

Normal ordering of the operators appearing in (2.2) is denoted by colons.

In addition to the quantities defined above, (2.2) and (2.3) contain geometrical factors reflecting the coupling between the gravitational field [described by the components of the matrix tensor $g_{\mu\nu}(x)$], and the distribution of matter in space-time [described in part by the fields $\psi(x)$ and $\varphi(x)$]. The determinant of the metric tensor is denoted by $g=\det(g_{\mu\nu})$. Partial differentiation is denoted by a comma:

$$A^\mu{}_{,\nu}\equiv\frac{\partial A^\mu}{\partial x^\nu}\,,\quad A_{\mu,\nu}\equiv\frac{\partial A_\mu}{\partial x^\nu}\,,$$

while covariant differentiation of tensor quantities is given by a semicolon:

$$A^\mu{}_{;\nu}\equiv A^\mu{}_{,\nu}+\Gamma^\mu_{\nu\lambda}A^\lambda\,, \tag{2.14}$$

$$A_{\mu;\nu}\equiv A_{\mu,\nu}-\Gamma^\lambda_{\mu\nu}A_\lambda\,. \tag{2.15}$$

The space-time affinity $\Gamma^\mu_{\nu\lambda}$ is given in terms of $g_{\mu\nu}$, and its derivatives by

$$\Gamma^\mu_{\nu\lambda}=\tfrac{1}{2}g^{\mu\sigma}\{g_{\nu\sigma,\lambda}+g_{\lambda\sigma,\nu}-g_{\nu\lambda,\sigma}\}\,. \tag{2.16}$$

The covariant derivative of a spinor of rank one is (the spinor indices $a,b=0,1,2,3$ are included in the definitions below)

$$\psi_a(x)_{;\mu}=\psi_a(x)_{,\mu}-\Gamma^b_{a\mu}\psi_b(x)\,, \tag{2.17}$$

$$\psi^a(x)_{;\mu}=\psi^a(x)_{,\mu}+\Gamma^a_{b\mu}\psi^b(x)\,, \tag{2.18}$$

where the spinor affinity Γ_μ (spinor indices omitted) is defined in terms of the space-time affinity and the metric tensor by

$$\Gamma_\mu=\tfrac{1}{8}\{\gamma^\lambda(x)\gamma_\lambda(x)_{,\mu}-\gamma_\lambda(x)_{,\mu}\gamma^\lambda(x)$$
$$-\Gamma^\rho_{\mu\lambda}[\gamma^\lambda(x)\gamma_\rho(x)-\gamma_\rho(x)\gamma^\lambda(x)]\}\,. \tag{2.19}$$

In general all spinor indices shall be omitted in the following sections. There should be no confusion between the affinity (Christoffel symbol) $\Gamma^\lambda_{\mu\nu}$ bearing three indices, which describes the properties of space-time, and the spinor affinity Γ_μ bearing one index (with two spinor indices understood), which describes the geometry of a fictitious spin space.

The quantity R appearing in (2.3) is the curvature scalar, and is related to the Ricci tensor $R_{\mu\nu}$ by contraction: $R=g^{\mu\nu}R_{\mu\nu}$. The Ricci tensor may be expressed in the form

$$R_{\mu\nu}=\Gamma^\alpha_{\mu\alpha,\nu}-\Gamma^\alpha_{\mu\nu,\alpha}-\Gamma^\alpha_{\beta\nu}\Gamma^\beta_{\mu\alpha}-\Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{\mu\nu}\,. \tag{2.20}$$

The equations of motion in covariant form are

$$\{i\gamma^\mu(x)[\partial_\mu-\Gamma_\mu(x)]-m\}\psi(x)$$
$$=\psi(x)\delta m+iZ_1{}^{-1}Z_3{}^{1/2}g_r\gamma^5(x)\psi(x)\varphi(x) \tag{2.21}$$

for fermions, and

$$\varphi(x)^{;\mu}{}_{;\mu}+\mu^2\varphi(x)=\varphi(x)\delta\mu^2-iZ_1{}^{-1}Z_2g_r\bar{\psi}(x)\gamma^5(x)\psi(x) \tag{2.22}$$

for boson fields. Variation of the metric tensor yields Einstein's field equations

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-8\pi\kappa T^{\text{av}}_{\mu\nu}\,. \tag{2.23}$$

In the following development the renormalization coefficients Z_1, Z_2, and Z_3 will not be retained. Their appearance does not alter the discussion at this point. It is to be emphasized that renormalization has not been neglected. The significant contributions to these constants resulting from the thermodynamic nature of the system are discussed in Sec. V.

In the equations above it has been assumed that the fields $\psi(x)$ and $\varphi(x)$ are q numbers. The metric tensor has not been quantized, and is to be considered as a set of c numbers. Equations (2.21)–(2.23) therefore represent a mixture of quantum and classical fields – quantized matter in classical space-time. Because of this mixture of quantum and classical fields, the energy momentum density tensor $T^{\text{av}}_{\mu\nu}$ must be a c number, since the left-hand side of (2.23) contains only c numbers. Defining the energy-momentum density tensor $T_{\mu\nu}$ which is a functional of the fields $\psi(x)$ and $\varphi(x)$, we shall take its average (over a suitably defined set of states) as the definition of $T^{\text{av}}_{\mu\nu}$. The existence of a suitable set of eigenstates will be assumed. Denoting an arbitrary quantum-mechanical average by $\langle\,\rangle$, (2.23) becomes

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=-8\pi\kappa\langle T_{\mu\nu}\rangle\,. \tag{2.24}$$

The specification of the average appearing in (2.24) will occupy the next section.

Before proceeding, it may be mentioned that (2.21)–(2.23) are reminiscent of the semiclassical second-quantized theory of radiation encountered in quantum mechanics, where the equations describing particle dynamics involve q-number fields, while the electromagnetic field is described by Maxwell's equations, which represent c-number fields. Furthermore, as in the Hartree-Fock model of the atom, where the potential is determined by wave functions which in turn are determined by

TABLE II. The major elements of the formalism are compared with their flat-space-time limits. The second column summarizes the general (curved-space-time) behavior of each element. The last column gives its behavior in the limit of flat space-time.

	Curved space-time	Flat space-time
Metric tensor	$g_{\mu\nu}$	Reducible throughout space-time to $\eta_{\mu\nu}=\mathrm{diag}(1, -1, -1, -1)$.
Gravitation	Replaced by curvature of space-time.	Absent.
Space-time affinity (Christoffel symbol)	Generally nonzero in any coordinate frame.	Vanish identically for a special choice of coordinates (inertial frames).
Spinor affinity	Generally nonzero in any coordinate frame.	Coordinate-independent representation in special inertial frame.
Mechanics of material objects	Relativistic including contributions from the curvature of space-time.	Relativistic – no gravitational contributions.
Second-quantized matter fields $\psi(x)$ and $\varphi(x)$	Functionals of $g_{\mu\nu}$ in addition to other matter fields.	Functionals of other matter fields.
Matter Lagrangian density $\mathcal{L}_E$	Functional of $\psi(x)$, $\varphi(x)$ and $g_{\mu\nu}$.	Functional of $\psi(x)$, $\varphi(x)$.
Gravitational Lagrangian density $\mathcal{L}_G$	Functional of $g_{\mu\nu}$.	Functional of $\eta_{\mu\nu}$; vanishes identically.
Equations of motion for $g_{\mu\nu}$	Einstein's equations $g_{\mu\nu}$ determined dynamically by matter content of space-time through fields $\psi(x)$ and $\varphi(x)$.	Einstein's equations in absence of matter – special relativity. Geometry as determined by $\eta_{\mu\nu}$ is absolute.
Dirac gamma matrices	$\gamma^\mu(x)$ generally coordinate-dependent, as determined by curvature of space-time.	γ^μ may be chosen to be coordinate-independent in special inertial frame.
Energy-momentum density tensor $T_{\mu\nu}$ (operator)	Determined by $\mathcal{L}_E$; contains contributions of curvature to interactions.	Determined by $\mathcal{L}_E$; no contributions from gravitation.
Energy-momentum density tensor $T^{\mathrm{av}}_{\mu\nu}$	Expectation value of $T_{\mu\nu}$ over quantum states of $\hat{f}_i(x)$.	Expectation value of $T_{\mu\nu}$ over quantum states of $\hat{f}_i(x)$.
$\hat{f}_i(x)$	Operator functional of $\psi(x)$ and $\varphi(x)$ and the c-numbers $g_{\mu\nu}$ whose ensemble average gives known observables over a hypersurface Σ or for all x^μ. Contains contributions due to curvature.	Operator functional of $\psi(x)$ and $\varphi(x)$ whose ensemble average yields known observables over Σ or for all x^μ. No gravitational effects.

TABLE II *(Continued)*

	Curved space-time	Flat space-time
$\lambda_i(x)$	Lagrange multipliers introduced with $\hat{f}_i(x)$. Determined in terms of ensemble averages. Contain contributions due to curvature.	Operator functional of $\psi(x)$ and $\varphi(x)$ whose ensemble average yields known observables over Σ or for all x^μ. No gravitational effects. Reduce to relativistic functions of such thermodynamic quantities as temperature, chemical potential, etc.
Statistical operator $\hat{\rho}$	Maximizes the "information entropy" subject to constraints introduced through averages of $\hat{f}_i(x)$. Contains contributions due to curvature.	Operator functional of $\psi(x)$ and $\varphi(x)$ whose ensemble average yields known observables over Σ or for all x^μ. No gravitational effects. Reduces to ensemble average from relativistic statistical mechanics.

the potential, (2.21)–(2.24) consist of a set of coupled equations that must be solved self-consistently.

In the limit of vanishing curvature (no gravitation), the metric $g_{\mu\nu}$ is reducible by means of a general coordinate transformation to the Minkowski form $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ valid throughout space-time. In this limit a Cartesian coordinate system exists such that the space-time and spinor affinities $\Gamma^\mu_{\alpha\beta}$ and Γ_μ vanish identically. As a consequence, the generally covariant Dirac γ matrices reduce to the coordinate-independent values γ^μ, which satisfy the anticommutation relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{2.25}$$

and all covariant derivatives reduce to partial derivatives. The Riemann curvature R and the Ricci tensor also vanish, so that the Lagrangian $\mathcal{L} = \mathcal{L}^0_E$ reduces to the usual flat-space-time pseudoscalar coupling in quantum field theory. The relation between the above element in curved and in flat space-time is summarized in Table II.

B. N-Point Functions

A compact method of investigating the general properties of a system of particles (macroscopic or microscopic) may be obtained by constructing the τ or Green's functions for that system. It will be recalled that having once constructed the Green's functions for a system it is possible in principle to obtain expressions for any (and all) observables in terms of them.

In order to discuss Green's functions within the framework of (2.21)–(2.23) it will be necessary to define three quantum-mechanical averages of a "time-ordered" product of N second-quantized field operators (N-point functions) which are in fact the τ or Green's functions for the system. A general N-point function will be defined in the Heisenberg representation. Consider N Heisenberg field operators $A(x_1), B(x_2), \ldots, N(x_n)$, where some or all of the operators may represent the same field, defined at the space-time points $x_1^\mu, \ldots, x_n^\mu$ (some or all of the events x_i^μ may be identical). A "time-ordering" operator T is next defined which, when acting on the set of operators above, places them in "chronometric" order. By "chronometric" order we mean that if a parameter s, defined along a possible timelike world line of a particle relative to an arbitrary hypersurface, is associated with each operator, then T places the operators in order of increasing s from right to left. A factor of (-1) is to be associated with each interchange of any two fermion operators. In terms of the definitions above, the N-point function is given by the quantum-mechanical average of the time-ordered (quotes dropped) product

$$\langle TA(x_1)B(x_2)\cdots N(x_n)\rangle .$$

The states entering into the expectation value will in general depend on the nature of a specific problem. Three possibilities will be considered: (1) a small number of particles in vacuum (elementary particle theory); (2) a many-body system assumed to lie in its ground state ($T=0$); and (3) a many-body system at finite temperature.

In the relativistic quantum field theory of elementary particles (including gravitation) the average is taken over the vacuum state $|0\rangle$, and the N-point function is defined by

$$\frac{\langle 0|TA(x_1)B(x_2)C(x_3)\cdots N(x_n)|0\rangle}{\langle 0|0\rangle}. \tag{2.26}$$

For a many-body system (including gravitation), described by the ground-state wave function $|\Psi_{N_1N_2\cdots N_n}\rangle$ and consisting of N_i particles of the first type, N_2 particles of the second type, ..., and

N_n particles of the nth type, the N-point function of the Heisenberg field operators is defined by

$$\frac{\langle\Psi_{N_1N_2\cdots N_n}|TA(x_1)B(x_2)\cdots N(x_n)|\Psi_{N_1N_2\cdots N_n}\rangle}{\langle\Psi_{N_1N_2\cdots N_n}|\Psi_{N_1N_2\cdots N_n}\rangle}\,. \tag{2.27}$$

Finally, for the many-body system not necessarily in its ground state, the average is to be taken over a grand canonical ensemble, and the N-point function is of the following form:

$$\langle TA(x_1)B(x_2)\cdots N(x_n)\rangle_\beta = \frac{1}{Z_g}\,\mathrm{tr}\{\hat\rho TA(x_1)B(x_2)\cdots N(x_n)\}\,. \tag{2.28}$$

The partition function Z_g is

$$Z_g = \mathrm{tr}\hat\rho\,, \tag{2.29}$$

where the density operator $\hat\rho$ is

$$\hat\rho = \exp-\sum_{i=0}^{k-1}\int d^4x(-g)^{1/2}\lambda_i(x)f_i(x)\,. \tag{2.30}$$

The terms used above to describe Z_g, $\hat\rho$, and the ensemble average in (2.28) are motivated by the fact that they describe exactly the flat space-time limit of these quantities, as will be seen below. The subscript β on the left-hand side of (2.28) is shorthand for the ensemble average (2.28)–(2.30). The trace (tr) appearing above is to be taken over a complete set of eigenstates of the operators entering into the definition of $\hat\rho$.

The approach outlined above is motivated by the original work of Shannon and Jaynes[23] in nonrelativistic statistical mechanics, and is a direct extension of that work to include relativistic dynamics and gravitation. The original work, based upon the concept of information entropy offers a convenient axiomatic starting point for a relativistic generalization of statistical mechanics. In arriving at (2.28)–(2.30) it has been assumed that the probabilities

$$p_i \equiv Z_g^{-1}\langle i|\hat\rho|i\rangle \tag{2.31}$$

are non-negative and normalizable:

$$p_i \geq 0\,,$$

$$\sum_i p_i = 1$$

and that they be consistent with all known constraints placed on the system. The latter are represented by the average values $F_i(x)$, $i=1,2,\ldots,k$, of an arbitrary (though in most practical instances small) number of functions of operators $f_i(x)$. The operators entering into the functions $f_i(x)$ are assumed to correspond to physical observables. Therefore

$$\langle f_i(x)\rangle_\beta = F_i(x) \tag{2.32}$$

in the notation of (2.28). It is to be emphasized that the $F_i(x)$ may be specified either over a hypersurface, $\Sigma(x)$ (such as the one used in defining the commutation relations earlier), or for all values of the coordinates x^μ. The first instance corresponds to an initial-value problem, while the latter describes situations where several of the constraints may be time-dependent. In any case the constraints are imposed in a covariant manner.

Although Jaynes emphasizes that the basic features of the information entropy approach provide a starting point for a theory of irreversible processes,[24] we shall restrict our attention in this paper to equilibrium processes in curved space-time. Since the nonrelativistic limit of (2.28)–(2.30) yields an N-point function formally identical to the Gibbs grand canonical ensemble average, it is reasonable to expect, in equilibrium, that one of the parameters will contain a generalized temperature.[24] We shall therefore tentatively set $\lambda_0(x) = \beta(x)$. Consequently the function $f_0(x)$ will be expected to contain the Hamiltonian density operator. The chemical potential and number operator may be introduced in the same tentative manner.

In order that the integrand of (2.30) be covariant, the products $\lambda_i(x)f_i(x)$ must transform as scalars. For this reason the Hamiltonian density $\mathcal{H}(x)$ is not equal to the function $f_0(x)$, since it is only one component of the energy-momentum density tensor: $\mathcal{H}(x) = T^{(00)}(x)$. To get around this difficulty, $T_{\mu\nu}$ must be contracted with a second-rank tensor to make a scalar. The natural choice, which introduces the physics of the problem, is to define an orthonormal set of tetrads $\lambda^{(a)\mu}$ $(a=0,1,2,3)$ in terms of the observers frame of reference, and a second set $\bar\lambda^{(a)\mu}$ in terms of the reference frame of the system. The two sets of tetrads evaluated at the same point x^μ are then used to form the scalar

$$T^{(ab)}(x) = \lambda^{(a)\mu}(x)\bar\lambda^{(b)\nu}(x)T_{\mu\nu}(x) \tag{2.33}$$

which represents the energy-momentum density tensor of the system with respect to an observer. The $a=b=0$ component, defined as the projection of $T_{\mu\nu}$ onto the timelike axis $\lambda^{(0)\mu}$ and $\bar\lambda^{(0)\nu}$ of the system and observer will be taken as the observed energy density, so that

$$f_0(x) = T^{(00)}(x) = \lambda^{(0)\mu}(x)\bar\lambda^{(0)\nu}(x)T_{\mu\nu}(x)\,. \tag{2.34}$$

In terms of (2.34), the first term in the argument of $\hat\rho$ will be

$$\int d^4x(-g)^{1/2}\beta(x)T^{(00)}(x)$$

$$= \int d^4x(-g)^{1/2}\beta(x)\lambda^{(0)\mu}(x)\bar{\lambda}^{(0)\nu}(x)T_{\mu\nu}(x) \tag{2.35}$$

which, by construction, is a scalar with respect to arbitrary coordinate transformation.

C. Special-Relativistic Limit

Several examples will be considered which are illustrative of the method outlined above. It will be assumed that the system of interest in each case is described by a classical energy-momentum density tensor. A quantum system could equally well have been considered, since the methods apply to them with the obvious modification that observable quantities be replaced by operators, and the average be taken with respect to a complete set of eigenstates.

The integrand in (2.35) is particularly simple in the case of flat space-time. The result, which yields the special-relativistic form[25] of the density operator $\hat{\rho}$ illustrates the procedure used in constructing ensemble averages in general, and serves as a check of our method. Considering flat space-time, the metric tensor $g_{\mu\nu}$ may be diagonalized $g_{\mu\nu}=\eta_{\mu\nu}$ where $\eta_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$, from which $(-g)^{1/2}=1$ follows. Setting $f_i=\delta_i^0 f_0(x)$, the scalar

$$f_0(x)=\lambda_\mu^{(0)}\bar{\lambda}_\nu^{(0)}T^{\mu\nu}(x)$$

represents the projection of the energy-momentum density tensor onto two timelike unit vectors tangent to the world lines of the observer S and of the system S' under consideration. The first unit vector, representing the observer, is taken to be of the form

$$\lambda^{(0)\mu}=\lambda_\mu^{(0)}=(1,0,0,0)\,. \tag{2.36}$$

The second unit vector, representing the system S', may be taken as the tangent to the world line of the center of mass of S', which will be assumed to travel along the x' axis with velocity v relative to S. The Lorentz transformation

$$\bar{\lambda}^{\mu(0)}=\partial\bar{x}^\mu/\partial x^\nu\lambda^{\nu(0)}$$

from S to S' yields the unit vector

$$\bar{\lambda}^{(0)\mu}=(\gamma,-\gamma v,0,0)\,, \tag{2.37}$$

where $\gamma=(1-v^2)^{-1/2}$. Inserting $\lambda^{(0)\mu}$ and $\bar{\lambda}^{(0)\mu}$ into $f_0(x)$ one finds

$$f_0=\lambda_0^{(0)}\bar{\lambda}_0^{(0)}T^{00}(x)+\lambda_0^{(0)}\bar{\lambda}_1^{(0)}T^{01}(x)$$
$$=\gamma T^{00}(x)+\gamma v T^{01}(x)\,.$$

Since $T^{00}(x)=\mathcal{H}(x)$, the Hamiltonian density, and $T^{01}(x)=\mathcal{P}_1(x)$, the momentum density along the x^1 axis, it follows that

$$f_0(x)=\gamma[\mathcal{H}(x)+v\mathcal{P}_1(x)]\,. \tag{2.38}$$

The density operator $\hat{\rho}$ is therefore given by[26]

$$\hat{\rho}(v)=\exp\left(-\int d^3x\,\beta(x)\gamma\{\mathcal{H}(x)+v\mathcal{P}_1(x)\}\right). \tag{2.39}$$

Noting that the parameter $\beta(x)$ plays the role of a temperature in the above expression, we examine [assuming $\lambda_i(x)$ and $g_{\mu\nu}$ to be constant]

$$\left[-\frac{\delta}{\delta\beta(\vec{x})}\ln\mathrm{tr}\hat{\rho}(v)\right]_{\lambda_i(x),g}=\frac{1}{Z_v}\,\mathrm{tr}\{\hat{\rho}(v)\gamma(H+vP_1)\}\,,$$

$$Z_v=\mathrm{tr}\hat{\rho}(v)\,,$$

where $\delta/\delta\beta(\vec{x})$ denotes the functional derivative with respect to $\beta(\vec{x})$. In arriving at the expression above, it has been assumed that the Lagrangian multiplier $\beta(x)=\beta(\vec{x},t)$ is independent of time in thermodynamic equilibrium, and the identity

$$\delta\beta(\vec{x}')/\delta\beta(\vec{x})=\delta^3(\vec{x}-\vec{x}')$$

has been used. Finally the Hamiltonian H and momentum P_1 (along the x^1 axis) are defined by

$$H=\int d^3x\,\mathcal{H}(x)\,,$$

$$P_1=\int d^3x\,\mathcal{P}_1(x)\,,$$

respectively. Integrating over three-space yields

$$-\int d^3x\left[\frac{\delta\ln Z_v}{\delta\beta(\vec{x})}\right]_{\lambda_i(x),g}=\frac{1}{Z_v}\,\mathrm{tr}\hat{\rho}(v)\gamma(H+vP_1)$$
$$=\langle\gamma(H+vP_1)\rangle_\beta\,. \tag{2.40}$$

The dependence on $\beta(x)$ of the average above is denoted by the β subscript, but (2.40) is independent of position for $g_{\mu\nu}=\eta_{\mu\nu}$.

It will be noted that (2.40) bears a strong formal resemblance to the average energy density of a system moving with velocity v (apart from the factor γ), and reduces for $v=0$ to the classical average energy with $\hat{\rho}(0)$ given by the Gibbs distribution function. Since the system is assumed homogeneous, we shall set $\beta(\vec{x})=\beta_0$ independent of x. The observations above strongly suggest that the Lagrange multiplier β_0 be associated with the temperature parameter of the system, and we therefore set (k = Boltzmann's constant)

$$\beta(\vec{x})=\beta_0=(kT)^{-1}\,.$$

The density operator $\hat{\rho}$ of a homogeneous system S', in motion along the x' axis with velocity v relative to an observer S, is therefore given in the absence of gravitational fields ($g_{\mu\nu}=\eta_{\mu\nu}$) by[25]

$$\hat{\rho}=\exp[-\beta_0\gamma(H+vP_1)]\,. \tag{2.41}$$

D. Arbitrary Curvature

The extension of the discussion above to include arbitrary gravitational fields ($g_{\mu\nu} \neq \eta_{\mu\nu}$) follows similar lines, although new features result from the nonzero curvature of space-time. By way of illustration, let a background metric $g_{\mu\nu}$ be assumed given. The first departure from the discussion above results from the coordinate dependence of $f_0(x)$, not only through the energy-momentum density tensor, but through $[-g(x)]^{1/2}$ and the unit vectors $\lambda^{(0)\mu}(x)$ and $\bar{\lambda}^{(0)\mu}(x)$. If the observer and the system are located at different points x^μ and $\bar{x}^\mu$, respectively, then it will in general be necessary to transport the members of the two tetrads to a common point when evaluating $f_0(x)$. For example, the timelike unit vector $\bar{\lambda}^{(0)\mu}(x')$ may be parallel transported to the event x^μ:

$$\bar{\lambda}^{(0)}_\mu = g_{\mu\nu'}\bar{\lambda}^{(0)\nu'} ,$$

where the prime on the index means that it depends on the coordinates x'^μ. The two-point tensor $g_{\mu\nu'}$ is a function of x^α and x'^β. In such circumstances f_0 will be given by

$$f_0 = \lambda^{(0)}_\mu \bar{\lambda}^{(0)\nu'} g_{\nu'\rho} T^{\mu\rho}$$

and is no longer dependent on x^μ alone, but on the path connecting x^μ and x'^μ as well.

The examples above make evident the inapplicability in curved space-time of an earlier assumption that $\beta(x)$ (or indeed any of the Lagrange multipliers) is independent of x^μ; although in the case of certain symmetries the independence of $\beta(x)$ or one or more of the coordinates may be established. An example of the latter is afforded by spherically symmetric time-independent space-time, where $\beta(x)$ is expected to depend only upon one coordinate $r = x^1$. This would yield, in the case of a star in quasistatic equilibrium, a $\beta(x) = \beta(r)$ which varies in the radial direction and is, on the time scale defined by the system, independent of x^0. However, it will be recalled that at any point x^μ, space-time may be described by the Minkowski metric and $f_0(x)$ reduces to the special-relativistic form discussed above. Therefore in a small enough neighborhood of x^μ it is still possible to associate $\beta(x)$ with a temperature parameter.[27]

The partition function $Z(g)$ will now depend upon the geometry of space-time. In analogy with (2.40) the thermodynamic quantities will be given in terms of functional derivatives of $Z(g)$, and will also depend upon gravitation. Consider, for example, the average of $T^{(00)}(x)$ [in the sense of (2.28)–(2.30)]: In curved space-time we have

$$-\left[\frac{\delta \ln Z(g)}{\delta\beta(x)}\right]_{\lambda_i(x),g} = \frac{1}{Z(g)}\ \mathrm{tr}\hat{\rho}(g)T^{(00)}(x) . \qquad (2.42)$$

In flat space-time ($g = \eta = \det\eta_{\mu\nu}$), (2.42) reduces to the usual thermodynamic expression for the average energy density of a system at temperature $T = (\beta_0 k)_g{}^{-1}$, further motivating the association of $\beta(x)$ with a temperature parameter. Similar techniques apply to any remaining Lagrange multipliers $\lambda_i(x)$.

In the functional derivatives with respect to $\beta(x)$ above it has been assumed that $g_{\mu\nu}$ is held constant [as well as the $\lambda_i(x)$ for $i \neq 0$]. Consider the functional derivative of $Z(g)$ with respect to $g_{\mu\nu}$ for fixed $\lambda_i(x)$ [including $\beta(x)$] and fixed $f_i(x)$. Then it is trivial to show, using

$$\delta(-g)^{1/2} = -\tfrac{1}{2}(-g)^{1/2} g^{\mu\nu}\delta g_{\mu\nu} ,$$

that the parameter $\beta(x)$ is given by

$$\beta(x) = -\frac{1}{2F_0(x)}\left[\frac{\delta \ln Z(g)}{\delta g_{\mu\nu}}\,\delta g_{\mu\nu}\right]_{\lambda_i(x) f_i(x)} .$$

The average energy $F_0(x)$ of matter at the point x^μ is defined in accordance with (2.32):

$$F_0(x) = \frac{1}{Z_g}\ \mathrm{tr}\hat{\rho} f_0(x) .$$

E. Newtonian Limit

Returning to the general form of $f_0(x)$ and $\hat{\rho}$, we mention two problems which may be explored in terms of the formalism outlined above. As long as attention is restricted to curvature of space-time which is small compared to macroscopic dimensions of the system under consideration, the linearized version of Einstein's theory may be used. The results would be of interest for two reasons: (1) They permit the calculation of relativistic corrections to the equations of statistical mechanics; (2) the leading-order correction $M\kappa/rc^2$ is proportional to the Newtonian potential due to a gravitating mass M. Consequently the nonrelativistic limit includes the effects of classical (Newtonian) gravity as the correct limit of the theory of gravitation. Such an approach may shed light on the difficulties associated with gravitational fields in classical statistical mechanics.

Basically the approach to either problem above begins with the assumption that the metric tensor may be written as[28]

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} ,$$

neglecting terms of order κ^2 and higher. The deviation of $g_{\mu\nu}$ from flat space-time is given by $\gamma_{\mu\nu}$, which satisfies the linearized equation

$$\gamma_{\mu\nu}{}^{,\alpha}{}_{,\alpha} = -2\kappa(T_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}T^\alpha{}_\alpha) . \qquad (2.43)$$

The timelike unit vector describing an inertial observer is given by (2.36),[29]

$$\lambda^{(0)\mu} = \lambda_{(0)\mu} = (1, 0, 0, 0)\,,$$

while the timelike unit vector $\bar{\lambda}^{(0)\mu}$ describing the reference frame of the system may be taken as the timelike solution of

$$\bar{\lambda}^{(a)}_{\mu}(x)\bar{\lambda}_{\nu(a)}(x) = \eta_{\mu\nu} + \gamma_{\mu\nu}(x) + O(\kappa^2)\,.$$

Expanding in powers of κ,

$$\bar{\lambda}^{\mu(a)} = \eta^{\mu(a)} + \kappa h^{\mu(a)}(x) + O(\kappa^2)\,,$$

$$\eta_{\mu(a)}\eta^{(a)}_{\nu} = \eta_{\mu\nu}\,.$$

In terms of $\bar{\lambda}^{(0)\mu}$ and $\lambda^{(0)\nu}$ above, (2.34) becomes

$$f_0(x) = T^{00}(x) + \kappa h^{(0)}_{\mu} T^{0\mu} + O(\kappa^2)$$
$$= \{1 + \kappa h^{(0)}_{0}(x)\}\mathcal{H}(x) + \kappa h^{(0)}{}_{i}(x)\mathcal{P}^{i}(x) + O(\kappa^2)\,, \tag{2.44}$$

where $\mathcal{P}^i(x) = T^{i0}(x)$ $(i = 1, 2, 3)$. It will now be assumed that the velocity of the system v (with respect to the observer) is small $v^2/c^2 \ll 1$ and will in fact be set equal to zero. It could be included, but it does not change the over-all approach below. The $\bar{\lambda}^{(a)}_{\mu}$ then describes a system at rest at x^μ with respect to the inertial observer $\lambda^{(b)}_{\mu}$. As a consequence no special-relativistic effects will appear in (2.44). The corrections due to small departures from flatness are seen to be proportional to κ. The correction to the Hamiltonian density, proportional to $h^{(0)}_0(x)$, will be discussed below. The second correction, proportional to the momentum density $\mathcal{P}^i(x)$, is multiplied by $h^{(0)}_i(x)$ which by a proper choice of coordinates may be made to vanish for a static, spherically symmetric matter distribution. It is expected, however, to contribute for a nonstatic field such as that produced by a rotating body.

It is straightforward to show that (2.44) takes the following form for space-time described by the interior Schwarzschild solution:

$$f_0^{\text{int}}(x) = \mathcal{H}(x)\left[1 - \frac{\kappa M}{2c^2 r_0}\left(3 - \frac{r^2}{r_0{}^2}\right)\right] + O(\kappa^2)\,,$$

where the matter distribution is of radius $r_0 > r$, density $\rho_0 = \text{const}$, and mass $M = 4\pi\rho_0 r_0{}^3/3$. We recognize in the term proportional to κ the ratio

$$\frac{\kappa M}{c^2 r_0}\left(3 - \frac{r^2}{r_0{}^2}\right) = \left(\frac{\text{gravitational potential energy}}{\text{rest energy}}\right)_{\text{Newtonian}}$$

in the Newtonian limit for a system inside a spherically symmetric distribution of uniform density ρ_0 and mass M.

Repeating the analysis above for the exterior Schwarzschild metric, it is trivial to show that

$$f_0^{\text{int}}(x) = \mathcal{H}(x)\left(1 - \frac{\kappa M}{rc^2}\right) + O(\kappa^2)\,,$$

where $r = (x^2 + y^2 + z^2)^{1/2}$ is the distance from the center of the mass M to the system. Denoting the Newtonian potential by $\varphi(r) = -M\kappa/r$, the density operator $\hat{\rho}$ may be written

$$\hat{\rho} = \exp\left[-\int d^3x\, \beta(x)\mathcal{H}(x)\left(1 + \frac{\varphi(r)}{c^2}\right) + O(\kappa^2)\right].$$

Notice that since the metric is static and $\mathcal{H}(x)$ is assumed to describe a system at rest, the integral appearing in $\hat{\rho}$ is over three dimensions. Then the Hamiltonian density is $T^{00} = \mathcal{H}(x) = \rho c^2$ with $\rho(r)$ the density of the system. If $\rho(r)$ is assumed to be nearly constant over a small region Ω located a distance R from the mass M and zero elsewhere, then $\hat{\rho}$ is approximately

$$\hat{\rho} \approx \exp\left\{-\rho c^2[1 + \varphi(R)]\int_\Omega d^3x\, \beta(\bar{x}) + \rho c^2 \frac{\varphi(R)}{R}\int_\Omega d^3x\, \beta(\bar{x}) r\right\}.$$

The first term in the exponential is the rest energy density plus the gravitational potential energy per unit volume at R. We might interpret the integral

$$\frac{k}{\Omega}\int_\Omega d^3x\, \beta(\bar{x}) = (\bar{T})^{-1}$$

as the inverse average temperature. The second term in the exponential above would then represent corrections due to the inhomogeneity of the system due to the gravitating mass M.

An alternate interpretation results if the integral appearing in $\hat{\rho}$ is written as

$$\int d^3x\, \mathcal{H}(x)\left(1 + \frac{\varphi(r)}{c^2}\right)\beta(\bar{x})$$
$$\approx \int_\Omega d^3x\, \rho(\bar{x})\left(1 + \frac{\varphi(R)}{c^2}\right)c^2\beta_0 - \int d^3x\, \rho(\bar{x}) r \frac{\varphi(R)}{R}\beta_0$$
$$= \beta_0 m c^2\left(1 + \frac{\varphi(R)}{c^2}\right) - \frac{\varphi(R)}{R}\beta_0 \int_\Omega d^3x\, \rho(\bar{x}) r\,,$$

having required that $\beta(\bar{x}) = \beta_0$, independent of $\bar{x}$. The leading term is the product of the rest energy of the system times the temperature T_g defined by

$$T_g{}^{-1} = k\beta_0\left(1 + \frac{\varphi(R)}{c^2}\right).$$

Defining the temperature in the absence of gravitation T_0 by $T_0 = (k\beta_0)^{-1}$, it follows that

$$T_g = T_0/(1 - M\kappa/Rc^2)$$

which will be recognized as the Newtonian limit of the temperature[30]

$$T_0 = [g_{00}(R)]^{1/2} T_g\,,$$

where $g_{00} = 1 - 2M\kappa/Rc^2$.

It is interesting to note that the two formally different definitions of temperature result from the same initial approach. This should serve to emphasize the operational nature of any definition of temperature in the presence of gravitational fields.

The thermodynamic average (2.28)–(2.30) has been postulated for systems in curved space-time, and its validity suggested by the nonrelativistic and flat space-time limits. Nevertheless it would be comforting to construct other arguments which would compare its general-relativistic character with existing theories. One such approach might be to find a connection between quantities contained in the formalism above and the results of general relativistic kinetic theory. It might then be possible to relate such parameters as $\beta(x)$ [and the remaining $\lambda_i(x)$] to quantities in kinetic theory which are physically defined in curved space-time, rather than in the flat-space-time limit only. This, it is felt, would lend solidity to the definitions above, and may go a long way towards the formulation of a general-relativistic statistical mechanics.

III. RELATIVISTIC MANY-BODY THEORY: FLAT SPACE-TIME

The equations of the preceding section constitute the starting point for the construction of a relativistic quantum many-body theory in curved space-time. These equations lend themselves easily to the construction of integral expressions for the Green's functions. However, efforts to include boundary conditions are frustrated by the relative complexity of Fourier decompositions of the noninteracting curved-space-time Green's functions compared to their flat-space-time counterparts. Fortunately it is possible to circumvent this difficulty by working in terms of expansions of $g_{\mu\nu}$ about $\eta_{\mu\nu}$, which contain functions of the geometry and flat-space-time propagators. As will be seen in subsequent sections, the thermodynamic boundary conditions may be imposed through these flat-space-time propagators. For this reason it will be necessary to construct the flat-space-time thermodynamic Green's functions. The complete curved-space-time Green's functions will then be presented.

The requirement that space-time be flat is satisfied if

$$g_{\mu\nu} = \eta_{\mu\nu} \tag{3.1}$$

and

$$\gamma^\mu(x) = \gamma^\mu , \tag{3.2}$$

where γ^μ satisfies (2.25). For simplicity a Cartesian coordinate system

$$x^\mu = (t, x, y, z)$$

will be used throughout this section, in which case $\eta_{\mu\nu}$ is the Minkowski metric

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.3}$$

and the γ^μ will be coordinate-independent. The equations of motion (in terms of unrenormalized fields) follow from (2.21) and (2.22), where covariant derivatives reduce to partial derivatives[31]:

$$(i\gamma^\mu \partial_\mu - m + \delta m)\psi(x) = ig_0\gamma^5\psi(x)\varphi(x) , \tag{3.4}$$

$$(\partial^\mu \partial_\mu + \mu^2 - \delta\mu^2)\varphi(x) = -ig_0\bar{\psi}(x)\gamma^5\psi(x) . \tag{3.5}$$

A. Fermion Two-Point Function

The immediate purpose will be to construct from (3.4) and (3.5), for the case of a noninteracting system ($g_0 = 0$), the thermodynamic two-point functions for relativistic fermion and boson fields. By thermodynamic we mean that the system is at finite density and finite temperature.

Consider first the two-point function for spinor fields:

$$S(x - x') = -i\langle T\psi(x)\bar{\psi}(x')\rangle , \tag{3.6}$$

where the fact that the system is homogeneous has been used. Equation (3.6) satisfies the equation of motion

$$(i\gamma^\mu \partial_\mu - m_0)S(x - x') = \delta^4(x - x') , \tag{3.7}$$

where m_0 is the bare mass of the fermion. In order to construct the Green's function, one follows the usual procedure, defining the Fourier transform of $S(x - x')$:

$$S(x - x') = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} S(p) , \tag{3.8}$$

where $p \cdot x = p_\mu x^\mu = \eta_{\mu\nu} p^\mu x^\mu$. Substituting $S(x - x')$ as given by (3.8) into the equations of motion, it follows that

$$S(p) = \frac{\not{p} + m_0}{2E_{\vec{p}}} \left(\frac{1}{p_0 - E_{\vec{p}}} - \frac{1}{p_0 + E_{\vec{p}}} \right) . \tag{3.9}$$

We have defined the 4-momentum $p^\mu = (p^0, \vec{p})$ with $p^0 = p_0$, denoted the energy by $E_{\vec{p}} = (p^2 + m_0{}^2)^{1/2}$, and set

$$\not{p} = \gamma^\mu p_\mu = \eta_{\mu\nu} p^\mu \gamma^\mu .$$

The construction is complete as soon as the boundary conditions defining the behavior of $S(p)$ in the neighborhood of the singularities $p_0 = \pm E_{\vec{p}}$, and the nature of the states entering into the expectation value of (3.6) have been specified. The three examples mentioned in the preceding section will be considered.

If the definition (2.26) is adopted, with $N=2$, $A(x_1)=\psi(x)$, and $B(x_2)=\bar{\psi}(x')$ and the vacuum is assumed normalizable, then one choice of boundary conditions results from the addition of an infinitesimal imaginary part $-i\epsilon$ ($\epsilon>0$) to the energy

$$E_{\vec{p}} \to E_{\vec{p}} - i\epsilon \,.$$

The result, $S_F(p)$, is the familiar Feynman (or causal) propagator which describes the motion of free particles and antiparticles forward in time.[31] The poles of $S_F(p)$, which depend on m_0 and $\vec{p}$, are shown in Fig. 2.

For a many-body system in its ground state, the finite density of the medium through which the particles propagate may be introduced in the same manner that finite density is introduced in the nonrelativistic construction of zero-temperature Green's functions.[17] The ground state, for the case of N_F particles and $\bar{N}_F$ antiparticles corresponding to the field $\psi(x)$, is $|\Psi_{N\bar{N}}\rangle$ which will be assumed normalizable. In the ground state all fermions and antifermions will lie in their lowest states consistent with the Pauli exclusion principle (assumed to hold in the absence of interactions for particles and antiparticles separately).[32] The maximum momentum, the Fermi momentum, of each noninteracting system is given, at zero temperature, by the expressions

$$p_F^3 = 3\pi^2 n_0 = 3\pi^2 N_F/V \,, \tag{3.10}$$

$$\bar{p}_F^3 = 3\pi^2 \bar{n}_0 = 3\pi^2 \bar{N}_F/V \,. \tag{3.11}$$

The number density of particles and antiparticles has been denoted by n_0 and $\bar{n}_0$, respectively. Equations (3.10) and (3.11) hold for fermions of spin one-half. The magnitude of the Fermi momentum (which for free particles in flat space-time is isotropic) has been denoted by p_F and $\bar{p}_F$ for particles and antiparticles, respectively. Note that there are two distinct Fermi seas which are filled.

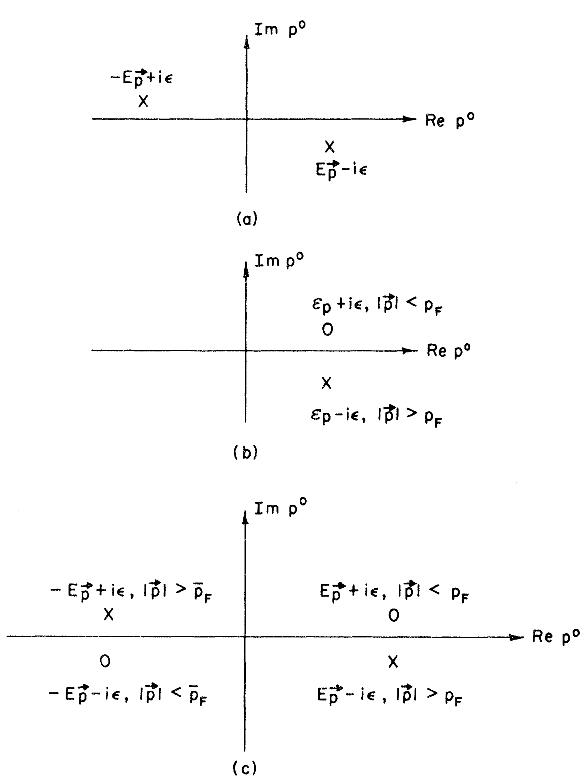

FIG. 2. The poles in the complex p_0 plane of the two-point function $S(p)$ are shown for three cases: (a) the relativistic elementary particle causal propagator $S_F(p)$, where positive- (particle) and negative- (antiparticle) energy states appear; (b) the nonrelativistic causal propagator for a thermodynamic system, where the antiparticle states are absent. The position of the pole depends upon the sign of $|\vec{p}|-p_F$, lying below the real axis (X) for $|\vec{p}|>p_F$ and above (0) for $|\vec{p}|<p_F$ (holes); (c) the relativistic thermodynamic causal propagator $S_F(p, p_F, \bar{p}_F)$, where both particle and antiparticle states appear as in (a). However, the finite density of the system results in two new states corresponding to holes. The position of positive- and negative-energy poles with respect to the real axis depends as in (b) on the sign of $|\vec{p}|-p_F$ and $|\vec{p}|-\bar{p}_F$.

The Pauli principle dictates that no particle with momentum $|\vec{p}|<p_F$ may be added to the system, with a similar constraint for antiparticles with momentum $|\vec{p}|<\bar{p}_F$; nor may particles with $|\vec{p}|>p_F$ be removed from the system, etc. In order that these constraints may be included as boundary conditions on $S(p)$, as well as the requirements of causality which lead to the particle-antiparticle interpretation of the poles $\mathrm{Re}\,p_0=\pm E_{\vec{p}}$, the infinitesimal imaginary functions $-i\delta_p$ and $-i\bar{\delta}_p$ of the momentum p and Fermi momenta p_F and $\bar{p}_F$ shall be added to the positive- and negative-energy poles respectively of (3.9):

$$E_{\vec{p}} \to E_{\vec{p}} - i\delta_p \,,$$
$$-E_{\vec{p}} \to -E_{\vec{p}} + i\bar{\delta}_p \,.$$

The functions δ_p and $\bar{\delta}_p$ are defined such that

$$\delta_p = \begin{cases} 1, & |\vec{p}|>p_F \\ -1, & |\vec{p}|\leq p_F \end{cases}$$

and

$$\bar{\delta}_p = \begin{cases} 1, & |\vec{p}|>\bar{p}_F \\ -1, & |\vec{p}|\leq \bar{p}_F \,. \end{cases}$$

The momentum and density dependent poles which result are shown in Fig. 2. The consequence of these restrictions is a relativistic noninteracting Green's function describing the behavior of an added fermion (particle or antiparticle) to a system of N_F particles and $\bar{N}_F$ antiparticles in its ground state, and is easily shown to have the momentum-space representation

$$S_F(p, p_F, \bar{p}_F) = \frac{\not{p}+m_0}{2E_{\vec{p}}} \left\{ \frac{1-n_F(p)}{p_0 - E_{\vec{p}} + i\epsilon} + \frac{n_F(p)}{p_0 - E_{\vec{p}} - i\epsilon} - \frac{1-\bar{n}_F(p)}{p_0 + E_{\vec{p}} - i\epsilon} - \frac{\bar{n}_F(p)}{p_0 + E_{\vec{p}} + i\epsilon} \right\} . \tag{3.12}$$

The density of particles $\rho = m_0 n_0$ and of antiparticles $\bar{\rho} = m_0 \bar{n}_0$ are parametrized by the Fermi momenta p_F and $\bar{p}_F$ through (3.10) and (3.11). At zero temperature the noninteracting fermion functions $n_F(p)$ and $\bar{n}_F(p)$ are defined by[33]

$$n_F(p) = \begin{cases} 1, & |\vec{p}| < p_F \\ 0, & |\vec{p}| > p_F \end{cases} \tag{3.13}$$

and

$$\bar{n}_F(p) = \begin{cases} 1, & |\vec{p}| < \bar{p}_F \\ 0, & |\vec{p}| > \bar{p}_F . \end{cases} \tag{3.14}$$

Finally, $S_F(x - x'; \rho, \bar{\rho})$ is given by the Fourier transform of (3.12) according to (3.8).

Physical arguments have been used to motivate the definition of the fermion two-point function. It is possible to derive (3.12) from (3.6) using the average (2.27), and the solutions to the Dirac equation. The procedure is straightforward and will not be given here. The general method is identical to that used in the case of bosons, which is outlined in Appendix A. Such derivations serve as guides to the construction of Green's functions. However it is to be emphasized that (3.12) may be taken as the definition of $S_F(p, p_F, \bar{p}_F)$, thus liberating it from any limitations of quantum field theory.

The final step in developing a relativistic noninteracting thermodynamic Green's function in flat-space-time is the generalization of (3.12) to cover systems which are not constrained to lie in their ground state. This is most easily effected by examining the boundary conditions imposed on $S_F(p)$ in the discussion above. Noting that the step functions $n_F(p)$ and $\bar{n}_F(p)$ defined by (3.13) and (3.14) may be interpreted as particle and antiparticle distribution functions at zero temperature, and recalling that the principle effect of finite temperatures results in assigning states with momenta $|\vec{p}| > p_F$ and $|\vec{p}| > \bar{p}_F$ a nonzero probability of occupancy, it is obvious that $n_F(p)$ and $\bar{n}_F(p)$ should be generalized to the temperature-dependent Fermi-Dirac distribution functions:

$$n_F(p;\beta) = \frac{1}{\exp[\beta(E_{\vec{p}} - \mu)] + 1} , \tag{3.15}$$

$$\bar{n}_F(p;\beta) = \frac{1}{\exp[\beta(E_{\vec{p}} - \bar{\mu})] + 1} . \tag{3.16}$$

The temperature is related to β by Boltzmann's constant k through the relation $\beta = (kT)^{-1}$. The particle chemical potential μ is related to the number density in the usual way:

$$n_0 = \frac{N_F}{V} = 2\int \frac{d^3p}{(2\pi)^3}\, n_F(p,\beta) . \tag{3.17}$$

A similar result relates the antiparticle chemical potential $\bar{\mu}$ to $\bar{n}_0$:

$$\bar{n}_0 = \frac{\bar{N}_F}{V} = 2\int \frac{d^3p}{(2\pi)^3}\, \bar{n}_F(p,\beta) . \tag{3.18}$$

The factors of 2 above account for spin. The relation between $n_F(p,\beta)$ and $\bar{n}_F(p,\beta)$ are shown in Fig. 3. The energies $E_{\vec{p}}$ and $\bar{E}_{\vec{p}}$, and the chemical potentials μ and $\bar{\mu}$ in the exponentials of (3.15) and (3.16) are physical quantities which are positive. For free particles with energies $E_{\vec{p}} = (|\vec{p}|^2 + m_0^2)^{1/2}$, the chemical potentials are

$$\mu(p_F) = (p_F^2 + m_0^2)^{1/2} , \tag{3.19}$$

$$\bar{\mu}(\bar{p}_F) = (\bar{p}_F^2 + m_0^2)^{1/2} . \tag{3.20}$$

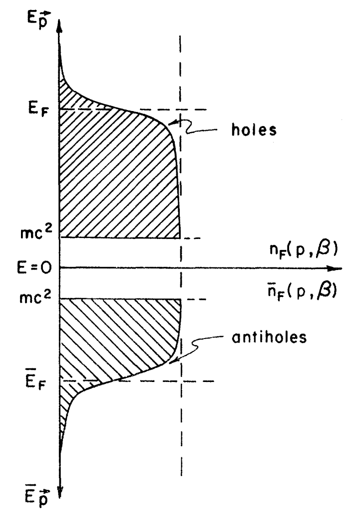

FIG. 3. The particle and antiparticle distribution functions $n_F(p, \beta)$ and $\bar{n}_F(p, \beta)$ are shown as a function of the physical energy E_p for a given temperature. There is a gap for $0 < E_p < mc^2$. The physical energy is positive for both particle and antiparticle states. In general, the Fermi energy E_F need not equal $\bar{E}_F$. The unfilled states with $|\vec{p}| < p_F$ correspond to holes in the particle Fermi sea. The unfilled states with $|\vec{p}| < \bar{p}_F$ in the antiparticle Fermi sea correspond to what might be termed "antiholes." In the absence of interactions the particle and antiparticle systems behave as separate independent components, each of which contribute to the pressure, density, etc. of the system.

Replacing the step functions in (3.12) by the temperature-dependent distribution functions $n_F(p,\beta)$ and $\bar{n}_F(p,\beta)$, we arrive at the relativistic thermodynamic two-point function with momentum representation:

$$S_F(p,\mu,\bar{\mu};\beta)=\frac{\not p+m_0}{2E_{\vec{p}}}\left\{\frac{1-n_F(p,\beta)}{p_0-E_{\vec{p}}+i\epsilon}+\frac{n_F(p,\beta)}{p_0-E_{\vec{p}}-i\epsilon}-\frac{1-\bar{n}_F(p,\beta)}{p_0+E_{\vec{p}}-i\epsilon}-\frac{\bar{n}_F(p,\beta)}{p_0+E_{\vec{p}}+i\epsilon}\right\}. \tag{3.21}$$

At finite temperature the density of the system is parametrized by the chemical potentials μ and $\bar{\mu}$ through (3.17) and (3.18). Equation (3.21) will be denoted in coordinate space by $S_F(x-x';\rho,\bar{\rho};\beta)$, where ρ and $\bar{\rho}$ are the particle and antiparticle densities, respectively.

The physical significance of the poles in (3.21) should be obvious. The singularities $p_0=E_{\vec{p}}\pm i\epsilon$ correspond to quasiparticles of energy $E_{\vec{p}}=(|\vec{p}|^2+m_0{}^2)^{1/2}$, momentum $\vec{p}$, and spin $\frac{1}{2}$. At zero temperature $n_F(p,\beta)$ reduces to $n_F(p)$ given by (3.13). The first term is then nonzero, corresponding to the addition of a particle, with momentum outside the Fermi sea $|\vec{p}|>p_F$, to the N-particle system. For $|\vec{p}|<p_F$ the numerator of the first term vanishes, and the second term describes the removal of a particle ($E_{\vec{p}}>0$) from the N-particle system. But this is just the relativistic extension of the well-known quasiparticle state of many-body theory. The states resulting from the removal of a particle with energy $E_{\vec{p}}<E_F$ will therefore be called "holes." These states are not to be confused with the particle-hole interpretation of the Dirac sea of negative-energy particles, since we are talking only about the positive-energy part of the propagator. The difference, it will be recalled, between "particle" and "hole" states in the quasiparticle sense is measured with respect to the Fermi sea, and no energy gap exists between the two; whereas the difference between particle and antiparticle states in the Dirac theory is measured with respect to zero energy, and an energy gap of $2mc^2$ does exist between the two. Therefore the first two terms of (3.12) describe the quasiparticles of a noninteracting system of N relativistic fermions at zero temperature. A similar analysis shows that the remaining two terms of (3.12) with poles $p_0=-E_{\vec{p}}\pm i\epsilon$ correspond to antiparticles of energy $E_{\vec{p}}>0$, momentum $-\vec{p}$, and spin $\frac{1}{2}$. For $|\vec{p}|>\bar{p}_F$, the third term describes the addition of an antiparticle to the system of $\bar{N}$ antiparticles. If $|\vec{p}|<\bar{p}_F$, the fourth term describes the removal of an antiparticle of energy $E_{\vec{p}}<\bar{E}_F$ from the system of $\bar{N}$ antiparticles, which is to be interpreted as the addition of an "antihole," in analogy with the quasiparticle convention established above. The quasiparticles of an antiparticle system are seen to be contained in the negative energy part of $S_F(p,p_F,\bar{p}_F)$. Relaxing the constraint on the temperature, the distribution functions $n_F(p,\beta)$ and $\bar{n}_F(p,\beta)$ make allowance for holes and antiholes with momenta above the Fermi momenta p_F and $\bar{p}_F$, respectively, and the addition of particles or antiparticles of momenta less than p_F and $\bar{p}_F$, respectively. The four possible quasiparticle states of an N particle and $\bar{N}$ antiparticle system are summarized in Table III.

Examination of the poles of $S_F(p,\mu;\bar{\mu};\beta)$ in Fig. 2 shows that there exists a fundamental physical distinction between the singularities resulting from the relativistic nature of the theory ($p_0=\pm E_{\vec{p}}\mp i\epsilon$), and those resulting from its thermodynamic or finite-density nature ($p_0=\pm E_{\vec{p}}\pm i\epsilon$). In the nonrelativistic limit the negative-energy poles do not contribute, and the well-known fermion thermodynamic Green's function results:

$$\lim_{m_0\to\infty} S_F(p,\mu,\bar{\mu};\beta)=G_F^{(0)}(\vec{p},\omega;\beta)=\frac{1-n_F(p,\beta)}{\omega-\mathcal{E}(\vec{p})+i\epsilon}+\frac{n_F(p,\beta)}{\omega-\mathcal{E}(\vec{p})-i\epsilon},$$

where the energy $\mathcal{E}(\vec{p})=p^2/2m$, $n_F(p,\beta)$ is given by (3.15) with $E_{\vec{p}}\to\mathcal{E}(\vec{p})$, and the chemical potential is

TABLE III. Collective states of the fermion two-point function as determined by the poles of $S_F(p;\mu,\bar{\mu})$ in the complex p_0 plane. From left to right the columns give (1) the pole in the complex plane; (2) the half-plane in which the integration is closed, with the real axis as part of the contour; (3) the restrictions on the momentum; (4) the resulting state in terms of the quasiparticle nomenclature of many-body theory. The third column applies only to the case of zero temperature, and would be replaced at finite temperature by a probability assignment through $n_F(p,\beta)$ or $\bar{n}_F(p,\beta)$.

p_0	Contour closed in	Momentum	Quasiparticle state
$E_{\vec{p}}-i\epsilon$	lower half plane	$\lvert\vec{p}\rvert>p_F$	"particle"
$E_{\vec{p}}+i\epsilon$	upper half plane	$\lvert\vec{p}\rvert<p_F$	"hole"
$-E_{\vec{p}}+i\epsilon$	upper half plane	$\lvert\vec{p}\rvert>\bar{p}_F$	"antiparticle"
$-E_{\vec{p}}-i\epsilon$	lower half plane	$\lvert\vec{p}\rvert<\bar{p}_F$	"antihole"

measured with respect to zero energy. The relativistic zero-density limit, in which the terms containing the singularities $p_0 = \pm(E_{\vec{p}} + i\epsilon)$ vanish, reduces to the fermion propagator from relativistic quantum field theory:

$$\lim_{N,\bar{N}\to 0} S_F(p,\mu,\bar{\mu};\beta) = S_F(p) = \frac{\not{p}+m_0}{p^2 - m_0^2 + i\epsilon}\,.$$

It is easily shown that the Fermi momentum and energy transform as the components of the 4-momentum $p_F^\mu = (E_F, p_F)$, with a similarly defined $\bar{p}_F^\mu$ for antiparticle components of the system. It follows therefore that if in one frame of reference a quasiparticle with 4-momentum $p^\mu = (E_{\vec{p}}, \vec{p})$ exists such that $E_{\vec{p}} > E_F$ and $|\vec{p}| > p_F$, then under the Lorentz transformation which takes $p^\mu \to p'^\mu = (E'_{\vec{p}}, \vec{p}')$ and $p_F^\mu \to p_F'^\mu = (E'_F, p'_F)$, one finds $E'_{\vec{p}} > E'$ and $|\vec{p}'| > p'_F$. Consequently an elementary excitation remains an elementary excitation of the same type under a Lorentz transformation. Since the distinction between noninteracting particles and antiparticles is well known to be invariant, it follows that the concept and type of quasiparticles for a noninteracting system of particles and antiparticles is Lorentz-invariant. The problems associated with the transformation and invariance of thermodynamic quantities, such as temperature, free energy, etc., in flat-space-time have been discussed in the literature.[25,34] The reader is referred to the article by Balescu for a historical review of this problem, as well as a discussion of relativistic statistical mechanics in flat-space-time.

B. Boson Two-Point Function

The two-point function describing the propagation of a spin-zero boson in a relativistic system of N_B particles and $\bar{N}_B$ antiparticles will now be discussed. For noninteracting systems, only the number of bosons present will enter into the argument, so that N_B and $\bar{N}_B$ will be understood to denote bosons only. The presence of fermions will enter only when interactions between fields are considered in the next section.

In the interests of generality, a complex pseudoscalar field $\varphi(x)$ will be considered, whose equation of motion in flat-space-time is given by (3.5):

$$(\partial^\mu \partial_\mu + \mu_0^2)\varphi(x) = 0\,,$$

with a similar equation for the complex conjugate filled $\varphi^*(x)$. The unrenormalized mass is denoted by μ_0, which is the same for both fields. The boson two-point function is defined by

$$\Delta(x-x') = -i\langle T\varphi(x)\varphi^*(x')\rangle \tag{3.22}$$

and is a solution of the equation

$$(\partial^\mu \partial_\mu + \mu_0^2)\Delta(x-x') = -\delta^4(x-x')\,. \tag{3.23}$$

The Fourier transform $\Delta(k)$ is defined according to earlier conversion by

$$\Delta(x-x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')}\Delta(k)\,. \tag{3.24}$$

As in the previous section translational invariance has been assumed. The 4-momentum is $k^\mu = (k^0, \vec{k})$. The solution to (3.23) in momentum space is easily shown to be

$$\Delta(k) = \frac{1}{2\omega_{\vec{k}}}\left(\frac{1}{k_0 - \omega_{\vec{k}}} - \frac{1}{k_0 + \omega_{\vec{k}}}\right) \tag{3.25}$$

in terms of the single-particle energy $\omega_{\vec{k}} = (k^2 + \mu_0^2)^{1/2}$, subject to suitable boundary conditions at $k^2 = \mu_0^2$.

The boundary conditions which are to be introduced in (3.25) may be deduced (with a bit more difficulty, due to the nature of the statistics imposed on fields of integer spin) along lines similar to those of the preceding section. Such an argument is, however, indirect. The relative simplicity of the pseudoscalar field makes it an ideal vehicle for a digression into the general method whereby finite-temperature and finite-density Green's functions may be derived within the framework of relativistic quantum field theory. The method is of significance in that it is simple, straightforward, and may be applied to fields of arbitrary spin and charge. The details are given in Appendix A.

The average appearing in (3.22) is given by (2.28)–(2.30). The trace is over a complete set of states labeled in part by the energy $\omega_{\vec{k}}$ and the number of bosons N_B and antibosons $\bar{N}_B$. Define the chemical potentials ζ and $\bar{\zeta}$ of the boson and antiboson systems through the relations

$$n_B(k,\beta) = \frac{1}{\exp[\beta(\omega_{\vec{k}} - \zeta)] - 1}\,, \tag{3.26}$$

$$\bar{n}_B(k,\beta) = \frac{1}{\exp[\beta(\omega_{\vec{k}} - \bar{\zeta})] - 1}\,. \tag{3.27}$$

At a critical temperature, the chemical potentials equal the lowest energy of the particles present in the system and Bose-Einstein condensation occurs. Above this temperature, ζ and $\bar{\zeta}$ are determined by the relations

$$\frac{N_B}{V} = \int \frac{d^3k}{(2\pi)^3} n_B(k,\beta)\,, \tag{3.28}$$

$$\frac{\bar{N}_B}{V} = \int \frac{d^3k}{(2\pi)^3} \bar{n}_B(k,\beta)\,, \tag{3.29}$$

where V is the volume. Below the critical temperature the chemical potentials are given in terms of the number of particles in the condensate.[30]

In terms of the quantities defined above, the boson two-point function for a finite density and temperature system is given (Appendix A) by[35]

$$\Delta_F(k,\zeta,\bar{\zeta};\beta)=\frac{1}{2\omega_{\vec{k}}}\left\{\frac{1+n_B(k,\beta)}{k_0-\omega_{\vec{k}}+i\epsilon}-\frac{n_B(k,\beta)}{k_0-\omega_{\vec{k}}-i\epsilon}-\frac{1+\bar{n}_B(k,\beta)}{k_0+\omega_{\vec{k}}-i\epsilon}+\frac{\bar{n}_B(k,\beta)}{k_0+\omega_{\vec{k}}+i\epsilon}\right\},$$

whose Fourier transform is denoted by $\Delta_F(x-x';N,\bar{N};\beta)$. The number density or chemical potential of each subsystem may be used to parametrize the thermodynamic nature of the Green's function.

In the limit of zero density, the terms $n_B(k,\beta)$ and $\bar{n}_B(k,\beta)$ vanish, and (3.30) reduces to the boson propagator in relativistic quantum field theory:

$$\lim_{N_B,\bar{N}_B\to 0}\Delta_F(k,\zeta,\bar{\zeta};\beta)=\Delta_F(k)=\frac{1}{k^2-\mu_0{}^2+i\epsilon}.$$

Next consider the nonrelativistic limit, in which

$$\omega_{\vec{k}}=(|\vec{k}|^2+\mu_0{}^2)^{1/2}\to\mathcal{E}(\vec{k})+\mu_0.$$

The negative-energy terms vanish in the limit $\mu_0\to\infty$, leaving

$$\lim_{\mu_0\to\infty}\Delta_F(k,\zeta,\bar{\zeta};\beta)=\frac{1}{2\mu_0}\left\{\frac{1+n_B(k)}{\omega-\mathcal{E}(\vec{k})+i\epsilon}-\frac{n_B(k)}{\omega-\mathcal{E}(\vec{k})-i\epsilon}\right\}.$$

This may be compared with the nonrelativistic limit of $S_F(p,\mu,\bar{\mu};\beta)$ given earlier. It will be seen that although the denominators are identical, apart from a difference in rest mass, the numerators reflect the different statistics which the two fields obey.

C. Interactions

The noninteracting relativistic Green's functions developed in this section contain the boundary conditions on density and temperature for a many-body system of fermions, bosons, and their antiparticles. It is now a simple matter to introduce interactions (in flat space-time) between these particles relativistically. Recalling that the functional-derivative approach developed by Schwinger has been applied in a Lorentz-invariant manner to the theory of elementary particles,[36] and that it has also been applied to nonrelativistic many-body system,[37] it is natural to consider the generalization of this technique to the thermodynamic region discussed above. Just such an extension has been developed by Fradkin.[18] The extension to include gravitational fields (in a manner consistent with Einstein's equations) is straightforward, and will be found in the next section. Its flat-space-time limit may trivially be taken, yielding results in agreement with those in the literature. In the case of flat space-time it is found that the N-point functions are determined by an infinite hierarchy of coupled integral equations which contain higher N-point functions (consistent with a given interaction). The thermodynamic nature of the problem is then introduced through the homogeneous terms in these integral equations.

The resulting set of equations for the Green's functions (in flat space-time) may be attacked by conventional methods of perturbation theory. Renormalization to physical quantities (which are independent of density and temperature) is achieved in the usual manner, although slight complications occur if renormalization to effective masses, etc. is desired. This will be discussed in more detail in Sec. V. The point to be emphasized, however, is that the thermodynamic nature of the equations results in no fundamental changes in the approach to the equations, although it does complicate the algebra considerably.

The equations have been investigated to second order in g for a system of baryons with a Yukawa coupling, the results of which will be published in a later paper.

IV. RELATIVISTIC MANY-BODY THEORY: CURVED SPACE-TIME

The previous restriction on the metric will now be lifted, and the general case of a relativistic system of interacting elementary particles in curved space-time considered. As will be shown, the curvature of space-time is expected to make significant contributions to the strong interactions and to the thermodynamic nature of the system as a result of its finite density. It is to be emphasized that only c-number metrics will be considered, and the quantum nature of space-time will be ignored.

The equations determining the exact interacting thermodynamic two-point functions in curved space-time will be presented first. Curved-space-time two-point functions in the absence of elementary particle interactions will be examined, and the question of boundary conditions considered. It will be shown that the finite density and temperature of the many-body system (including particles and antiparticles) may be introduced through flat-space-time propagators. It is partly for this reason that we have dealt so extensively with the special case of Minkowski space in the preceding section. Physically it is not surprising that the thermodynamic boundary conditions enter through flat-space-time propagators, for they are a consequence of the local dynamics that have been assumed, as well as of the local nature of the statis-

TABLE IV. Existing methods of constructing Green's functions are summarized in the table above. From left to right the columns list (1) a representative reference to the approach; (2) the dynamics which are included; (3) the thermodynamic nature of the system, determining the type of states that enter into the definition of the Green's function; (4) type of Green's function as determined by the way in which boundary conditions are imposed; (5) whether or not gravitation is included within the formalism.

Reference	Dynamics	Thermodynamic nature of system	Green's function	Space-time
Bogoliubov and Shirkov [a]	Relativistic	Zero density	Time dependent	Flat
Fradkin [b]	Relativistic	Zero density	Time dependent	Flat
Fradkin [b]	Relativistic	Finite density and temperature	Imaginary time	Flat
Martin and Schwinger [c]	Nonrelativistic	Finite density and temperature	Time dependent	Flat
Utiyama [d]	Relativistic	Zero density	Time dependent	Curved

[a] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959).
[b] E. S. Fradkin, in *Quantum Field Theory and Hydrodynamics,* edited by D. V. Strobel'tsyn (Consultants Bureau, New York, 1967).
[c] P. C. Martin and J. Schwinger, Phys. Rev. 115, 1343 (1959).
[d] R. Utiyama, Phys. Rev. 125, 1727 (1962).

tics (Fermi-Dirac or Bose-Einstein) imposed on identical particles. Since it is always possible to find a coordinate transformation which eliminates the effects of gravitation at a point, the local effects discussed above may be introduced in flat space-time. This will be shown in detail below. This does not imply that gravitation will not contribute to interactions, for the simple reason that the latter give extent to point particles. It is therefore impossible by coordinate transformations to eliminate gravitational contributions to the interactions of physical particles.

A. Interacting Two-Point Functions

Equations for the exact interacting two-point functions corresponding to the fermion fields $\psi(x)$ and the boson field $\varphi(x)$ will now be given for a relativistic thermodynamic system in curved space-time. The equations of motion for the fields $\psi(x)$ and $\varphi(x)$ are given by (2.21) and (2.22), and are coupled to Einstein's field equations through (2.23). In the pressure of interactions, the curved-space-time fermion two-point function will be denoted by $G(x, x')$, while the corresponding boson two-point function will be denoted by $D(x, x')$.

Equations defining $G(x, x')$ and $D(x, x')$ may be derived by a modification of the method of functional derivatives developed originally by Schwinger. Table IV summarizes the areas in which this approach has been used. Restrictions on the metric which enable one to generalize the flat-space-time derivation to include gravitation are discussed in the literature. The fundamental difference between the zero-density curved-space-time equations and those needed to describe many-body systems lies in the use of (2.28)–(2.30) for the N-point functions. The generalization to include many-body effects in curved space-time is straightforward. For this reason the results will be stated below. The derivation of $G(x, x')$ is outlined in Appendix B, from which the extension to describe other fields and higher N-point functions is obvious.

Starting with (2.21) and (2.22) and the definition of an N-point function (2.28)–(2.30), it is found that $G(x, x')$ and $D(x, x')$ satisfy the following equations, where the particle and antiparticle densities of the entire system are denoted by ρ and $\bar{\rho}$. Other thermodynamic parameters will enter as well; for example, a generalized temperature which will be position-dependent. We denote it symbolically by β. The fermion two-point function is the solution of

$$(-g)^{1/2}\{i\gamma^{\mu}(x)[\partial_{\mu}-\Gamma_{\mu}(x)]-m+\delta m\}\,G_F(x,x';\rho,\bar{\rho};\beta)=\delta^4(x-x')-(-g)^{1/2}\int d^4z(-g)^{1/2}\Sigma_C(x,z)G_F(z,x';\rho,\bar{\rho};\beta). \tag{4.1}$$

The self-energy operator, whose density and temperature dependence is understood, is given by

$$\Sigma_C(x,x') = ig^2\gamma^5(x)\int d^4z\int d^4y\, G_F(x,y;\rho,\bar{\rho};\beta)[-g(y)]^{1/2}\Gamma_5(x',y\,|\,z)D_F(x,z;\rho,\bar{\rho};\beta)[-g(z)]^{1/2}\ , \tag{4.2}$$

where the vertex function is defined in terms of the three-point function by the relation

$$\langle T\psi(x)\bar{\psi}(y)\varphi(z)\rangle_\beta = \int d^4\xi\int d^4\eta\int d^4\zeta\, G_F(x,\xi;\rho,\bar{\rho};\beta)[-g(\xi)]^{1/2}\Gamma_5(\xi,\eta\,|\,\zeta)[-g(\eta)]^{1/2}$$
$$\times G_F(y,\eta;\rho,\bar{\rho},\beta)[-g(\zeta)]^{1/2}D_F(z,\zeta;\rho,\bar{\rho};\beta)\,. \tag{4.3}$$

The temperature and density dependence of the vertex function is understood.

The boson Green's function $D(x,x')$ satisfies the equation

$$(-g)^{1/2}\{\partial^\nu\partial_\nu - \Gamma^\lambda_\nu\lambda\partial^\nu + \mu^2 - \delta\mu^2\}D_F(x,x';\rho,\bar{\rho},\beta) = -\delta^4(x-x') - i(-g)^{1/2}\int d^4z[-g(z)]^{1/2}\Pi_C(x,z)D_F(z,x';\rho,\bar{\rho};\beta)\,. \tag{4.4}$$

The polarization operator is defined in analogy with the self-energy by

$$\Pi_C(x,x') = -ig^2\,\mathrm{tr}\gamma^5(x)\int d^4y\int d^4z\, G_F(x,y;\rho,\bar{\rho},\beta)[-g(y)]^{1/2}\Gamma_5(y,z\,|\,x')[-g(z)]^{1/2}G_F(z,x';\rho,\bar{\rho};\beta)\,, \tag{4.5}$$

where the density and temperature dependence is understood.

The dependence of the two-point functions on the higher N-point functions is contained in the vertex function appearing in (4.2) and (4.5).

Equations (4.1) and (4.4) may be written in the form of integral equations containing the noninteracting two-point functions in curved space-time. They are

$$G_F(x,x';\rho,\bar{\rho};\beta) = G_F^{(0)}(x,x';\rho,\bar{\rho};\beta) + \int d^4y\int d^4z\, G_F^{(0)}(x,y;\rho,\bar{\rho};\beta)[-g(y)]^{1/2}\Sigma_C(y,z)[-g(z)]^{1/2}G_F(z,x';\rho,\bar{\rho};\beta) \tag{4.6}$$

for fermions, and

$$D_F(x,x';\rho,\bar{\rho};\beta) = D_F^{(0)}(x,x';\rho,\bar{\rho};\beta) + \int d^4y\int d^4z\, D_F^{(0)}(x,y;\rho,\bar{\rho};\beta)[-g(y)]^{1/2}\Pi_C(y,z)[-g(z)]^{1/2}D_F(z,x';\rho,\bar{\rho};\beta) \tag{4.7}$$

for bosons. The equations above serve as a convenient starting point for iterative solutions for a given metric tensor. Such an approach may be simplified by diagrammatic analysis as in flat space-time. Graphical representations of the integral equations for $G(x,x')$ and $D(x,x')$ (4.6) and (4.7) and for the self-energy and polarization operators (4.2) and (4.5) are given in Fig. 4.

The results above, together with the noninteracting curved-space-time functions $D_F^{(0)}(x,x';\rho,\bar{\rho};\beta)$ and $G_F^{(0)}(x,x';\rho,\bar{\rho};\beta)$ to be discussed below, constitute a relativistic finite-density and finite-temperature many-body formalism describing strongly interacting elementary particles in curved space-time. These equations allow in principle for the determination of the single-particle Green's functions $D_F(x,x';\rho,\bar{\rho};\beta)$ and $G_F(x,x';\rho,\bar{\rho};\beta)$, which contain all relevant information about such matters as particle spectrum, elementary excitation spectrum, effective masses, pressure, density, and others.

The system of equations (4.1)–(4.5) represents a complete set of coupled equations for the two-point functions $G_F(x,x';\rho,\bar{\rho};\beta)$ and $D_F(x,x';\rho,\bar{\rho};\beta)$, in terms of the three-point function and the metric tensor $g_{\mu\nu}$. In order to complete the description of a many-body system in curved space-time, we show how the equations for $g_{\mu\nu}(x)$ (Einstein's equations) depend upon the matter distribution through the Green's functions. In the usual approach to Einstein's equations, the microscopic (nongravitational) behavior of matter is introduced through the energy-momentum density tensor $T^{\mathrm{av}}_{\mu\nu}$, whose form is assumed *a priori*. Equations (2.23) are then solved for $g_{\mu\nu}$, subject to suitable boundary conditions on the curvature of space-time. The distribution of matter, as given by $T^{\mathrm{av}}_{\mu\nu}$, may therefore be considered to determine, through Einstein's equations, the structure of space-time. In the approach of this paper $T^{\mathrm{av}}_{\mu\nu}$ is not assumed, but must be calculated in terms of $g_{\mu\nu}$. In fact what is given is a description of allowed elementary particle interactions through a Lagrangian as well as the amount of matter present in space-time, from which the actual distribution ($T^{\mathrm{av}}_{\mu\nu}$) and $g_{\mu\nu}$ are to be determined. The upshot, from a mathematical

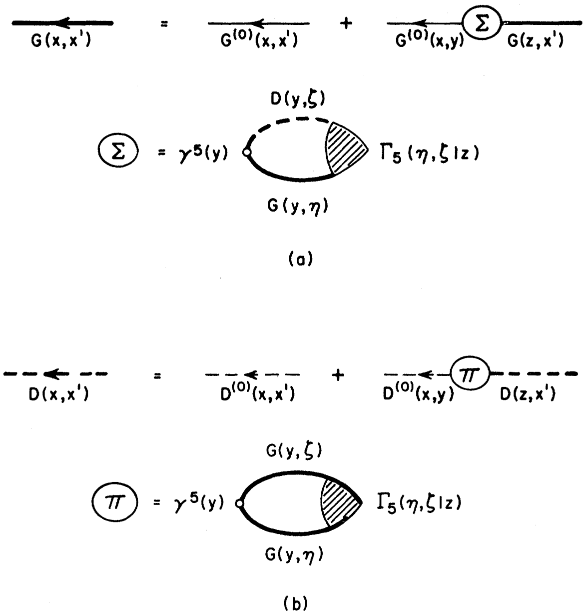

FIG. 4. Diagrammatic representation of the integral equations (a) for $G(x, x')$ and (b) $D(x, x')$ and for the self-energy and polarization operators $\Sigma(x, x')$ and $\Pi(x, x')$. Heavy lines correspond to exact Green's functions; light lines to noninteracting curved-space-time Green's functions. The exact function $\Gamma_5(x, y|z)$ is represented by a shaded vertex; the bare vertex by ○.

point of view, is that the equations determining the metric and those determining the matter fields are coupled through $T^{\text{av}}_{\mu\nu}$, and must be solved self-consistently; from the standpoint of physics, the result is a more nearly fundamental description of matter in space-time.

B. Equation of State

A less exact, though interesting situation is one where contributions to the interactions between particles due to curvature are negligible,[38] but where it is desirable to include the relativistic nature of particle dynamics and couplings in constructing the energy-momentum density tensor. The procedure is equivalent to calculating the equation of state for the system, given the relativistic particle interactions. Consider for simplicity a collection of fermions of mass m and ignore the possibility of real bosons in the system. Such will be the case if it is assumed that the curvature of space-time is small across a particle, although it need not be small over a sample containing a large number of particles. In the following we shall, however, assume that it is small over a region containing many particles. The restrictions above result in a significant simplification of the equations of statistical mechanics, in that they admit the following approximations:

(i) the subvolume of the entire system is macroscopically large, but small compared to the curvature of space-time;
(ii) the subsystem is in equilibrium;
(iii) the first law of thermodynamics is assumed to hold in each subsystem.

It is then possible to express the pressure, volume, temperature, and partition function in the form

$$\ln Z_g = \beta(x)P(x)V\,.$$

The notation $\beta(x)$ and $P(x)$ is understood to signify the location, within the entire system, of the subvolume V, and should not be interpreted as coordinate dependence of these quantities within that subvolume. Furthermore Z_g is evaluated within V, and its parametric dependence on x will be understood. If we next define the chemical potential of the system by $\mu(p_F; x)$ – this may be done in terms of the two-point function – it is possible to show that the pressure, within the subvolume at the point x, is

$$P(\mu, x) - P(0, x) = \int_0^{\mu(p_F)} d\mu' n_0(x, \mu')$$
$$= -i \lim_{x'\to x+0} \int_0^{\mu(p_F, x)} d\mu' \times \operatorname{tr}\gamma^0(x) G_F(x, x'; \mu', \beta).$$

In the limit, $x' \to x+0$ specifies that the hypersurface containing x'^μ occurs "after" that containing x^μ. The term $P(0, x)$ represents the pressure at zero density, and is assumed to be zero. Next consider the density $\rho_0(x)$ of matter (in the subvolume located at x) as seen by an observer moving with 4-velocity $u^\mu(x)$. Then in terms of the number density $n_0(x)$, as seen by a comoving observer

$$\begin{aligned}\rho_0(x) &= m n_0(x)\\ &= m\, \operatorname{tr}\lambda^{(0)}_\mu \langle T\overline{\psi}(x)\gamma^\mu(x)\psi(x)\rangle_\beta\\ &= -im \lim_{x'\to x+0} \operatorname{tr}\gamma^0(x) G_F(x, x'; \mu', \beta)\,.\end{aligned}$$

Finally we assume that it is possible to characterize the system by the equation of state for a perfect fluid

$$P = P(\rho_0, \beta)\,.$$

The energy-momentum density tensor is then given by

$$T^{\text{av}}_{\mu\nu}(x) = \rho_0(x) u_\mu(x) u_\nu(x) + P(x)[u_\mu(x) u_\nu(x) - g_{\mu\nu}]\,,$$

which is expressible in terms of the two-point function through the relations above. Consequently, Einstein's equations become

$$R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R=8\pi i\kappa\lim_{x'\to x+0}\left\{m\,\mathrm{tr}\gamma^0(x)G_F(x,x';\mu,\beta)u_\mu(x)u_\nu(x)+\int_0^{\mu(p_F)}d\mu'\,\mathrm{tr}\gamma^0(x)G_F(x,x';\mu,\beta)[u_\mu(x)u_\nu(x)-g_{\mu\nu}]\right\}.$$

It is to be emphasized that this procedure, as a result of the restrictions mentioned above, does not contain the contribution of gravitation to interactions, but does describe relativistic interacting particles through the two-point function $G_F(x,x',\mu,\beta)$. An analysis which includes curvature on the microscopic level is possible, though more difficult because of the possible coordinate dependence of the Lagrange multipliers $\lambda_i(x)$ appearing in the partition function. The most general approach would involve the construction of $T_{\mu\nu}$ from a variational principle and the Lagrangian density for matter fields in covariant form. The expectation value $T_{\mu\nu}=T^{\mathrm{av}}_{\mu\nu}$ would then be decomposed in terms of various N-point functions and the result coupled to Einstein's equations.

C. Noninteracting Curved Space-Time Two-Point Functions

In writing expressions for the interacting curved-space-time two-point functions (4.6)–(4.7), one has in mind the possibility of iterative solutions based upon a knowledge of the homogeneous two-point functions $G_F^{(0)}(x,x';\rho,\bar{\rho},\beta)$ and $D_F^{(0)}(x,x';\rho,\bar{\rho},\beta)$, plus an ansatz for the vertex function $\Gamma_5(x,y\,|\,z)$. It is assumed that the homogeneous term is just the noninteracting curved-space-time two-point function which results from setting the strong-interaction coupling constant $g=0$. The solution of the noninteracting equations for the two-point functions in flat space-time is trivial; however, the presence of gravitation makes the equivalent problem in curved space-time more difficult due to the requirement that (2.23) also be satisfied for $T^{\mathrm{av}}_{\mu\nu}(g=0)$. Throughout the following discussion it will be assumed that the metric is given, and that together with $D^{(0)}(x,x')$ and $G^{(0)}(x,x')$ it is a self-consistent solution for $g=0$ of (2.21)–(2.23).

Using the results of previous sections, the noninteracting boson two-point function in curved space-time is defined by

$$D^{(0)}(x,x')=-iZ_g^{-1}\,\mathrm{tr}\{\hat{\rho}T\varphi(x)\varphi(x')\}. \tag{4.8}$$

The noninteracting field $\varphi(x)$ is assumed for simplicity to be real. The density operator $\hat{\rho}$ is given in general by (2.30) and need not be specified in more detail for the following discussion, except to note that it is independent of coordinates. Similarly the noninteracting fermion two-point function in curved space-time is

$$G^{(0)}(x,x')=-iZ_g^{-1}\,\mathrm{tr}\{\hat{\rho}T\psi(x)\bar{\psi}(x')\}, \tag{4.9}$$

where $\psi(x)$ is the noninteracting spinor field. The partition function $Z_g=\mathrm{tr}\hat{\rho}$. The two-point functions $D^{(0)}(x,x')$ and $G^{(0)}(x,x')$, subject to suitable boundary conditions to be examined below, are solutions of the noninteracting equations of motion, which from (2.21)–(2.22) are easily shown to be

$$(-g)^{1/2}\{\partial_\mu\partial^\mu-\Gamma^\lambda_{\mu\lambda}\partial^\mu+\mu_0^{\,2}\}D^{(0)}(x,x')=-\delta^4(x-x'), \tag{4.10}$$

$$(-g)^{1/2}\{i\gamma^\mu(x)[\partial_\mu-\Gamma_\mu(x)]-m_0\}G^{(0)}(x,x')=\delta^4(x-x'), \tag{4.11}$$

respectively. It is immediately obvious that the two-point functions depend upon curvature, and in the absence of gravitation ($g_{\mu\nu}=\eta_{\mu\nu}$), that they may be reduced to (3.7) and (3.23). The latter suggests that the two-point functions be expanded in terms of the space-time interval $\sigma(s,s')$ which is defined as the geodesic measure connecting $x^\mu(s)$ and $x'^\mu(s')$. Such an expansion has been discussed in the literature[22] for the vacuum Green's function. Denote $\sigma(s,s')$ by

$$\sigma(s,s')=\tfrac{1}{2}(s-s')\int_{s'}^{s}ds\,g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}, \tag{4.12}$$

where s is the path parameter, and set

$$\Delta(x,x')=-\frac{\det(\sigma_{;\mu\nu})}{[-g(x)]^{1/2}[-g(x')]^{1/2}}. \tag{4.13}$$

In the last line the convention used is as follows: covariant differentiation with respect to x^μ is denoted by $(;\mu)$ as usual; covariant differentiation with respect to x'^μ is denoted by $(;\mu')$. It then follows that the two-point function in curved space-time is given by

$$D^{(0)}(x,x')=\Delta(x,x')^{1/2}\sum_{n=0}^{\infty}A_n(x,x')\left(\frac{\partial}{\partial\mu_0^{\,2}}\right)^n\Delta(x-x')\times\frac{\eta^{\mu\nu}(x_\mu-x'_\mu)(x_\nu-x'_\nu)}{2\sigma(x,x')} \tag{4.14}$$

subject to suitable boundary conditions yet to be specified. The coefficients $A_n(x,x')$ are given in terms of $\sigma(x,x')$ and the geodesic parallel displacement matrix $I(x,x')$ by recurrence relations.[22] The term $\Delta(x-x')$ is just the flat-space-time boson two-point function satisfying (3.23) and given by (3.24) and (3.25). Lumping all factors, except $\Delta(x-x')$, appearing in the right-hand side of (4.14) into the term $C_n(x, x'; \mu_0)$, we write

$$D^{(0)}(x, x') = \sum_{n=0}^{\infty} C_n(x, x'; \mu_0^2)\Delta(x - x'; \mu_0).$$

The coefficient C_n contains, apart from the mass differentiation operator, geometrical factors describing the structure of space-time. The factor $\Delta(x - x')$ is independent of the geometry. Since the flat-space-time propagator is independent of curvature, we may go one step further and define the symbolic operator $\mathcal{C}(x, x'; \mu_0)$. When operating on the flat-space-time propagator $\mathcal{C}(x, x', \mu_0)$ generates its curved-space-time equivalent:

$$D^{(0)}(x, x') \equiv \mathcal{C}(x, x', \mu_0)\Delta(x - x'). \tag{4.15}$$

We hasten to emphasize that the above relation is purely formal, and that $\mathcal{C}(x, x'; \mu_0)$ must be determined self-consistently through Einstein's equations and $D^{(0)}(x, x')$, which it serves to define. Setting aside the question of geometrical singularities in $\mathcal{C}(x, x', \mu_0)$, it will be observed that the analytic behavior of $D^{(0)}(x, x')$ is determined primarily through the behavior of $\Delta(k)$ at $k^2 = \mu_0^2$. The finite density and finite temperature of the system are consequently introduced into $D^{(0)}(x, x')$ through the boundary conditions on $\Delta(k)$. From the discussion of Sec. III it will be seen that the causal propagator describing the behavior of a boson in a many-body system in curved space-time is given by

$$D_F^{(0)}(x, x', \rho, \bar{\rho}; \beta) = \mathcal{C}(x, x'; \mu_0)\Delta_F(x - x'; \rho, \bar{\rho}; \beta). \tag{4.16}$$

In order to break away from quantum field theory, (4.10) and (4.11) may be taken as defining relations for the two-point functions in place of (4.8) and (4.9). The boundary conditions are then imposed through (4.16), which are a result of Einstein's equations and (4.10) and (4.11).

Once the boson propagator $D^{(0)}(x, x')$ is known it is straightforward to construct the spinor two-point function $G^{(0)}(x, x')$. From the tensor equations (4.10) and (4.11) and a knowledge of their properties in flat space-time it is possible to generate $G^{(0)}(x, x')$ from $D^{(0)}(x, x')$ by operating on the latter with

$$\{i\gamma^\mu(x)[\partial_\mu - \Gamma_\mu(x)] + m_0\},$$

where μ_0 is replaced by m_0 in $D^{(0)}(x, x')$. Therefore

$$G^{(0)}(x, x'; m_0) = \{i\gamma^\mu(x)[\partial_\mu - \Gamma_\mu] + m_0\}D^{(0)}(x, x'; m_0), \tag{4.17}$$

which, by (4.15), may be written as

$$G^{(0)}(x, x'; m_0) = \{i\gamma^\mu(x)[\partial_\mu - \Gamma_\mu] + m_0\}\,\mathcal{C}(x, x'; m_0) \times \Delta(x - x'; m_0).$$

As in the case of $D^{(0)}(x, x')$ above, the analytic structure of $G^{(0)}(x, x')$ will be determined by the poles of $\Delta(x - x'; m_0)$ in momentum space. Examination of (3.9) for $S(p)$ and (3.25) for $\Delta(k)$ shows, apart from a trivial change of integration variable and mass, that the analytic structure of the two is identical. It is therefore possible to introduce fermion statistics and the finite-density and finite-temperature boundary conditions through the term $\Delta(x - x'; m_0)$. The result may be denoted formally in terms of the operator $\mathcal{S}(x, x', m_0)$ which when acting on the flat-space-time fermion propagator $S_F(x - x'; m_0)$ converts it into the curved-space-time propagator:

$$G^{(0)}(x, x', \rho, \bar{\rho}, \beta) = \mathcal{S}(x, x'; m_0)S_F(x - x'; \rho, \bar{\rho}; \beta). \tag{4.18}$$

Equation (4.18) formally denotes the introduction of thermodynamic boundary conditions through the flat-space-time Green's function. The operator $\mathcal{S}(x, x'; m_0)$ is to be determined self-consistently through Einstein's equations and $G^{(0)}(x, x')$ which it serves to define.

The usefulness of the covariant Taylor series expansion (4.14) is obviously limited to small values of the space-time interval $\sigma(x, x')$. This in no way limits its applicability in treating boundary conditions, and is aptly suited to the discussion of renormalization.[39] It will be noticed that (4.16) and (4.18) contain unrenormalized (bare) masses.

The expansions above allow us to write the integral equations for $G^{(0)}(x, x')$ and $D^{(0)}(x, x')$ in terms of flat-space-time propagators. For the case of fermions, (4.18) and (4.6) yield

$$G_F(x, x'; \rho, \bar{\rho}; \beta) = \mathcal{S}(x, x'; m_0)S_F(x - x'; \rho, \bar{\rho}; \beta) + \int d^4y \int d^4z\, \mathcal{S}(x, y; m_0)S_F(x - y; m_0)[-g(y)]^{1/2}\Sigma_C(y, z)[-g(z)]^{1/2}\, G_F(z, x'; \rho, \bar{\rho}; \beta),$$

with similar expressions for the boson two-point function, the polarization and self-energy operator, and the vertex function. It is evident from these equations that the many-body effects may in fact be introduced through the homogeneous part of the flat-space-time noninteracting Green's functions.

V. RENORMALIZATION

A feature basic to all many-body theories, including those dealing with virtual particles, is the process of renormalization, by which is meant the replacement of the mass and charge of each particle entering into the theory by its observed value. It is encountered both in elementary particle physics, as well as in nonrelativistic many-body theory. Basically renormalization results in a shift in the values of the mass and charge as a result of interactions. Before turning to a discussion of the effects of curved space-time on the renormalization procedure, it is advisable to clarify a point concerning the removal of divergences in relativistic quantum field theory. As mentioned above, all theories (whether or not they contain divergences) involve renormalization. For a many-body system this means that the relevant masses are the effective masses, which depend not only upon coupling constants, but also upon density and temperature. For nonthermodynamic systems (e.g., small numbers of elementary particles interacting in vacuum) the relevant masses are the physical masses.

As is well known, many relativistic field theories exist, where infinities arising in the calculation of radiative corrections to any process may be eliminated in the course of renormalization to physical masses and charges. That this is by no means the rule is evidenced by pseudoscalar-meson theory involving derivative couplings, where regularization is not accomplished by renormalization. It is to be emphasized that the two processes, renormalization and regularization, are distinct.[16]

The renormalization of the many-body theory in curved space-time will be discussed in terms of the fermion and boson two-point functions and vertex operator discussed in the preceding section. The question of regularization will then be considered.

The formal renormalization procedure is basically identical to that used in flat space-time. Renormalization coefficients z_1, z_2, and z_3 in addition to mass counterterms δm and $\delta\mu$, are introduced, either in terms of fields or N-point functions. Defining renormalized quantities by a subscript R, one sets

$$\psi(x) = Z_2^{1/2}\psi_R(x)\,,$$

$$\varphi(x) = Z_1^{1/2}\varphi_R(x)\,,$$

$$g^R = Z_1^{-1}Z_2Z_3^{1/2}g_0\,.$$

The two-point functions are then

$$D(x,x') = Z_3D_R(x,x')\,,$$

$$G(x,x') = Z_2G_R(x,x')$$

and the vertex function is

$$\Gamma_5(x,y\,|\,z) = Z_1^{-1}\Gamma_5^R(x,y\,|\,z)\,.$$

In order to illustrate some of the consequences of renormalization in curved space-time, the specific example of mass renormalization will be used. Throughout the discussion it will be assumed that the metric is given, and that the results of renormalization are consistent with it. In flat-space-time renormalization is most easily discussed in momentum space, where it is trivial to solve formally for the wave-function renormalization coefficients in terms of the two-point function on the mass shell, and the vertex operator. The simplicity of this approach results in part from the fact that the noninteracting two-point function takes a trivially manageable form as a function of momentum. That this is apparently not the case in curved space-time is suggested by (4.14). Due to the absence of translational invariance in general, the Fourier transforms of $G_F(x,x')$ will be of the form $G_F(p,p')$. The effective mass $m_{\rm eff}$ is then defined as being the zero of the inverse Green's function on the mass shell:

$$\lim_{p^2,p'^2\to m_{\rm eff}} G_F(p,p';\alpha)^{-1} = 0\,.$$

Similarly the boson effective mass $\mu_{\rm eff}$ is given by the solution of

$$\lim_{k^2,k'^2\to \mu^2_{\rm eff}} D_F(k,k';\alpha)^{-1} = 0\,.$$

The parameter α in the argument of the two-point functions denotes the density of all particles and antiparticles in the system, as well as the temperature of the system. Any other thermodynamic variables which enter will also be understood to be included. It is to be emphasized that $m_{\rm eff}$ and $\mu_{\rm eff}$ as defined above are density- and temperature-dependent. Furthermore, as the density is determined by the metric $g_{\mu\nu}$, it follows that the effective masses will in general depend upon the curvature of space-time. This is a highly significant result for the study of superdense matter, where curvature may be expected to dominate elementary particle interactions.[15]

The renormalized coupling constant, determined in part by the vertex function, is also expected to be a function of density and temperature in general. This raises the intriguing question as to possible effects on the strength of interactions and bound states resulting from extreme curvature in superdense matter.

It is well known that the infinities arising from the radiative corrections in the relativistic quantum field theory of point interactions are unaffected by the inclusion of c-number gravitational fields. This has been shown in the literature for quantum electrodynamics.[34] The divergences persist in the presence of c-number gravitational fields and the regularization procedure is identical to that used in flat space-time. Since the proof depends upon the equivalence principle and the point nature of the interaction, it is expected to hold for the pseudoscalar coupling discussed above. Consequently, the curved-space-time many-body theory may be regularized by the introduction of three quantities: two wave-function renormalization coefficients and a vertex renormalization coefficient (we ignore the effect associated with the term[16] $\lambda\varphi^4$, where λ is a coupling constant, which is of importance for π-π scattering). The bare quantities m_0, μ_0, and g_0 may be eliminated in favor of their physical counterparts m_{phys}, μ_{phys}, and g_{phys}. The latter are independent of density and temperature, being just the flat-space-time vacuum parameters normally appearing in relativistic quantum field theory. It is then possible to express the physical parameters m_{eff}, μ_{eff}, and g_{eff} in terms of known quantities:

$$m_{\text{eff}} = m_{\text{eff}}(m_{\text{phys}}, g_{\text{phys}}; \alpha),$$

$$\mu_{\text{eff}} = \mu_{\text{eff}}(\mu_{\text{phys}}, g_{\text{phys}}, \alpha),$$

$$g_{\text{eff}} = g_{\text{eff}}(g_{\text{phys}}, m_{\text{phys}}, \mu_{\text{phys}}, \alpha),$$

where α denotes the density and temperature of the system. The procedure above yields a system of equations which are regularized and which depend upon physical parameters (m_{phys}, etc.). They constitute a reasonable starting point for any calculation involving the two-point functions of a system of strongly interacting superdense matter in curved space-time.

VI. CONCLUSION

The result of the previous sections constitute a relativistic many-body theory, which includes the strong interactions as given by a theory of elementary particle physics, as well as the curvature of space-time as determined by Einstein's theory of gravitation. The discussion has for simplicity been limited to the case of a single spin-one-half field coupled to a single real spin-zero field via a Yukawa coupling. The exact two-point functions $G(x, x'; \rho, \beta)$ and $D(x, x'; \rho, \beta)$ were defined as solutions of a set of coupled integral equations, higher N-point functions, and Einstein's equations, and depend upon a generalized temperature parameter β, as well as the density ρ of particles and antiparticles in the system. It was shown that the many-body effects could be introduced as boundary conditions on the homogeneous part of the Green's function $G^{(0)}(x, x'; \rho, \beta)$ and $D^{(0)}(x, x'; \rho, \beta)$. The latter are expressed in terms of the corresponding homogeneous part of the flat-space-time propagators, and geometrical factors which are functionals of the metric tensor.

Renormalization and regularization of the equations was discussed. The renormalization of observable parameters to their effective values was emphasized. The effective masses and charge will consequently depend upon the geometry of space-time.

In developing the N-point functions in curved space-time a generalized definition of the statistical density operator was given, which includes contributions due to gravitation. Its special-relativistic limit was shown to agree with previous results, and its nonrelativistic limit taken. The latter contains, as corrections to the interaction between particles, the Newtonian potential energy of the gravitational field. Furthermore, a position-dependent temperature parameter was maintained throughout. Further investigation of the physical significance of the temperature parameter, as well as other thermodynamic parameters, in the presence of gravitational fields (as introduced through the curvature of space-time) is needed. This may be approached either by examining specific thermodynamic models such as the exchange of energy between two systems, or by making contact with general-relativistic kinetic theory. Such work will be of interest not only as it affects the generally relativistic formalism, but also for any insight which it might yield concerning the position dependence of such concepts as temperature in the presence of gravitational fields in the Newtonian limit.

A major accomplishment of this work is reflected in the fact that it makes no fundamental distinction between interactions and interacting particles. At no point in the discussion is it necessary to separate self-interactions which give rise to particle structure from the interparticle interactions reflecting the many-body nature of the problem. Our approach takes into consideration not only the interactions between particles in determining the macroscopic behavior of a system, but contributions due to the particle's structure as well. Finally it does so in curved space-time, thereby including gravitational contributions to the interactions.

Although the entire discussion was developed around a spin-one-half and a real pseudoscalar field, the method is quite general and may be applied to systems composed of particles represented by other fields as well. The generalization to include more than one field of a given type is trivial. Other interactions than the Yukawa interaction may also be included through additional terms in the Lagrangian density. In fact, (2.2) may be generalized to include as many additional fields and interactions as may be of interest for problems of astrophysics or cosmology. As an example we have investigated the equation of state for superdense matter in flat space-time, using a Lagrangian density which includes fields representing the first SU(3)-symmetric baryon octet, and the first eight mesons in the SU(3)-symmetric pseudoscalar and vector octets. An SU(3)-invariant coupling is included. The results will be discussed in a subsequent paper.

The extension to include gravitation is formally straightforward, and results from writing the equations for the additional fields in generally covariant form, and coupling them to Einstein's equations through the energy-momentum density tensor.

The formalism of the previous sections in the flat-space-time limit represents a relativistic many-body theory which is applicable to systems where nonrelativistic theories fail. In particular, it offers a method of studying the equation of state of superdense matter in a realistic way, which takes into account in a field-theoretical way the interactions between elementary particles. This makes it possible to extend calculations of macroscopic properties of matter beyond the limit $\rho \sim 10^{15}$ g/cm^3, which marks the terminus of work based on potentials. There is every reason to expect that the macroscopic behavior of superdense matter will show collective effects analogous to those familiar at normal densities. Obvious examples are: effects of screening on the equation of state; plasma oscillations and vibrational modes in neutron stars and in gravitational collapse;[40] bounds placed in the equation of state due to causality;[41] superfluidity, and superconductivity due to charged constituents in the stellar cores; ferromagnetism[42]; and Bose-Einstein condensation of integer-spin particles such as pions in stellar interiors, to mention but a few of the effects which have been discussed in the literature. The treatment of such topics as these at densities $\rho > 10^{15}$ g/cm^3 requires a fully relativistic many-body theory of strongly interacting matter, such as the one presented above.

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APPENDIX A

The boson two-point function $\Delta_F(k,\zeta,\bar{\zeta};\beta)$ given by (3.30) will be derived within the framework of relativistic quantum field theory. The solutions of the Klein-Gordon equation in flat space-time are

$$\varphi(x)=\int\frac{d^3k}{[2\omega_{\vec{k}}(2\pi)^3]^{1/2}}\,[a(\vec{k})e^{-ik\cdot x}+b^\dagger(\vec{k})e^{ik\cdot x}]\,, \tag{A1}$$

$$\varphi^*(x)=\int\frac{d^3k}{[2\omega_{\vec{k}}(2\pi)^3]^{1/2}}\,[a^\dagger(\vec{k})e^{ik\cdot x}+b(\vec{k})e^{-ik\cdot x}]\,, \tag{A2}$$

where $k\cdot x=\omega_{\vec{k}}t-\vec{k}\cdot\vec{x}$. The operators $a(\vec{k})$, $b(\vec{k})$, etc. satisfy the commutation relations

$$[a(\vec{k}),\ a^\dagger(\vec{k}')]=\delta(\vec{k}-\vec{k}')=[b(\vec{k}),b^\dagger(\vec{k}')]\,, \tag{A3}$$

$$[a(\vec{k}),a(\vec{k}')]=[b(\vec{k}),b(\vec{k}')]=0\,. \tag{A4}$$

All other combinations have zero commutators. Substituting (A1)–(A2) into (3.22) gives

$$\begin{aligned}\Delta_F(x-x')&=-i\theta(t-t')\langle\varphi(x)\varphi^*(x')\rangle_\beta-i\theta(t'-t)\langle\varphi^*(x')\varphi(x)\rangle_\beta\\&=\int\frac{d^3k}{(2\pi)^3}\int\frac{d\omega}{2\pi}\,\frac{1}{2\omega_{\vec{k}}}\,\frac{1}{\omega+i\epsilon}\{\langle a(\vec{k})a^\dagger(\vec{k})\rangle_\beta e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-i\omega(t-t')}+\langle b^\dagger(\vec{k})b(\vec{k})\rangle_\beta e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{-i\omega(t-t')}\\&\qquad+\langle a^\dagger(\vec{k})a(\vec{k})\rangle_\beta e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{i\omega(t-t')}+\langle b(\vec{k})b^\dagger(\vec{k})\rangle_\beta e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}e^{i\omega(t-t')}\}\,.\end{aligned} \tag{A5}$$

In arriving at (A5) use has been made of the commutation relations, and the time-ordering operator has been written in terms of the step function $\theta(t-t')$. Finally an integral representation for $\theta(t-t')$ has been used to introduce the integration over ω. The expectation values appearing in (A5) are reducible, through (A3), to the two terms

$$\begin{aligned}\langle a^\dagger(\vec{k})a(\vec{k})\rangle_\beta&\equiv Z_g^{-1}\,\mathrm{tr}\{\exp[-\beta(\omega_{\vec{k}}-\zeta)]\,a^\dagger(\vec{k})a(\vec{k})\}\\&=n_B(k,\beta),\end{aligned} \tag{A6}$$

$$\begin{aligned}\langle b^\dagger(\vec{k})b(\vec{k})\rangle_\beta&\equiv \bar{Z}_g^{-1}\,\mathrm{tr}\{\exp[-\beta(\omega_{\vec{k}}-\bar{\zeta})]\,b^\dagger(\vec{k})b(\vec{k})\}\\&=\bar{n}_B(k,\beta)\,,\end{aligned} \tag{A7}$$

where

$$Z_g = \operatorname{tr} \exp[\beta(\omega_{\vec{k}}^{+} - \zeta)] \tag{A8}$$

$$\overline{Z}_g = \operatorname{tr} \exp[-\beta(\omega_{\vec{k}}^{+} - \overline{\zeta})]\,. \tag{A9}$$

In each term of (A5) change the variables of integration as follows: in the first term let $k_0 = \omega_{\vec{k}}^{+} + \omega$; in the second term let $k_0 = \omega - \omega_{\vec{k}}^{+}$ and $\vec{k} \to -\vec{k}$; etc. Then it is possible to write the right-hand side of (A5) in the form

$$\int \frac{d^3k dk_0}{(2\pi)^4 2\omega_{\vec{k}}^{+}} e^{-ik\cdot(x-x')} \left\{ \frac{1+n_B(k,\beta)}{k_0 - \omega_{\vec{k}}^{+} + i\epsilon} - \frac{n_B(k,\beta)}{k_0 - \omega_{\vec{k}}^{+} - i\epsilon} - \frac{1+\overline{n}_B(k,\beta)}{k_0 + \omega_{\vec{k}}^{+} - i\epsilon} + \frac{\overline{n}_B(k,\beta)}{k_0 + \omega_{\vec{k}}^{+} + i\epsilon} \right\}. \tag{A10}$$

The fact that $\omega_{\vec{k}}^{+} = \omega_{-\vec{k}}^{+}$ has been used. From (3.24) it follows that $\Delta_F(k, \zeta, \overline{\zeta}; \beta)$ is given by (3.30) as asserted.

Similar results obtain for any integer-spin field possessing at least one quantum number distinguishing the particle from its antiparticle. For fields possessing no internal symmetries (e.g., photon and neutral pion) the particle serves as its own antiparticle. Then $a(\vec{k}) = b(\vec{k})$, etc. and (A6)–(A7) reduce to $n_B(k,\beta) = \overline{n}_B(k,\beta)$. Furthermore, such particles may be created in arbitrary numbers so that the corresponding chemical potentials ζ and $\overline{\zeta}$ appearing in the distribution functions vanish identically.

A derivation along similar lines may be used to construct fermion two-point functions which include temperature and density effects. It should be obvious that fields of arbitrary spin may be treated in this way, in some instances trivially. For example, the propagator for a massive spin-one field which satisfies the equation of motion

$$(\partial_\nu \partial^\nu + m_1{}^2)\varphi^\mu(x) = 0\,,$$

$$\partial^\mu \varphi_\mu(x) = 0$$

has the momentum representation

$$D_{\mu\nu}(k) = -\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Delta(k)\,.$$

The introduction of thermodynamic boundary conditions for $D_{\mu\nu}(k)$ results from their introduction via $\Delta(k)$; specifically $\Delta(k) \to \Delta_F(k, \zeta, \overline{\zeta}, \beta)$ as given by (3.30).

APPENDIX B

The equation determining an N-point function characterizing a many-body system in curved space-time may be derived by the method of functional derivatives. As an example the construction of the equation for the fermion two-point function $G(x, x')$ will be outlined below. Wave-function and charge-renormalization coefficients will be ignored since their presence in no way alters the general approach.

Assume that $\psi(x)$ and $\varphi(x)$ satisfy (2.21)–(2.22), respectively, with $Z_1 = Z_2 = Z_3 = 1$, and define $G(x, x')$ and $D(x, x')$ according to (2.28)–(2.30). Next introduce the following external fields with appropriate statistics:

$\eta(x), \overline{\eta}(x)$, fermions

$A(x)$, bosons

from which the Lagrangian $\mathcal{L}_{\text{ext}}$ is constructed:

$$\mathcal{L}_{\text{ext}}(x) = \overline{\eta}(x)\psi(x) + \overline{\psi}(x)\eta(x) + A(x)\varphi(x)\,. \tag{B1}$$

In the Heisenberg representation, $\psi(x)$ satisfies

$$(-g)^{1/2}\{i\gamma^\mu(x)[\partial_\mu - \Gamma_\mu(x)] - m + \delta m - ig\gamma_5(x)\varphi(x)\}\,\psi(x) = \eta(x)\,, \tag{B2}$$

which couples $\psi(x)$ and $\varphi(x)$ to $\eta(x)$. In general, $\psi(x)$ and $\varphi(x)$ will depend on $\eta(x)$ and $A(x)$. One therefore transforms to a new representation defined by the requirement that in it the $\psi'(x)$ and $\varphi'(x)$ be independent of external fields. The transformation is accomplished by the operator

$$S(u_2, u_1) = T \exp\left[-i \int_{\Sigma(u_1)}^{\Sigma(u_2)} d^4x\, \mathcal{L}'_{\text{ext}}(x)\right], \tag{B3}$$

where u_1 and u_2 are real parameters measured along a timelike world line, and $\Sigma(u_1)$ and $\Sigma(u_2)$ are spacelike hypersurfaces. The time-ordering operator is defined in Sec. II. The parameters u_1 and u_2 are to be chosen such that

$$\psi'(x) = S(u_2, -\infty)\psi(x)S(u_2, -\infty)^{-1}\,, \tag{B4}$$

$$\varphi'(x) = S(u_2, -\infty)\varphi(x)S(u_2, -\infty)^{-1}\,, \tag{B5}$$

where $x^0 = x^0(u_2)$. Setting $S(\infty) = S(\infty, -\infty)$ it then follows that

$$\frac{\delta S(\infty)}{\delta\overline{\eta}(x)} = -iS(\infty, x^0)\psi'(x)S(x^0, -\infty) = -iS(\infty)\psi(x)\,. \tag{B6}$$

In a similar fashion it may be shown that

$$\frac{\delta^2 S(\infty)}{\delta A(x)\delta\overline{\eta}(x)} = -S(\infty)\varphi(x)\psi(x)\,. \tag{B7}$$

Consequently (B2) may be written in the form

$$(-g)^{1/2}\left\{i\gamma^\mu(x)[\partial_\mu - \Gamma_\mu(x)] - m + \delta m + g\gamma_5(x)\frac{\delta}{\delta A(x)}\right\}\frac{\delta S(\infty)}{\delta\overline{\eta}(x)} = -iS(\infty)\eta(x)\,. \tag{B8}$$

The two-point function may now be written in terms of functional derivatives:

$$G(x, x') = -i\langle T\psi(x)\overline{\psi}(x)\rangle_\beta = -i\frac{\langle T\psi'(x)\overline{\psi}'(x')S(\infty)\rangle_\beta}{\langle TS(\infty)\rangle_\beta} = \frac{i}{S_0}\frac{\delta^2 S_0}{\delta\eta(x')\delta\overline{\eta}(x)}\bigg|_{\eta=\overline{\eta}=0}, \tag{B9}$$

where the expectation value is defined according to (2.28)–(2.30) and $S_0 \equiv \langle TS(\infty)\rangle$. It should be noted that the many-body nature of the system enters the derivation at this point. Operating on (B8) with $iS_0^{-1}\delta/\delta\overline{\eta}(x')$ and then setting $\eta(x)=\overline{\eta}(x)=0$ yields

$$(-g)^{1/2}\left\{i\gamma^\mu(x)[\partial_\mu - \Gamma_\mu(x)] - m + \delta m + g\gamma_5(x)\frac{\delta}{\delta A(x)}\right\}G(x, x') = \delta^4(x, x'). \tag{B10}$$

It is next possible to show that the last term on the left-hand side of (B10) is proportional to the three-point function, which may be written as

$$\langle T\psi(x)\overline{\psi}(y)\varphi(z)\rangle_\beta = \int d^4\xi\int d^4\eta\int d^4\zeta\, G(x, \xi)[-g(\xi)]^{1/2}\Gamma_5(\xi\eta\,|\,\xi)[-g(\eta)]^{1/2}G(y, \eta)[-g(\zeta)]^{1/2}D(z, \zeta). \tag{B11}$$

Equation (B11) serves as a definition of the vertex function $\Gamma_5(\xi, \eta\,|\,\zeta)$. It is then convenient to define the self-energy operator by

$$\Sigma_c(x, x') = ig^2\gamma^5(x)\int d^4z\int d^4y\, G(x, y)[-g(y)]^{1/2}\Gamma_5(x'y\,|\,z)D(x, z)[-g(z)]^{1/2}. \tag{B12}$$

In terms of the self-energy operator $\Sigma_c(x, z)$, (B10) takes the form

$$(-g)^{1/2}\{i\gamma^\mu(x)[\partial_\mu - \Gamma_\mu] - m + \delta m\}G(x, x') = \delta^4(x - x') - (-g)^{1/2}\int d^4z\,\Sigma_c(x, z)G(z, x')[-g(z)]^{1/2}. \tag{B13}$$

A similar approach, starting with (2.22), leads to the equation of motion for the boson two-point function $D(x, x')$ and the polarization operator given in Sec. IV.

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[1]B. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (Univ. of Chicago Press, Chicago, 1965); I. Robinson, A. Schild, and E. L. Schucking, *Quasi-Stellar Sources and Gravitational Collapse* (Univ. of Chicago Press, Chicago, 1965); J. A. Wheeler, *Geometrodynamics* (Academic, New York, 1962); J. P. Cox and R. T. Giuli, *Principles of Stellar Structure* (Gordon and Breach, New York, 1968), Vol. 2; H.-Y. Chiu, *Stellar Physics*, (Ginn Blaisdell, Waltham, Mass., 1968), Vol. 1; K. S. Thorne, in *Proceedings of the International School of Physics "Enrico Fermi,"* Course 35, High-Energy Astrophysics, edited by L. Gratton (Academic, New York, 1966); Ya. B. Zel'dovich and I. D. Novikov, *Relativistic Astrophysics*, edited by K. S. Thorne and W. D. Arnet (Univ. of Chicago Press, Chicago, 1971), Vol. 1.

[2]V. A. Ambartsumyan, in *The Structure and Evolution of Galaxies*, Proceedings of the Thirteenth Conference on Physics at University of Brussels, 1964 (Wiley-Interscience, London, 1965); Dokl. Akad. Nauk. Arm. SSR 16, 73 (1953).

[3]V. A. Ambartsumyan and G. S. Saakyan, Sov. Astron. AJ 4, 187 (1960); 5, 601 (1962).

[4]C. W. Misner, Astrophys. J. 151, 431 (1968); R. V. Wagoner, W. A. Fowler, and F. Hoyle, Astrophys. J. 148, 3 (1967); Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. 41, 1609 (1961) [Sov. Phys. JETP 14, 1143 (1962).

[5]I. Robinson *et al.*, Ref. 1.

[6]L. D. Landau, Collected Papers of L. D. Landau, edited by D. Ter Haar (Pergamon, New York, 1965).

[7]W. Baade and F. Zwicky, Proc. Natl. Acad. Sci. U.S. 20, 2599 (1934).

[8]J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. 55, 374 (1939).

[9]A. G. W. Cameron, Astrophys. J. 130, 884 (1969).

[10]A. Hewish, S. J. Bell, J. D. H. Pilkington, P. F. Scott, and R. A. Collins, Nature 217, 709 (1968).

[11]F. Hoyle and J. V. Narlikar, Proc. Roy. Soc. (London) A278, 465 (1964).

[12]Ya. B. Zel'dovich, Sov. Phys. JETP, 16, 1102 (1963); Ya. B. Zel'dovich and I. D. Novikov, Sov. Phys. Usp. 7, 763 (1965); S. A. Bludman and M. A. Ruderman, Phys. Rev. 170, 1176 (1968); W. Israel, *ibid.* 153, 1388 (1967).

[13]L. Gratton and G. Szamosi, Nuovo Cimento 33, 1056 (1964); G. Marx and J. Nemeth, Acta Phys. Hung. 18, 77 (1964); G. S. Sahakian and Yu. L. Vartanian, Nuovo Cimento 30, 82 (1963); E. E. Salpeter, Ann. Phys. (N.Y.) 11, 393 (1960); H. Bethe, G. Borner, and S. Katsukiko, Astron. and Astrophys. 7, 279 (1970); C. G. Wang, W. K. Rose, and S. L. Schlenker, Astron. J. 160, 117 (1970); V. A. Ambartsumyan and G. S. Saakyan, Sov. Astron. AJ 5, 601 (1962); B. M. Barker, M. S. Bhatia, and G. Szamosi, Nuovo Cimento 44, 109 (1966); J. Nemeth and D. W. L. Sprung, Phys. Rev. 176, 1496 (1963); G. Szamosi, in *Proceedings of the International School of Physics "Enrico Fermi,"* Course 35, edited by L. Gratton (Academic, New York, 1966).

[14]K. S. Thorne, Ref. 1.

[15]J. A. Wheeler, Ref. 1.

[16]N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959); J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[17]A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics* (Prentice Hall, Englewood Cliffs, N. J., 1963).

[18]E. S. Fradkin, in *Quantum Field Theory and Hydrodynamics*, edited by D. V. Strobel'tsyn (Consultant Bureau, New York, 1967).

[19]W. D. Brown, R. D. Puff, and L. Wilets, Phys. Rev. D 2, 331 (1970).

[20]R. Ruffini and S. Bonazzola, Phys. Rev. 187, 1767 (1969); F. Capra, Acta Phys. Austriaca 26, 327 (1967).

[21]G. Szamosi, Ref. 13.

[22]B. S. DeWitt, in *Relativity Groups and Topology*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1964); R. Utiyama, Phys. Rev. 125, 1727 (1962).

[23]E. T. Jaynes, in *Brandeis Summer Institute in Theoretical Physics*, edited by K. W. Ford (Benjamin, New York, 1962), Vol. 3.

[24]The exact thermodynamic nature of such concepts as temperature, etc. within the curved-space-time formalism needs further investigation; we shall continue to use them with this understanding.

[25]R. Balescu, Physica 40, 309 (1968).

[26]The constraints in this case are time-independent; consequently the integration in (2.30) is over a three-space orthogonal to the timelike axis of S.

[27]R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford Univ. Press, Oxford, England, 1934). This assumes that gravitation does not contribute significantly to interactions between particles.

[28]H. P. Robertson and T. W. Noonan, *Relativity and Cosmology* (Saunders, Philadelphia, Pa., 1968); J. L. Synge, *Relativity – The General Theory* (Wiley, New York, 1960).

[29]The indices in parenthesis are raised and lowered by $\eta^{(ab)} = \eta_{(ab)} = \mathrm{diag}\,(1, -1, -1, -1)$. The use of tetrads is discussed extensively by Synge, Ref. 28.

[30]L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon, New York, 1964).

[31]The notation of this section follows that of Bjorken and Drell, Ref. 16.

[32]In general the ground state will contain bosons as well as fermions. It is possible to consider the bosons as composite particles corresponding to bound states of fermions. In this case there exist correlations between free and bound fermions (see Ref. 21) as well as between free fermions. For simplicity such effects will be ignored. It is to be emphasized nonetheless that they could be included within the formalism of this section.

[33]Equations (3.13)–(3.14) are not manifestly Lorentz-invariant, since they involve three components of p^μ and $\bar{p}^\mu$. The decomposition of excitations into "particle" and "hole" is invariant nevertheless. The possibility of differing densities of noninteracting particles and antiparticles breaks the manifest invariance with respect to positive- and negative-energy states (3.12). Furthermore the boundary conditions refer to an observer at rest with respect to the center of mass of the system.

[34]J. L. Anderson, in *Gravitation and Relativity*, edited by H.-Y. Chiu and W. F. Hoffman (Benjamin, New York, 1964).

[35]No attempt is made at manifest Lorentz invariance (see Ref. 30), the comments of which also apply to (3.30) and (3.26)–(3.27).

[36]N. N. Bogoliubov and D. V. Shirkov, Ref. 16.

[37]L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (Benjamin, New York, 1962); P. C. Martin and J. Schwinger, Phys. Rev. 115, 1343 (1959).

[38]This limit is closely related to general-relativistic kinetic theory which ignores gravitational contributions to particle interactions.

[39]R. Utiyama, Ref. 22.

[40]F. Capra, Acta Phys. Austriaca 26, 327 (1967).

[41]M. Ruderman, N.Y.U. Technical Report No. 6169, 1969 (unpublished).

[42]D. H. Brownell and J. Callaway, Nuovo Cimento 60B, 169 (1969).