Equivalence Theorem on Bogoliubov-Parasiuk-Hepp-Zimmermann-Renormalized Lagrangian Field Theories

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The equivalence theorem for Bogoliubov-Parasiuk-Hepp-Zimmermann-renormalized Lagrangian field theories is proved by means of Feynman graphs. The transformation of Green's functions under a transformation of the Lagrangian vertex is also derived.

I. INTRODUCTION

For a period of over three decades, the equivalence theorem on Lagrangian field theories has been discussed,[1] proved,[2-4] and widely employed.[5] In the modern version, it deals with the equality of the scattering matrices constructed from two Lagrangian densities, $\mathcal{L}_0$ and $\mathcal{L}'$, which are related by

$$\mathcal{L}'(\phi')=\mathcal{L}_0(\phi), \tag{1.1}$$

where the field $\phi'(x)$ is obtained from $\phi(x)$ by a local transformation

$$\phi'=f(\phi). \tag{1.2}$$

Treating the functions $f(\phi)$ of local operators $\phi(x)$ also as local operators, Kamefuchi *et al.*[2] showed that the canonical commutation relations,

$$\delta(x_0)\left[\frac{\delta\mathcal{L}}{\delta\dot{\phi}}(x),\ \phi(0)\right]=i\delta^4(x), \tag{1.3}$$

imply that ϕ' and $\delta\mathcal{L}/\delta\dot{\phi}'$ also satisfy the same canonical commutation relations, so that the Hamiltonian densities are equal:

$$\mathcal{H}\left(\phi,\ \frac{\delta\mathcal{L}}{\delta\dot{\phi}}\right)=\mathcal{H}'\left(\phi',\ \frac{\delta\mathcal{L}'}{\delta\dot{\phi}'}\right). \tag{1.4}$$

From this follows the equivalence theorem. On the other hand, Salam *et al.*[3] proved the theorem by employing the Feynman-path-integral representation of vacuum expectation values of time-ordered products. Both proofs are only formal in nature and not rigorous. The first proof suffers from the fact that products of distributions at equal space-time points do not exist in general, while for the second proof Feynman path integrals may diverge for quantized field theories.

In this communication, we will present a perturbational but rigorous proof of the equivalence theorem, based on Feynman-graph expansion of vacuum expectation values of time-ordered products, together with the Bogoliubov-Parasiuk-Hepp-Zimmermann[6,7] (BPHZ) renormalization. To describe the theorem exactly, it is inadequate to interpret the terms in a Lagrangian density as Zimmermann's normal products.[7] Rather, as in our earlier papers,[8,9] the Lagrangian density is treated as a *vertex*, which is a generalization of Zimmermann's normal products. Vertices have been defined and discussed at length for scalar fields in Ref. 8 (to which the reader will be referred time after time), and for Dirac fields in Ref. 9.

In Sec. II, the equivalence theorem will be stated and shown to follow from Lemma 1, which is a statement about how the Green's functions transform under (1.1). Lemma 1 will be proved in Sec. III and generalized in Sec. IV. The paper ends with a short discussion (Sec. V) on the power of the BPHZ renormalization in the proof of Lemma 1.

II. EQUIVALENCE THEOREM ON BPHZ-RENORMALIZED THEORIES

In the language of vertices (described in Ref. 8), the analog of a transformation of a classical function takes on a slightly more complicated form, because the information contained in the excess-subtraction functions of vertices must be transmitted. Let $w=[(f_1,\ldots,f_N),\alpha]$ and $v=[(g_1,\ldots,g_M),\beta]$ be two simple vertices and let j be an integer between 1 and N (the order of w), inclusive. Then we use the notation $[(f_1,\ldots,f_{j-1},v,f_{j+1},\ldots,f_N),\alpha]$ for the simple vertex $[h^{(j)},\gamma^{(j)}]$ which is defined in Ref. 8.[10] We will call this vertex *the replacement of the* jth *field in the simple vertex* w *by the simple vertex* v. Generalizing this to more than one replacement, we define

$$[(f_1,\ldots,f_{i-1},u,f_{i+1},\ldots,f_{j-1},v,f_{j+1},\ldots,f_N),\alpha]$$

to be the replacement of the ith field in $[(f_1,\ldots,f_{j-1},v,f_{j+1},\ldots,f_N),\alpha]$ by u. Further replacement can be defined inductively (proceeding from right to left). By performing formal sums, this concept is easily generalized to the case where the v's are *vertices*.

To express explicitly the fact that a vertex W in general involves the basic fields $\phi^1, \ldots, \phi^M$ and their derivatives, we use the notation $W(\phi)$. For any set of vertices $V^1, \ldots, V^M$, we denote by $W(V)$ the vertex obtained from $W(\phi)$ by performing the replacements $\phi^i \to V^i$, $\partial_\mu\phi^i \to \partial_\mu V^i$, etc. This is the analog of a transformation of a classical function.

Now we enunciate as follows.

Lemma 1. For any set of nonderivative vertices $V^1, \ldots, V^M$, a Lagrangian vertex $\mathcal{L}_0(\phi)$ determines another Lagrangian vertex

$$\mathcal{L}'(\phi) \equiv \mathcal{L}_0(\phi + V). \tag{2.1}$$

Then up to any fixed orders in V and $\mathcal{L}_0^{\text{int}}(\phi)$ [the interaction part of $\mathcal{L}_0(\phi)$], the BPHZ-renormalized vacuum expectation values of time-ordered products[11] $\langle 0|\, T[\cdots]\,|0\rangle^{\mathcal{L}'}$ constructed from $\mathcal{L}'$ are related to those constructed from $\mathcal{L}_0$ by

$$\langle 0|T[\{\phi^i + V^i\}_{\text{op}}(x)\{\phi^j + V^j\}_{\text{op}}(y)\cdots]|0\rangle^{\mathcal{L}'} = \langle 0|T[\phi^i_{\text{op}}(x)\phi^j_{\text{op}}(y)\cdots]|0\rangle^{\mathcal{L}_0}. \tag{2.2}$$

If the operators V^i_{op}, as defined by $\mathcal{L}'$, do not have single-particle states, that is,

$$\langle 0|V^i_{\text{op}}(x)|\Psi\rangle^{\mathcal{L}'} = 0 \tag{2.3}$$

for all single-particle states $|\Psi\rangle$, then, when we apply the Lehmann-Symanzik-Zimmermann[12] (LSZ) reduction formula to both sides of Eq. (2.2), the operators V^i_{op} on the left-hand side do not survive. Hence we obtain the *equivalence theorem* on BPHZ-renormalized theories: The scattering matrix constructed from $\mathcal{L}'$ is identical to that constructed from $\mathcal{L}_0$, if $\mathcal{L}'$ and $\mathcal{L}_0$ are related by Eq. (2.1).

In general Eq. (2.3) may not be valid; however, we can invoke a powerful property of Haag-Ruelle theory[13] to obtain the equivalence between $\mathcal{L}_0$ and $\mathcal{L}'$. Since the two theories defined by $\mathcal{L}_0$ and $\mathcal{L}'$ have identical asymptotic Hilbert spaces, we can identify the operators of these theories by their matrix elements; that is, two operators are equal if their matrix elements are equal. Then Eq. (2.2) and the LSZ reduction formula show that the operator $\{\phi^i\}^{\mathcal{L}_0}_{\text{op}}$ as defined by $\mathcal{L}_0$ is equal to the operator $\{\phi^i + V^i\}^{\mathcal{L}'}_{\text{op}}$ as defined by $\mathcal{L}'$:

$$\{\phi^i\}^{\mathcal{L}_0}_{\text{op}}(x) = \{\phi^i + V^i\}^{\mathcal{L}'}_{\text{op}}(x). \tag{2.4}$$

By definition, $\{\phi^i + V^i\}^{\mathcal{L}'}_{\text{op}}$ is a normal product of the operators $\{\phi^j\}^{\mathcal{L}'}_{\text{op}}$ and is therefore relatively local to $\{\phi^j\}^{\mathcal{L}'}_{\text{op}}$. Hence $\{\phi^i\}^{\mathcal{L}_0}_{\text{op}}$ is relatively local to $\{\phi^j\}^{\mathcal{L}'}_{\text{op}}$. Furthermore, they have the same single-particle spectrum. It then follows from the work[14] of Haag, Nishijima, and Zimmermann that $\{\phi^i\}^{\mathcal{L}'}_{\text{op}}$ and $\{\phi^i\}^{\mathcal{L}_0}_{\text{op}}$ describe two theories with identical scattering matrices.

III. PROOF OF LEMMA 1

We shall restrict ourselves to scalar fields, as the reader can easily extend the proof to include spin-$\frac{1}{2}$ Dirac fields described in Ref. 9.

Let the Lagrangian vertex $\mathcal{L}_0$ be of the form

$$\mathcal{L}_0(\phi) = \tfrac{1}{2}\sum_i [(\partial_\mu\phi^i, \partial^\mu\phi^i), 0] - \tfrac{1}{2}\sum_i m_i^2[(\phi^i, \phi^i), \alpha^{\text{mass}}] + \mathcal{L}_0^{\text{int}}(\phi). \tag{3.1}$$

(The reader is referred to Ref. 8 for the definition of α^{mass}.) $\mathcal{L}'$, as defined by (2.1), can be separated into four parts:

$$\mathcal{L}' = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3, \tag{3.2}$$

where

$$\mathcal{L}_1 \equiv \sum_i [(\partial_\mu\phi^i, \partial^\mu V^i), 0] - \sum_i m_i^2[(\phi^i, V^i), \alpha^{\text{mass}}], \tag{3.3}$$

$$\mathcal{L}_2 \equiv \tfrac{1}{2}\sum_i [(\partial_\mu V^i, \partial^\mu V^i), 0] - \tfrac{1}{2}\sum_i m_i^2[(V^i, V^i), \alpha^{\text{mass}}], \tag{3.4}$$

and

$$\mathcal{L}_3 \equiv \mathcal{L}_0^{\text{int}}(\phi + V) - \mathcal{L}_0^{\text{int}}(\phi). \tag{3.5}$$

The V's appearing in parts of vertices in $\mathcal{L}'$ will be called *brooms*.

The vertices in $\mathcal{L}_1$ are of two types, shown respectively in Figs. 1(a) and 1(b). In Fig. 1(a), we have already used Lemma 1 of Ref. 8 to transform the derivative on V^i to that on ϕ^i (analogously to

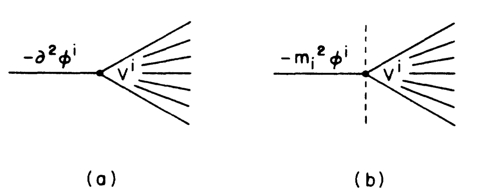


FIG. 1. (a) a broom with a momentum stick; (b) a broom with a mass stick.

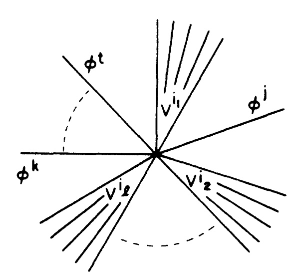


FIG. 2. An l-broom derivative of $\mathcal{L}_0^{\text{int}}(\phi)$.

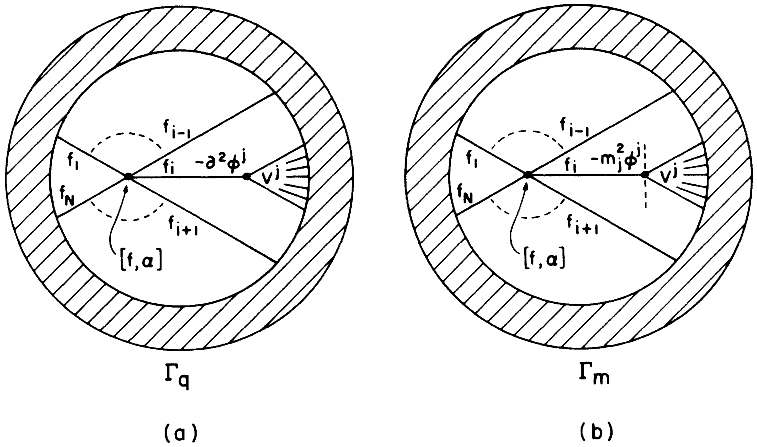


FIG. 3. Broomsticks in a graph.

partial integration). In Fig. 1(b), the dashed line, separating the vertex into two parts, indicates explicitly that the effect of α^{mass} is to increase the degree function[15] $\delta(\gamma)$ by 2 for any graph γ containing both the left and the right parts of the vertex. These vertices look like *brooms* with sticks, and so we will call the former a broom with a momentum stick, and the latter a broom with a mass stick. All the vertices in $\mathfrak{L}_3$ have brooms. Thy can be obtained by replacing and number l of basic fields (or their derivatives) of a simple vertex in $\mathfrak{L}_0^{\text{int}}(\phi)$ by l brooms without sticks (or their derivatives). We will call such a derived vertex an *l-broom derivative of* $\mathfrak{L}_0^{\text{int}}$. A typical one is shown in Fig. 2.

Replacing the momentum stick of a broom in a graph Γ_q [Fig. 3(a)] by the corresponding mass stick, we obtain the graph Γ_m of Fig. 3(b). Because of the presence of a dashed line, Γ_m has the same subtraction scheme as Γ_q. By this we mean that the degree function of any subgroup of Fig. 3(a) remains unchanged upon replacing the momentum stick by the mass stick. The BPHZ renormalization of the graph Γ_q is an operation $\mathcal{R}$ on the unrenormalized integrand I_{Γ_q} and yields the renormalized integrand $\mathcal{R}I_{\Gamma_q}$. Therefore this operation on I_{Γ_q} is identical to that on I_{Γ_m}; hence in the sum $\mathcal{R}I_{\Gamma_q}+\mathcal{R}I_{\Gamma_m}$ it is meaningful to factor out the operation $\mathcal{R}$,

$$\mathcal{R}I_{\Gamma_q}+\mathcal{R}I_{\Gamma_m}=\mathcal{R}(I_{\Gamma_q}+I_{\Gamma_m})\,, \qquad (3.6)$$

and denote the sum $I_{\Gamma_q}+I_{\Gamma_m}$ by I_{Γ_p}, where the graph Γ_p is shown in Fig. 4. The pair of short lines across the stick of the broom in Γ_p means $(-\partial^2-m_j^{\,2})$. Such a combination is called a *broom pair*. Therefore,

$$\mathcal{R}I_{\Gamma_q}+\mathcal{R}I_{\Gamma_m}=\mathcal{R}I_{\Gamma_p}\,. \qquad (3.7)$$

Abstracting away the common parts of these graphs, the definition of a broom pair takes on the symbolic form of Fig. 5. In a manner similar to the proof of Ward-Takahashi identities in Ref. 8, we find[16] that the factor $(-\partial^2-m_j^{\,2})$ on a broom pair cancels a similar factor in the propagator and

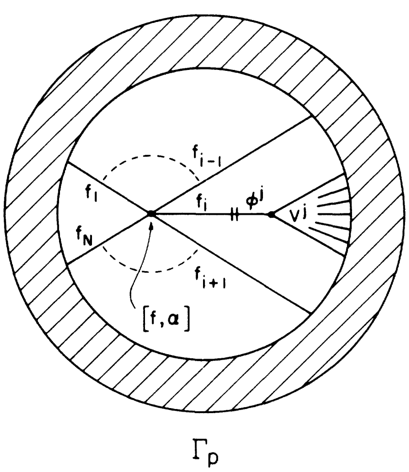


FIG. 4. A broom-pair vertex in a graph.

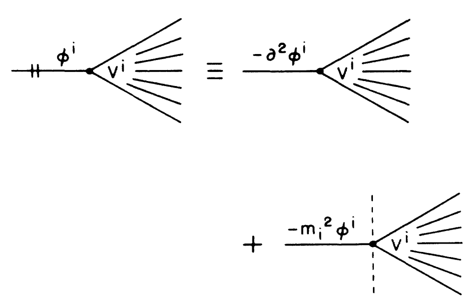


FIG. 5. Definition of a broom pair.

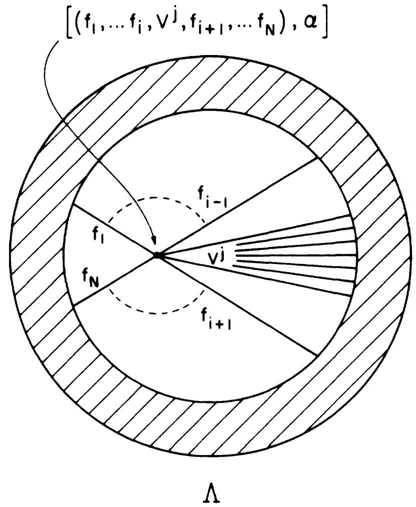


FIG. 6. Creation of the vertex $[(f_1, \ldots, f_{i-1}, V^j, f_{i+1}, \ldots, f_N), \alpha]$.

generates an over-all minus sign. Thus

$$\mathfrak{R} I_{\Gamma_p} = -\mathfrak{R} I_{\Lambda}, \tag{3.8}$$

where Λ (Fig. 6) is obtained from Γ_p by shrinking the broomstick of the broom pair, and generating the vertex $[(f_1, \ldots, f_{i-1}, V^j, f_{i+1}, \ldots, f_N), \alpha]$, which is the replacement of the ith field in $[f, \alpha]$ by V^j.

It may happen that, in Γ_q, Γ_m, and Γ_p, a field of $[f, \alpha]$ other than f_i is also joined to the broom. In this case, shrinking the stick will produce a graph with a "bubble" (Fig. 7). However, Ref. 8 shows that the BPHZ renormalization annihilates any graph with bubbles. Hence we can drop all such graphs from our considerations.

Abstracting away the common parts in Γ_p and Λ, we obtain the *broomstick identity* of Fig. 8, which must be understood to be valid only when "immersed" in a larger graph. This identity will be

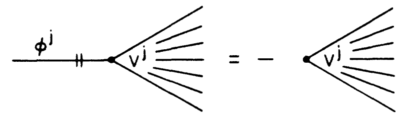


FIG. 8. The broomstick identity.

useful in proving Lemma 1.

For simplicity and essentially without loss of generality, we will assume that there is only one basic field ϕ in the theory and that $\mathfrak{L}_0^{\text{int}}(\phi)$ has the simple form

$$\mathfrak{L}_0^{\text{int}}(\phi) = g\phi^N, \tag{3.9}$$

where N is an integer, and g is a coupling constant.

The left-hand side of Eq. (2.2) is constructed from the interaction part of $\mathfrak{L}'$. Up to any order in g and V, we divide the contributing graphs into five mutually exclusive classes, as follows:

Class 0. A graph containing no brooms is a class-0 graph.

Class 1. In a class-1 graph, at least one external vertex[17] is either the head of a broom or a broomstick.

The rest of the graphs belonging to the left-hand side of Eq. (2.2) do not have brooms at external vertices, but each graph has at least one broom. We further divide them into three classes.

Class 2. In a class-2 graph, any broom has a stick, and any stick is attached to the tail of another broom, but these sticks form closed loops.

Class 3. A class-3 graph contains at least one broom which either is part of a simple vertex of $\mathfrak{L}_3$ or has a stick joined to $\mathfrak{L}_0^{\text{int}}(\phi + V)$.

Class 4. A class-4 graph does not contain any simple vertices of $\mathfrak{L}_3$ nor a broomstick joined to

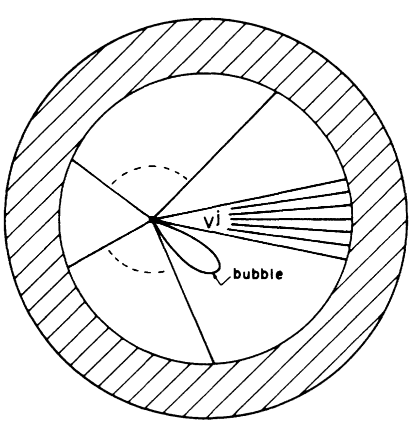


FIG. 7. A bubble graph.

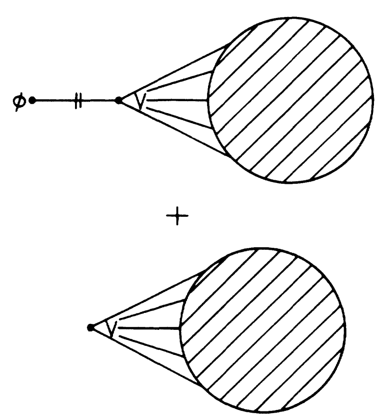


FIG. 9. A partial sum over class-1 graphs.

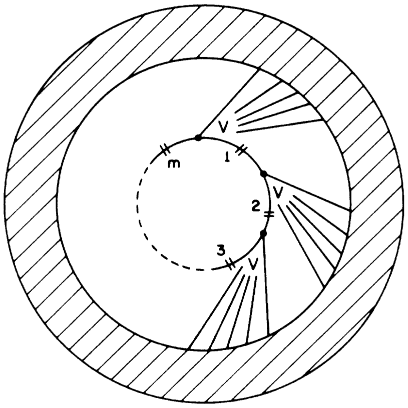


FIG. 10. A class-2 graph with a loop of m broom pairs.

$\mathcal{L}_0^{\text{int}}(\phi+V)$, but contains at least either a simple vertex of $\mathcal{L}_2$ or a stick with two brooms, one at each end.

We will now consider the contribution to the left-hand side of Eq. (2.2) from each of these classes.

Class 0. These graphs are exactly those contributing to the right-hand side of Eq. (2.2). Thus we prove (2.2) by showing that the other four classes of graphs do not contribute.

Class 1. Their contribution can be considered as partial sums, each as shown in Fig. 9. The broomstick identity then implies that each partial sum is zero.

Class 2. These diagrams can be partitioned into groups like the one shown collectively in Fig. 10, where a closed loop is formed from m sticks (of broom pairs). Upon applying the broomstick identity $(m-1)$ times onto the loop, we produce a graph containing a big broom with a loop replacing the stick (Fig. 11), whose integrand is annihilated by the BPHZ subtraction scheme. Hence the contribution from class-2 graphs is zero.

Class 3. Let l be an integer between 1 and N, inclusive, and consider a subset $\mathcal{C}_l^A$ of class-3 graphs, each with a common part A and a variable

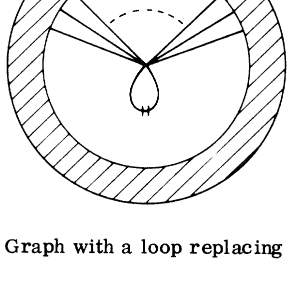
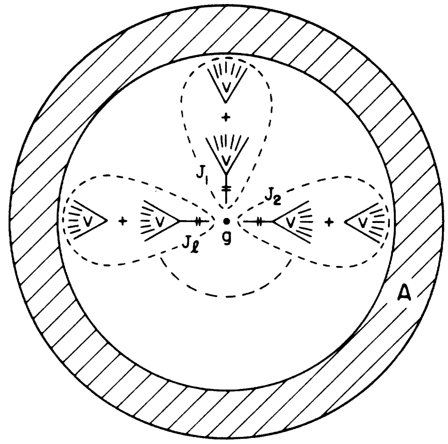


FIG. 11. Graph with a loop replacing the stick.



$\Pi(J_1 \ldots J_\ell)$

FIG. 12. Binomial representation of a partial sum over class-3 graphs. The $(N-l)$ ϕ propagators have been omitted for clarity.

internal part satisfying the following conditions: (i) It is of order g, and (ii) it has l brooms and $(N-l)$ other ϕ propagators. Such an internal part then contains an m-broom derivative of $\mathcal{L}_0^{\text{int}}(\phi)$ with $0 \leq m \leq l$, and the sum of the BPHZ-renormalized integrand over graphs of $\mathcal{C}_l^A$ can be expressed analogously as a sum of graphs $\Pi(J_1 \cdots J_l)$:

$$\sum_{\Gamma \in \mathcal{C}_l^A} \mathcal{R}I_\Gamma = \sum_{\{J_1, \ldots, J_l\} \subset \{1, \ldots, N\}} N! \mathcal{R}I_{\Pi(J_1 \cdots J_l)}, \tag{3.10}$$

where the graphs $\Pi(J_1 \cdots J_l)$ are expressed in Fig. 12 analogously to a binomial form. In this diagram, the $(N-l)$ ϕ propagators have been suppressed for the sake of clarity. In Eq. (3.10), $N!$

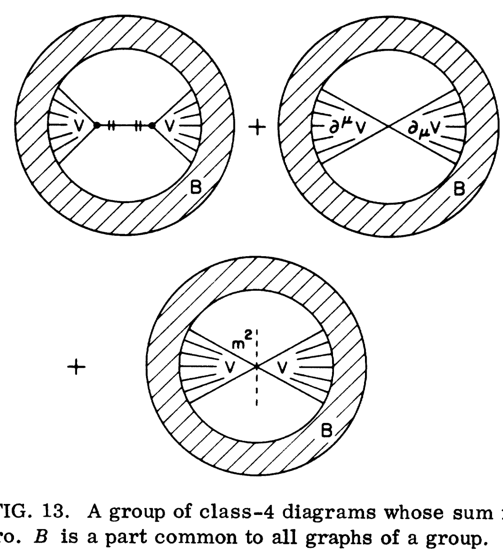


FIG. 13. A group of class-4 diagrams whose sum is zero. B is a part common to all graphs of a group.

is the multiplicity of each graph, and the summation is over all the different positions in which the "factors" (a "factor" is enclosed by a broken line in the diagram) can be attached to the central vertex labeled by g. [The ith combination replaces the J_ith field of $\mathcal{L}_0^{\rm int}(\phi)$.] Now the broomstick identity says that each factor in Fig. 12 is zero; hence

$$\sum_{\Gamma\in \mathcal{C}_l^A} \mathcal{R} I_\Gamma = 0 . \tag{3.11}$$

Now the class-3 diagrams can be partitioned into subsets $\mathcal{C}_l^A$, each characterized by a fixed part A and an integer l. Therefore it follows that the class-3 graphs also do not contribute.

Class 4. These graphs can be partitioned into groups like that of Fig. 13. The sum of integrands over a group is easily seen to be zero by application of the broomstick identity.

We have now proved Lemma 1 for the simple case of Eq. (3.9). Generalization to more than one basic field only involves unessential complication in bookkeeping. If $\mathcal{L}_0^{\rm int}(\phi)$ has more than one simple vertex, we prove Lemma 1 merely by repeating the above procedure once for each simple vertex of $\mathcal{L}_0^{\rm int}(\phi)$. Derivative coupling can be similarly treated. [When Dirac fields are included, we need not consider, in the definition of a replacement, the fermion signature factor in Eq. (1.4) of Ref. 9. Since $\mathcal{L}'$ and $\mathcal{L}_0$ are Lorentz scalars, those V's corresponding to Dirac fields must contain an odd number of Dirac fields, so that the fermion signature factor is +1.]

IV. GENERALIZATION OF LEMMA 1

Instead of ϕ, we may consider vertices in general in the right-hand side of Eq. (2.2). The graphs contributing to

$$\langle 0|T[A(\phi+V)_{\rm op}(x)B(\phi+V)_{\rm op}(y)\cdots]|0\rangle^{\mathcal{L}'},$$

where $A(\phi), B(\phi), \ldots$ are vertices, may be divided similarly to those in Sec. III into five mutually exclusive classes. The argument in Sec. III again applies here and shows that the sum of amplitudes over class-0 graphs is

$$\langle 0|T[A(\phi)_{\rm op}(x)B(\phi)_{\rm op}(y)\cdots]|0\rangle^{\mathcal{L}_0},$$

while graphs of classes 1–4 do not contribute. Hence a generalization of Lemma 1 is the following.

Lemma 2.

$$\langle 0|T[A(\phi+V)_{\rm op}(x)B(\phi+V)_{\rm op}(y)\cdots]|0\rangle^{\mathcal{L}'} = \langle 0|T[A(\phi)_{\rm op}(x)B(\phi)_{\rm op}(y)\cdots]|0\rangle^{\mathcal{L}_0}, \tag{4.1}$$

up to any fixed orders in $\mathcal{L}_0^{\rm int}(\phi)$ and V.

Both Lemmas 1 and 2 are statements about the transformation of Green's functions under the induced transformation (2.1) on the Lagrangian vertex.

V. DISCUSSION

We have rigorously proved Lemmas 1 and 2, and the equivalence theorem for BPHZ-renormalized theories. Besides yielding a finite value for each graph, the BPHZ renormalization scheme nicely puts away all bubble graphs, which are produced on applying the broomstick identity (Fig. 8) in the proof of Lemma 1. [There are no bubble graphs in the right-hand side of Eq. (2.2).] Since the essence of the proof is the property of graphs, we conjecture that if bubble graphs are allowed as formal expressions for integrals (even though they do not exist), and if not $i\mathcal{L}^{\rm int}$ but $i\mathcal{L}_{\rm eff}^{\rm int}$ (see footnote 18) is used to construct Feynman graphs, then Lemma 1 again holds, but only formally. Another impetus for this conjecture is that the Feynman-path-integral method formally proves Lemma 2 (and therefore also Lemma 1), and that the perturbational expansion for a Feynman path integral allows bubble graphs. To show that Lemma 2 follows formally from the Feynman-path-integral method, perform the transformation

$$\phi^t \to \phi^t + V^t(\phi) \tag{5.1}$$

on the integration variables in the Feynman path integral,[19]

$$\int\left(\prod_{t=1}^{M} d\phi^t\right) A(\phi(x))\cdots B(\phi(z)) \exp\left[i\int d^4u\, \mathcal{L}_{0\,\rm eff}(\phi(u))\right],$$

and formally obtain the relation

$$\int\left(\prod_{t=1}^{M} d\phi^t\right) A(\phi(x))\cdots B(\phi(z)) \exp\left[i\int d^4u\, \mathcal{L}_{0\,\rm eff}(\phi(u))\right] = \int\left(\prod_{t=1}^{M} d\phi^t\right) A(\phi(x)+V(\phi(x)))\cdots B(\phi(z)+V(\phi(z))) \times \exp\left[i\int d^4u\, \mathcal{L}'_{\rm eff}(\phi(u))\right]. \tag{5.2}$$

Upon treating these Feynman integrals as vacuum expectation values of time-ordered products, this relation is identical to Eq. (4.1).

Earlier, in Refs. 8 and 9, we have demonstrated the feasibility of using $i\mathcal{L}^{\text{int}}$ and not $-i\mathcal{H}^{\text{int}}$ in constructing Feynman graphs. While the use of $-i\mathcal{H}^{\text{int}}$ has been shown[20] to be equivalent to the use of $i\mathcal{L}_{\text{eff}}^{\text{int}}$, the difference between $\mathcal{L}^{\text{int}}$ and $\mathcal{L}_{\text{eff}}^{\text{int}}$ is a singular term proportional to $\delta^4(0)$. Charap[21] and Honerkamp and Meetz[4] have demonstrated the effect of this term on soft-pion amplitudes. What we can learn from theii papers is that this term is in fact the first subtraction term of a BPHZ renormalization on these amplitudes. Hence this term is a primitive imitation of BPHZ renormalization, and need not be included. Therefore, we do not use $i\mathcal{L}_{\text{eff}}^{\text{int}}$ in a BPHZ-renormalized theory; in any case, its singular nature precludes its use in any rigorous theory.

For the purpose of satisfying the renormalization conditions, let the sum of the counterterms of $\mathcal{L}_0$ be of the form given by Zimmermann,[7] but in the context of vertices:

$$\mathcal{L}_0^{\text{counter}}(\phi) = a[(\partial^\mu\phi, \partial_\mu\phi), 0] + b[(\phi, \phi), 0] \;. \quad (5.3)$$

Then in order that $\mathcal{L}_0$ and $\mathcal{L}'$ be equivalent (same scattering matrices), the counterterms of $\mathcal{L}'$ must be given by the transformation (2.1):

$$\mathcal{L}'^{\text{counter}} = \mathcal{L}_0^{\text{counter}}(\phi + V) \;. \quad (5.4)$$

Thus, there is no standard form of counterterms, and we are willing to conjecture that perhaps the scattering matrix of a Lagrangian theory depends also on the form of its counterterms.

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[10]If the reader does not feel like referring to Ref. 8 at this time, he may neglect the excess-subtraction functions and the BPHZ renormalization operator $\mathfrak{R}$ in the subsequent part of the paper, and he may still appreciate the graphical nature of the proof in Sec. III.

[11]The notation $W_{\text{op}}(x)$ means the local field operator defined by the vertex W. See Ref. 8.

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[15]See Ref. 8 for definition.

[16]For this purpose, one only needs to recall from Ref. 8 that (a) any internal simple vertex carries a factor $(2\pi)^4 i$ and (b) the propagator is $i(2\pi)^{-4}(p^2 - m_i^{\,2})^{-1}$ for the field ϕ^i (p is the momentum carried in the propagator).

[17]An external vertex in a graph has external momentum flowing into it.

[18]According to Refs. 3, 4, and 20, given a classical Lagrangian density,

$$\mathcal{L} = \tfrac{1}{2}\,\partial^\mu\phi^i\;\partial_\mu\phi^j g^{ij}(\phi) + \mathcal{L}_1\,,$$

where g^{ij} and $\mathcal{L}_1$ do not depend on derivatives of ϕ, $\mathcal{L}_{\text{eff}}^{\text{int}}$ is the interaction part of

$$\mathcal{L}_{\text{eff}} = \mathcal{L} - \tfrac{1}{2} i\,\delta^4(0)\ln\det g(\phi).$$

[19]Under the transformation of (5.1),

$$\left(\prod_{t=1}^{M} d\phi^t\right)\exp\left[\tfrac{1}{2}\delta^4(0)\int d^4u\,\ln\det g(\phi(u))\right]$$

is invariant.

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