Quantum Field-Theory Models in Less Than 4 Dimensions*

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The scalar $\lambda_0\phi^4$ interaction and the Fermi interaction $G_0(\bar\psi\psi)^2$ are studied for space-time dimension d between 2 and 4. An unconventional coupling-constant renormalization is used: $\lambda_0=u_0\Lambda^\epsilon$ ($\epsilon=4-d$) and $G_0=g_0\Lambda^{2-d}$, with u_0 and g_0 held fixed as the cutoff $\Lambda\to\infty$. The theories can be solved in two limits: (1) the limit $N\to\infty$ where ϕ and ψ are fields with N components, and (2) the limit of small ϵ, as a power series in ϵ. Both theories exhibit scale invariance with anomalous dimensions in the zero-mass limit. For small ϵ, the fields ϕ, ϕ^2, and $\phi\nabla_{\alpha_1}\cdots\nabla_{\alpha_n}\phi$ all have anomalous dimensions, except for the stress-energy tensor. These anomalous dimensions are calculated through order ϵ^2; they are remarkably close to canonical except for ϕ^2. The $(\bar\psi\psi)^2$ interaction is studied only for large N; for small ϵ it generates a weakly interacting composite boson. Both the ϕ^4 and $(\bar\psi\psi)^2$ theories as solved here reduce to trivial free-field theories for $\epsilon\to 0$. This paper is motivated by previous work in classical statistical mechanics by Stanley (the $N\to\infty$ limit) and by Fisher and Wilson (the ϵ expansion).

I. INTRODUCTION

The purpose of this paper is to discuss a number of quantum field theories in space-time dimension d less than 4. To be precise, the scalar field theory $\lambda_0\phi^4$ will be considered in 3.99 dimensions ($d=4-\epsilon$ with ϵ small). Both the scalar theory and the Fermi interaction $G_0(\bar\psi\psi)^2$ will be discussed for the larger range $2<d<4$, but with the restriction that ψ and ϕ have N internal components, with N large. A related problem, the ϕ^3 interaction in $6+\epsilon$ dimensions, has been discussed by Mack.[1]

These theories will be renormalized in an unconventional manner such that the renormalized theories are scale-invariant at short distances with anomalous dimensions.[2] In the case of the $\lambda_0\phi^4$ theory, the unconventional renormalization consists of writing $\lambda_0=u_0\Lambda^\epsilon$, where Λ is the cutoff, and holding u_0 (rather than λ_0) fixed as $\Lambda\to\infty$. This means $\lambda_0\to\infty$ as $\Lambda\to\infty$. Conventionally one would have held λ_0 fixed, resulting in a super-renormalizable theory (for $d<4$) with free-field behavior at short distances. In the case of the Fermi interaction, the unconventional renormalization consists of letting $G_0=g_0\Lambda^{2-d}$ and holding g_0 fixed as $\Lambda\to\infty$, i.e., letting $G_0\to 0$ as $\Lambda\to\infty$. This procedure eliminates nonrenormalizability problems. In both cases the fixed constant (u_0 or g_0) is dimensionless; the basic idea is to hold the dimensionless coupling constant fixed as $\Lambda\to\infty$.

In both theories there will be an eigenvalue condition of the Gell-Mann–Low type[3] for the dimensionless bare coupling constant. (For a review of the Gell-Mann–Low theory, see Ref. 4.) By satisfying this eigenvalue condition one obtains a theory which is scale-invariant with anomalous dimensions at short distances. There are no free coupling constants in the exactly scale-invariant theories (skeleton theories). In the case of the Fermi interaction there is an arbitrary renormalized coupling constant which is important at low momenta and serves to break the scale invariance . In the case of the $u_0\Lambda^\epsilon\phi^4$ theory, the eigenvalue for u_0 is *small* if ϵ is small: u_0 is of order ϵ. The theory can therefore be solved by perturbation theory in ϵ. What happens in practice is that one calculates an ordinary Feynman-graph expansion in u_0, with each individual graph extended to noninteger d in an obvious manner.[5,6] Then one gets logarithms as for $d=4$; additional logarithms are generated when individual graphs are expanded in powers of ϵ. At the eigenvalue for u_0 all these logarithms exponentiate. In the case of the ϕ^4 theory the eigenvalue condition for u_0 is important only to ensure exponentiation within the ϵ expansion. When the theory is solved nonperturbatively, as in the $N\to\infty$ limit, the unrenormalized constant u_0 can be arbitrary. Instead it is the *renormalized* coupling constant which has a fixed value. This possibility was already explained in a previous paper.[4]

It will not be proven here that the logarithms exponentiate. Some nontrivial consistency conditions will be verified, but a full proof would require modern renormalization-group methods[7] which are too complex to describe here. Alternatively one should be able to construct a proof using the Callan-Symanzik equations[8]; see Ref. 1.

In the case of the $g_0\Lambda^{2-d}(\bar\psi\psi)^2$ theory, g_0 itself will be small only for $d\simeq 2$. However, another trick enables one to solve the theory approximately for any d. Namely, for large N (e.g., ψ represents a multiplet of N spinor fields ψ_i) the bubble graphs dominate, and are easily summed. This trick will also be used for $u_0\Lambda^{\epsilon}(\phi^2)^2$ theory, where ϕ has N components.

The tricks used here to calculate the field theories (the ϵ expansion and the large-N limit) were both discovered in classical statistical mechanics. The simplicity of the large-N limit was discovered by Stanley.[9] (Stanley showed that the partition function for a spin-spin interaction with N-component spins reduces to the soluble "spherical model" of Berlin and Kac[10] in the limit $N\to\infty$. See Refs. 9 and 11 for details.) The ϵ expansion was discovered by Fisher and the author.[12] The mathematical equivalence of partition functions to the Feynman path integral (see Ref. 7 and references cited therein) makes it evident that any method for analyzing partition functions (especially near a critical point) will have field-theoretic applications. However, no background in statistical mechanics is assumed in this paper. For a review of the ϵ expansion and the renormalization group applied to statistical mechanics, see Ref. 7.

All of the theories discussed here become trivial free-field theories in the limit $d\to 4$. The $g_0\Lambda^{2-d}(\bar{\psi}\psi)^2$ theory, after renormalization, generates a free scalar field in addition to a free spinor field in the limit $d\to 4$. The generation of bosons by a Fermi interaction is well known; see Ref. 13 for a review. The interest in studying these theories for $d<4$ is that they provide a variety of examples of field theories which scale with anomalous dimensions at short distances, and which involve a Gell-Mann–Low eigenvalue condition. These (as well as the ϕ^3 theory in $6+\epsilon$ dimensions[1]) are the first examples of field theories where the eigenvalue can be calculated explicitly.

The result of most practical interest reported here is a calculation of the anomalous dimensions of the tensor operators $\phi\nabla_{\mu_1}\cdots\nabla_{\mu_n}\phi$ (for even n) using the ϵ expansion. These tensors correspond to the tensors in the real world which determine the behavior of the Callan-Gross and Cornwall-Norton integrals over deep-inelastic structure functions. See Ref. 14 or 15 for a review. The Bjorken scaling theory requires all these tensors to have canonical dimensions. All the tensors are found here to have anomalies, except for the second-rank stress tensor.

Consider the case of a theory with isospin ($N=3$). Then there are both isospin-singlet tensors (the second-rank isosinglet tensor being the stress-energy tensor) and isospin-two tensors. The isosinglet tensors in the real world govern the sum of proton and neutron deep-inelastic cross sections; the isospin-two tensors ($I=1$ in the real world) govern the proton-neutron difference in deep-inelastic scattering. The anomalous dimensions of these tensors have been computed to order ϵ^2, with the result

$$d_A(n,I=0)=\frac{\epsilon^2}{121}\left[\frac{5}{2}-\frac{15}{n(n+1)}\right]\quad(n\text{ even}),\quad(1.1)$$

$$d_A(n,I=2)=\frac{\epsilon^2}{121}\left[\frac{5}{2}-\frac{9}{n(n+1)}\right]\quad(n\text{ even}),\quad(1.2)$$

where $d_A(n,I)$ is the anomalous part of the dimension of the nth-rank tensor of isospin I. The precise connection of these dimensions to deep-inelastic scattering is in the behavior of sum rules for large q^2 (see Refs. 14, 15):

$$\int_1^\infty \frac{\nu W_2}{\omega^n}(\omega,q^2)\bigg|_{p+n} d\omega \propto (q^2)^{-d_A(n,I=0)/2},\qquad(1.3)$$

$$\int_1^\infty \frac{\nu W_2}{\omega^n}(\omega,q^2)\bigg|_{p-n} d\omega \propto (q^2)^{-d_A(n,I\neq 0)/2},\qquad(1.4)$$

for $q^2\to\infty$, where ω is the scaling variable $2M\nu/q^2$, ν the electron energy loss, q the electron momentum transfer, M the proton mass, and $W_2(\omega,q^2)$ the structure function.

What is remarkable about these results is the small anomaly in the dimensions. For instance, $d_A(2,I=2)$ is $\epsilon^2/121$. If one could ignore higher orders in ϵ, this anomaly would be very small for both $d=3$ ($\epsilon=1$) and $d=2$ ($\epsilon=2$). One has some weak information about higher orders (see Sec. III) which suggests that the results in order ϵ^2 are correct within perhaps a factor of 3 for $\epsilon=1$, which means the exact anomalies are small, at least for $\epsilon=1$. If such small anomalies are present in strong interactions they would be undetectable in present deep-inelastic scattering experiments.

The stress-energy tensor has canonical dimensions, so the Bjorken scaling law still applies to the Callan-Gross integral ($n=2$) for the sum of proton and neutron structure functions [Eq. (1.3)]. Thus experimentally it is important to study the Cornwall-Norton sum rule with $n=4$ to look for a possible breakdown of scaling, especially at very high q^2.

The $N\to\infty$ calculations are interesting apart from the specific field theories discussed here. Particles in strong interactions group into multiplets, such as SU(3) octets or Regge trajectories. Hence any technique which simplifies the large-N case could be of great practical value. The methods used here should be studied further, in S-matrix theory as well as field theory, for possible practical applications.

This paper is arranged as follows. The $N\to\infty$ calculations for both $(\phi^2)^2$ and $(\bar{\psi}\psi)^2$ theories are reported in Sec. II. The calculations involved are very simple; much of Sec. II consists of conclusions from these calculations. In Sec. III the calculation of anomalous dimensions for small ϵ for the ϕ^4 theory is reported. In Sec. IV two incomplete arguments are given to motivate the assumptions used in Sec. III. Corrections to the $N\to\infty$ limit are discussed in Sec. V. Conclusions are reported in Sec. VI.

The Appendix contains a discussion of spaces with nonintegral dimensions.

II. LARGE N, $2<d<4$

In this section the large-N approximation will be calculated for the $\lambda_0(\phi^2)^2$ and $G_0(\bar\psi\psi)^2$ theories. The approximation will be studied for $2<d<4$. The existence of a renormalized theory which scales will be demonstrated. It will be shown that there is an eigenvalue condition for the dimensionless bare coupling constant in the Fermi case. It will be shown that ϕ and ψ have canonical dimensions for $N\to\infty$. (There are anomalies in order $1/N$; see Sec. V.) The renormalized composite fields ϕ^2 and $\bar\psi\psi$ have anomalous dimensions for $N\to\infty$. The anomalous dimension of ϕ^2 is 2 independent of d (canonically it should have been $d-2$); the anomalous dimension of $\bar\psi\psi$ is 1 independent of d.

The analysis will not be thorough. The aim is to bring out highlights of the theory with brief arguments. So far as the author knows, a more careful analysis does not change any of the results reported here. The original calculation of Stanley[9] for large N was a stationary-phase calculation of a partition-function integral. The graphical argument presented here is much simpler. It is not rigorous.

Consider either a scalar field theory with an interaction Lagrangian density $-\lambda_0(\phi^2)^2$ or a spinor theory with interaction $-G_0(\bar\psi\psi)^2$. Let ϕ and ψ be multiplets of fields ϕ_i and ψ_i, with $1\le i\le N$, N being large. (ϕ^2 means $\sum_i\phi_i{}^2$, and $\bar\psi\psi$ means $\sum_i\bar\psi_i\psi_i$.) Suppose these theories are solved by Feynman perturbation theory. It is logical to consider first those graphs with the largest power of N for a given order in G_0 or λ_0. A power of N is generated for each closed loop in a graph (because each loop involves a sum over a field index i). No other N dependence occurs. Therefore the most important graphs are those with the maximum number of loops for a given order. One maximizes the number of loops by minimizing the number of lines per loop. The minimum number of lines in a loop is two. (All graphs with one-line loops are eliminated by mass renormalization.) The graphs containing only two-line loops are the "bubble graphs" (sometimes called "parquet" graphs), which are summed easily and often. The advantage of the large-N case is that it is legitimate to consider only these graphs.

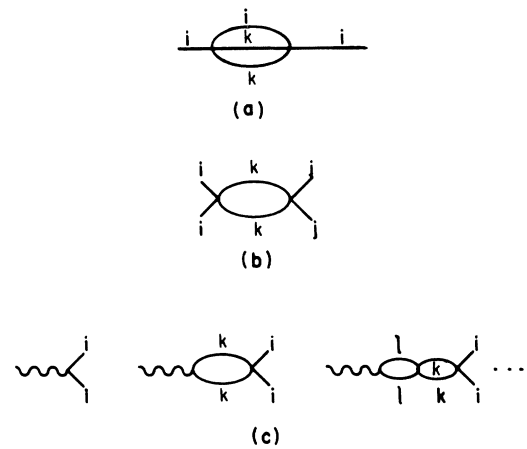

FIG. 1. (a) Diagram giving the lowest-order correction to the propagator. The two lines with internal index k form a loop; the sum over k gives a factor of N. (b) Diagram giving the leading correction to the four-point function (k is summed over). (c) Bubble graphs for vertex function involving ϕ^2 or $\bar\psi\psi$. The wavy line represents ϕ^2 or $\bar\psi\psi$; the straight lines refer to the elementary fields ϕ, ψ, or $\bar\psi$. The indices k and l are summed over.

The bubble graphs provide one power of N for every power of λ_0 or G_0. To compensate for this one considers values of λ_0 and G_0 of order $1/N$. This makes other graphs besides the bubble graphs negligible.

If one now looks at vacuum expectation values of individual ϕ or ψ fields, all diagrams vanish like $1/N$ (at least), except for the free-field terms. For example, the leading corrections to the propagator and four-point functions are the graphs shown in Figs. 1(a) and 1(b). Each of these graphs has one loop, giving one power of N, but two coupling constants, giving two inverse powers of N.

One gets nontrivial results by looking at graphs for the composite operators ϕ^2 and $\bar\psi\psi$. It is most convenient to study the vertex functions $\Gamma(q,q_1)$ involving these operators:

$$\delta_{ij}D(q)\Gamma_B(q,q_1)D(q_1)=\int_x e^{iq\cdot x}\int_y e^{iq_1\cdot y}\langle\Omega|T\phi_i(x)\phi_j(y)\phi^2(0)|\Omega\rangle\,, \tag{2.1}$$

$$\delta_{ij}S(q)\Gamma_F(q,q_1)S(q_1)=\int_x e^{iq\cdot x}\int_y e^{iq_1\cdot y}\langle\Omega|T\psi_i(x)\bar\psi_j(y)\bar\psi(0)\psi(0)|\Omega\rangle\,. \tag{2.2}$$

[The operators $\phi^2(0)$ and $\bar{\psi}(0)\psi(0)$ will be made finite with the help of cutoffs.] Here $D(q)$ and $S(q)$ are the propagators for ϕ and ψ, respectively. The bubble diagrams for Γ_B and Γ_F are shown in Fig. 1(c); these diagrams are all independent of N.

Let $k=q+q_1$. Then the bubble graph sums give

$$\Gamma_B(q,q_1)=\frac{1}{1+4\lambda_0 N v_B(k^2)}, \tag{2.3}$$

$$\Gamma_F(q,q_1)=\frac{1}{1+2G_0 N v_F(k^2)}, \tag{2.4}$$

where

$$v_B(k^2)=-i\int_p \frac{1}{p^2-m^2+i\epsilon}\,\frac{1}{(p+k)^2-m^2+i\epsilon}\,\frac{-\Lambda^2}{p^2-\Lambda^2+i\epsilon}, \tag{2.5}$$

$$v_F(k^2)=+i\,\mathrm{Tr}\int_p \frac{1}{\not p-m+i\epsilon}\,\frac{1}{\not p+\not k-m+i\epsilon}\left(\frac{-\Lambda^2}{p^2-\Lambda^2+i\epsilon}\right)^2; \tag{2.6}$$

$\int_p$ means $(2\pi)^{-d}\int d^dp$. Λ is a cutoff; m is the (renormalized) mass. The function $v_B(k^2)$ is finite without a cutoff when $d<4$, but the cutoff dependence will be needed to compare $d<4$ with $d=4$.

The integral $\int d^dp$ for nonintegral d is most easily handled by the Schwinger trick which converts the propagators to Gaussian form.[1] For example, one writes

$$v_B(k^2)=-\Lambda^2\int_0^\infty dy_1\int_0^\infty dy_2\int_0^\infty dy_3\int_p \exp\{i(p^2-m^2+i\epsilon)y_1+i[(p+k)^2-m^2+i\epsilon]y_2+i(p^2-\Lambda^2+i\epsilon)y_3\}. \tag{2.7}$$

The p integration reduces to

$$\int_p \exp\left[i(y_1+y_2+y_3)\left(p+\frac{ky_2}{y_1+y_2+y_3}\right)^2\right] \tag{2.8}$$

apart from factors independent of p. A translation (assumed valid for nonintegral d) reduces this to

$$\int_p \exp[i(y_1+y_2+y_3)p^2]. \tag{2.9}$$

For integral d this integral is

$$\frac{-i}{(2\pi)^d}(i\pi)^{d/2}(y_1+y_2+y_3)^{-d/2} \tag{2.10}$$

(the explicit factor $-i$ is due to the Lorentz metric). One now assumes that this formula is valid for noninteger d. The definition of γ matrices for nonintegral d presents no problem here: All one needs is to calculate $\mathrm{Tr}\gamma_\mu\gamma_\nu$ and $\mathrm{Tr}I$. Apart from a multiplicative constant these are $g_{\mu\nu}$ and 1, respectively, and the constant can be absorbed into a redefinition of G_0. The constant is set equal to 4 here. See the Appendix for more discussion of spaces with nonintegral dimensions. The rest of the calculation of $v_B(k^2)$ and $v_F(k^2)$ is straightforward, apart from a final integration (assuming $\Lambda^2\gg k^2$ and m^2). The results will be given only for the case $m=0$ and $k^2\ll\Lambda^2$, keeping terms through order $\Lambda^{-\epsilon}$. The $\Lambda^{-\epsilon}$ term is calculated only to order ϵ^{-1} for small ϵ. The results are

$$v_B(k^2)=C_B(\epsilon)(-k^2-i\epsilon)^{-\epsilon/2}-\frac{\Lambda^{-\epsilon}}{8\pi^2\epsilon}, \tag{2.11}$$

$$v_F(q^2)=-C_F(\epsilon)(-k^2-i\epsilon)^{1-\epsilon/2}+C_1(\epsilon)\Lambda^{2-\epsilon}-\frac{1}{4\pi^2}\frac{k^2\Lambda^{-\epsilon}}{\epsilon}, \tag{2.12}$$

with $C_B(\epsilon)$, $C_F(\epsilon)$, and $C_1(\epsilon)$ all positive constants ($i\epsilon$ is the usual infinitesimal; it will be omitted in future formulas). The constants $C_B(\epsilon)$, etc. are

$$C_B(\epsilon)=\frac{2(\epsilon/2)!}{(4\pi)^{d/2}\epsilon}\int_0^1 dx[x(1-x)]^{-\epsilon/2}, \tag{2.13}$$

$$C_F(\epsilon)=\frac{16(\epsilon/2)!(3-\epsilon)}{(4\pi)^{d/2}\epsilon(2-\epsilon)}\int_0^1 dx[x(1-x)]^{1-\epsilon/2}, \tag{2.14}$$

$$C_1(\epsilon)=\frac{8(\epsilon/2)!}{(4\pi)^{d/2}(2-\epsilon)}. \tag{2.15}$$

Note that for ϵ small (d near 4)

$$C_B(\epsilon)\simeq\frac{1}{8\pi^2\epsilon},\quad C_F(\epsilon)\simeq\frac{1}{4\pi^2\epsilon},\quad C_1(\epsilon)\simeq\frac{1}{4\pi^2}, \tag{2.16}$$

while for $2-\epsilon$ small (d near 2)

$$C_B(\epsilon)\simeq\frac{1}{\pi(2-\epsilon)},\quad C_F(\epsilon)\simeq\frac{2}{\pi(2-\epsilon)}\simeq C_1(\epsilon). \tag{2.17}$$

For $d=4$ exactly one gets

$$v_B(k^2)=\frac{1}{16\pi^2}\ln\left(\frac{\Lambda^2}{-k^2}\right)+C_2\,, \tag{2.18}$$

$$v_F(k^2)=\frac{\Lambda^2}{4\pi^2}+\frac{1}{8\pi^2}\,k^2\ln\left(\frac{\Lambda^2}{-k^2}\right)+C_3k^2\,, \tag{2.19}$$

where C_2 and C_3 are constants.

Now consider the scalar vertex function $\Gamma_B(q,q_1)$. First study the case $d=4$ exactly. Then as $\Lambda\to\infty$ (holding λ_0, q and q_1 fixed)

$$\Gamma_B(q,q_1)\simeq\frac{4\pi^2}{\lambda_0N\ln\Lambda^2}\,, \tag{2.20}$$

i.e., $\Gamma_B(q,q_1)\to 0$ logarithmically as $\Lambda\to\infty$. This is a well-known result; it occurs because λ_0 is held fixed (no charge renormalization is performed). One can obtain a nonzero limit for $\Gamma_B(q,q_1)$ if one performs both a charge and a wave-function renormalization. (The wave-function renormalization is a renormalization of the composite field ϕ^2; there is no wave-function renormalization of ϕ.) The charge renormalization that is required can be written

$$\lambda_0=-\frac{4\pi^2}{N\ln\Lambda^2}-\left(\frac{4\pi^2}{N\ln\Lambda^2}\right)^2\frac{1}{\lambda_R}\,, \tag{2.21}$$

where λ_R is to be held fixed as $\Lambda\to\infty$. [Persons who pay attention to dimensional analysis will shudder at this equation. To make it dimensionally correct, substitute $\ln(\Lambda^2/q_R^{\,2})$ for $\ln\Lambda^2$, where q_R is an arbitrarily chosen "reference momentum." This only changes the definition of λ_R.] Wave-function renormalization consists of dividing Γ_B by $\ln\Lambda^2/8\pi^2$. For $m=0$ the renormalized Γ_B is

$$\Gamma_B(q,q_1)(\text{renormalized})=\frac{\lambda_R}{1-N\lambda_R\ln(-k^2)/4\pi^2}\,. \tag{2.22}$$

This is a well-known formula from renormalization group calculations. It is not scale-invariant. Furthermore, since λ_0 is negative, one expects the theory to have no ground state (due to the interaction Hamiltonian having no lower bound). This difficulty does not seem to show up, however, when one sums only bubble graphs. Finally, it has a ghost pole for large negative k^2 (spacelike k) if λ_R is positive. In a word, this theory is unacceptable.

Suppose now that $2<d<4$ $(0<\epsilon<2)$. Let $\Lambda\to\infty$ holding λ_0 fixed. For $m=0$ the vertex function is

$$\Gamma_B(q,q_1)=\frac{1}{1+4N\lambda_0C_B(\epsilon)(-k^2)^{-\epsilon/2}}\,. \tag{2.23}$$

For large k^2 the λ_0 term is negligible and $\Gamma_B(q,q_1)$ has its free-field value. This is what one expects, since $\lambda_0\phi^4$ theory is super-renormalizable for $d<4$.

Now let $2<d<4$, $\lambda_0=u_0\Lambda^\epsilon$, and let $\Lambda\to\infty$ holding u_0 fixed. Then one obtains (for $m=0$)

$$\Gamma_B(q,q_1)=\frac{\Lambda^{-\epsilon}}{4Nu_0C_B(\epsilon)(-k^2)^{-\epsilon/2}}\,. \tag{2.24}$$

This becomes finite after a wave-function renormalization [division by $\Lambda^{-\epsilon}/(4Nu_0)$; the finite part of this constant is chosen by whim]:

$$\Gamma_B(q,q_1)(\text{renormalized})=\frac{(-k^2)^{\epsilon/2}}{C_B(\epsilon)}\,. \tag{2.25}$$

The renormalized vertex is scale-invariant (for $m=0$). The power of k^2 corresponds to ϕ^2 having an anomalous dimension 2 for any ϵ. This is easily seen: Γ_B would be a constant (as in the free-field theory) if ϕ^2 had its canonical dimension $d-2$; to change Γ_B to the behavior k^ϵ one must increase the dimension of ϕ^2 by $\epsilon=4-d$, yielding dimension 2.

If the mass is not neglected the renormalized vertex function is

$$\Gamma_B(q,q_1)(\text{renormalized})=\frac{1}{v_B(k^2)}\,. \tag{2.26}$$

The function v_B is easily seen to be finite for $\Lambda\to\infty$ and positive for $k^2<0$. Thus $\Gamma_B(q,q_1)$ is a well-behaved vertex. It has no ghost poles, in contrast to the case of positive λ_R for $d=4$. One can choose u_0 positive, thus avoiding the difficulties associated with negative λ_0.

The renormalized vertex function is independent of u_0, and in fact contains no arbitrary coupling constants. The only free parameter is the mass m. Nevertheless the vertex is not zero and even has a nontrivial imaginary part. So the theory is a field-theoretic example of a theory that bootstrappers dream of. There is no eigenvalue condition for u_0 here; in Sec. III an eigenvalue condition will be obtained, but only for technical reasons.

Now the vertex of the four-fermion theory will be discussed. Consider first the range $2<d<4$, Λ very large (but not ∞), $m=0$, and let $G_0=g_0\Lambda^{\epsilon-2}$. Then one has

$$\Gamma_F(q,q_1)=\frac{1}{1+2Ng_0C_1(\epsilon)-2Ng_0C_F(\epsilon)\Lambda^{\epsilon-2}(-k^2)^{1-\epsilon/2}}\,. \tag{2.27}$$

The k^2 term has been included, although at first sight it is negligible compared to the other terms. Now let

$$g_0=-\frac{1}{2NC_1(\epsilon)}\,. \tag{2.28}$$

There is no reason to forbid this choice. The coupling constant of the four-fermion interaction can be negative with impunity, as far as is known at present. With this particular choice,

$$\Gamma_F(q, q_1) = \frac{\Lambda^{2-\epsilon} C_1(\epsilon)}{C_F(\epsilon)(-k^2)^{1-\epsilon/2}} . \tag{2.29}$$

After a wave-function renormalization,

$$\Gamma_F(q, q_1)(\text{renormalized}) = (-k^2)^{\epsilon/2-1} . \tag{2.30}$$

This is again scale-invariant and corresponds to the anomalous dimension 1 for the renormalized operator $\bar{\psi}\psi$. A more general renormalized form (still with $m=0$) results if one chooses

$$g_0 = -\frac{1}{2NC_1(\epsilon)} + g_R \frac{\Lambda^{\epsilon-2}}{2N} \frac{C_F(\epsilon)}{C_1^2(\epsilon)} \tag{2.31}$$

and holds g_R fixed as $\Lambda \to \infty$. Then

$$\Gamma_F(q, q_1)(\text{renormalized}) = \frac{1}{g_R + (-k^2)^{1-\epsilon/2}} . \tag{2.32}$$

Now one has an eigenvalue condition for the bare coupling constant, namely Eq. (2.28) [this is valid for $\Lambda \to \infty$ even if Eq. (2.31) is used for finite Λ], but an arbitrary renormalized coupling constant g_R. As long as $g_R > 0$ there are no ghost states in the vertex function. (This remains true even if m is not zero.) The theory is scale-invariant for $|k^2|^{1-\epsilon/2} \gg g_R$. [The constant g_R has dimensions $(\text{mass})^{2-\epsilon}$, as is evident from its definition, Eq. (2.31).] For ϵ near zero the vertex function looks much like the propagator of a scalar particle of mass $\sqrt{g_R}$. This is not surprising; the appearance of scalar particles (or vector particles when appropriate) in four-fermion theories has been found previously.[13]

Further insight is obtained by studying the four-point functions of the scalar and spinor theories. These functions are described by the same graphs as the vertex functions, except for an extra factor of the bare coupling constant and the addition of crossed terms. (The four-point function is the proper four-point function; disconnected graphs and self-energy insertions on external lines are dropped.)

The four-point function for the scalar theory depends on four momenta q_1, q_2, q_3, q_4, and a corresponding set of internal indices i_1, i_2, i_3, i_4. It has the form

$$H_B(q_1, q_2, q_3, q_4)\delta_{i_1 i_2}\delta_{i_3 i_4} + H_B(q_1, q_3, q_2, q_4)\delta_{i_1 i_3}\delta_{i_2 i_4} + H_B(q_1, q_4, q_2, q_3)\delta_{i_1 i_4}\delta_{i_2 i_3} ,$$

where

$$H_B(q_1, q_2, q_3, q_4) = \frac{8u_0\Lambda^\epsilon}{1 + 4Nu_0\Lambda^\epsilon v_B(k^2)} , \tag{2.33}$$

with $k = q_1 + q_2$. There is an analogous expression for the four-point function for the Fermi interaction in terms of a function $H_F(q_1, q_2, q_3, q_4)$:

$$H_F(q_1, \ldots, q_4) = \frac{2g_0\Lambda^{\epsilon-2}}{1 + 2Ng_0\Lambda^{\epsilon-2}v_F(k^2)} . \tag{2.34}$$

There can be no wave-function renormalization for these amplitudes, because only the wave-function renormalization constants for ϕ and ψ are involved and these are 1 to leading order in $1/N$.

Now take the limit $\Lambda \to \infty$ holding u_0 and g_0 fixed. The result will be quoted for $m=0$ and ϵ small. One obtains for H_B

$$H_B(q_1, \ldots, q_4) = \frac{16\pi^2\epsilon}{N} (-k^2)^{\epsilon/2} . \tag{2.35}$$

The explicit factor ϵ means that if one takes the limit $\epsilon \to 0$ holding N and k^2 fixed, one gets zero. So the theory becomes a free-field theory in the limit $d \to 4$.

The same result is true for the four-fermion case. Let $\Lambda \to \infty$, with g_0 expressed in terms of g_R through Eq. (2.31) and holding g_R fixed. For ϵ small and $m=0$ one obtains

$$H_F(q_1, \ldots, q_4) = \frac{-4\pi^2\epsilon}{N[g_R + (-k^2)^{1-\epsilon/2}]} . \tag{2.36}$$

Once again $H_F \to 0$ as $\epsilon \to 0$, holding g_R, k^2, and N fixed. Thus the theories being discussed here become trivial free-field theories in the limit of four dimensions.

Consider further the form of H_F for ϵ small but finite. Suppose terms of order ϵ^2 are neglected; then

$$H_F(q_1, \ldots, q_4) = \frac{4\pi^2\epsilon}{N(k^2 - g_R)} , \tag{2.37}$$

which is the same as the Born approximation for the exchange of a scalar boson, with ϵ substituting for the coupling constant squared. This suggests that the four-fermion interaction is identical with a Yukawa interaction $h_0 \sum_{i=1}^N \bar{\psi}_i\psi_i\phi$, where h_0 is a new coupling constant of order $\epsilon^{1/2}$ and ϕ is a single scalar field. This is in fact the case. The bubble graphs are again the leading graphs for the Yukawa interaction; keeping only the bubbles gives

$$H_F(q_1, \ldots, q_4)(\text{Yukawa}) = \frac{h_0^2}{k^2 - \mu_0^2 + h_0^2 N v_F(k^2)} , \tag{2.38}$$

where μ_0 is the bare mass of the Yukawa field. The renormalization procedure is more complicated in this case because a mass renormalization is needed. The following procedure gives a finite H_F for $\Lambda\to\infty$: Let $h_0=\Lambda^{\epsilon/2}h_0'/N$. The constant h_0' is dimensionless; hold it fixed as $\Lambda\to\infty$. Also let

$$\frac{\mu_0^2}{N}-h_0^2C_1(\epsilon)\Lambda^{2-\epsilon}=\mu_R^2h_0^2C_F(\epsilon) \tag{2.39}$$

and hold μ_R^2 fixed. Then in the limit $\Lambda\to\infty$, with ϵ small and $m=0$, one has

$$H_F(q_1,\ldots,q_4)=\frac{-4\pi^2\epsilon}{N[\mu_R^2+(-k^2)^{1-\epsilon/2}]}. \tag{2.40}$$

This is identical with Eq. (2.36) if one identifies μ_R^2 with g_R.

This is only a very limited demonstration of the equivalence of $\bar\psi\psi\phi$ and $(\bar\psi\psi)^2$ theories. Nevertheless, the author conjectures that the equivalence is exact and is true for arbitrary N, not just $N\to\infty$. Previous work on the equivalence of electrodynamics and the vector Fermi interaction makes this conjecture at least plausible. In the renormalization-group language of Ref. 7 the two theories should correspond to the same fixed point for large N; it is difficult for a single fixed point for large N to become two separate fixed points for small N.

Consider now the four-fermion interaction for $d=3$ ($\epsilon=1$). Renormalized as in Eq. (2.31), the solution for k^2 positive for H_F is

$$H_F(q_1,\ldots,q_4)=\frac{-1}{NC_F(\epsilon)}\,\frac{1}{g_R-i(k^2)^{1/2}}. \tag{2.41}$$

This means the imaginary part of the fermion scattering amplitude is as large as the real part, despite the fact that the whole amplitude is of order $1/N$. This amplitude is nevertheless consistent with unitarity. The amplitude squared is of order $1/N^2$, but the sum over intermediate states includes a sum over particle species which produces a factor N.

Finally, consider the case $d\to 2$ ($\epsilon\to 2$). In this case

$$H_F(q_1,\ldots,q_4)\simeq-\frac{1}{N}\left(\tfrac{1}{2}\pi\right)\frac{2-\epsilon}{g_R+(-k^2)^{1-\epsilon/2}}. \tag{2.42}$$

To a first approximation for ϵ near 2, H_F is a small constant, of order $(2-\epsilon)/N$. In the limit $\epsilon\to 2$, holding N and k^2 fixed, $H_F\to 0$. Thus the four-fermion interaction (with our renormalization) becomes a trivial free-field theory both in 2 and 4 dimensions. However, near 2 dimensions the theory still looks like a four-fermion interaction after renormalization (the fermion scattering amplitude is almost constant), while near 4 dimensions the renormalized theory looks like a Yukawa theory with weak coupling.

III. ϕ^4 THEORY FOR SMALL ϵ

In this section the $\lambda_0\phi^4$ theory will be studied for $d\simeq 4$ ($d=4-\epsilon$ with ϵ small) and arbitrary N. The theory will be renormalized by letting $\lambda_0=u_0\Lambda^\epsilon$ and holding u_0 fixed. The crucial property of the theory which makes it soluble is that the coupling strength of the renormalized theory is of order ϵ, so calculations can be carried out using only low-order Feynman graphs. The anomalous dimensions will be calculated for the fields ϕ, ϕ^2 and the tensor fields $\phi\nabla_{\alpha_1}\cdots\nabla_{\alpha_n}\phi$ (with even n). Except for ϕ^2, these anomalous dimensions become canonical in the large-N limit, so it is especially interesting to compute them for small N.

There is one tricky feature of the calculation, which is best explained by studying further the approximation of Sec. II for large N. Consider the four-point function $H_B(q_1,\ldots,q_4)$. For small ϵ, large Λ, and $m=0$, and keeping the leading cutoff-dependent terms, H_B is

$$H_B(q_1,\ldots,q_4)=\frac{16\pi^2\epsilon u_0\Lambda^\epsilon}{2\pi^2\epsilon+Nu_0\Lambda^\epsilon[(-k^2)^{-\epsilon/2}-\Lambda^{-\epsilon}]}. \tag{3.1}$$

For $\Lambda\to\infty$ with u_0 fixed this gives a scaling form for H_B independent of Λ:

$$H_B(q_1,\ldots,q_4)=\frac{16\pi^2\epsilon}{N}\,(-k^2)^{\epsilon/2}. \tag{3.2}$$

However, if one expands in powers of u_0 one obtains

$$H_B(q_1,\ldots,q_4)=8u_0\Lambda^\epsilon-\frac{4Nu_0^2\Lambda^{2\epsilon}}{\pi^2\epsilon}[(-k^2)^{-\epsilon/2}-\Lambda^{-\epsilon}]+\frac{2N^2u_0^3\Lambda^{3\epsilon}}{(\pi^2\epsilon)^2}[(-k^2)^{-\epsilon/2}-\Lambda^{-\epsilon}]^2-\cdots. \tag{3.3}$$

If one takes the limit $\Lambda\to\infty$ order by order in u_0, there is no longer a simple limit. Consider now the double expansion in u_0 and ϵ:

$$H_B(q_1,\ldots,q_4)=8u_0+8\epsilon u_0\ln\Lambda+\frac{2u_0^2N}{\pi^2}\ln\left(\frac{-k^2}{\Lambda^2}\right)+\cdots. \tag{3.4}$$

One now has a jumble of $\ln\Lambda$'s and $\ln k$'s, and there again will be no limit for $\Lambda\to\infty$ in general. The remarkable fact is that there is a unique choice for u_0, namely $u_0=2\pi^2\epsilon/N$, for which the cutoff dependence of (3.4) disappears order by order in ϵ, and the $\ln k$ terms sum up to the simple power $(-k^2)^{-\epsilon/2}$ of Eq. (3.2). This can be seen from Eq. (3.1). Substituting $u_0=2\pi^2\epsilon/N$ in Eq. (3.1) gives Eq. (3.2) for any value of Λ and k. The only question left is whether one is allowed to expand Eq. (3.1) in powers of u_0 and ϵ before making the substitution for u_0. It is not difficult to show that the expansion is legitimate provided $\epsilon \ln\Lambda$ and $|u_0 \ln(k^2/\Lambda^2)|$ are small; these restrictions can be satisfied for a sufficiently large range of values of ϵ and k^2 to justify substituting for u_0 after expanding in u_0 and ϵ. Thus, although Eq. (3.3) contains all powers of both Λ^ϵ and $(-k^2)^\epsilon$, it reduces to Eq. (3.2), which has only one power of $(-k^2)^\epsilon$ and no Λ dependence provided one expands in ϵ and then substitutes $u_0 = 2\pi^2\epsilon/N$. One can of course verify this directly.

This introduces the essential idea of the ϵ expansion. For any N, one can solve the interaction $u_0\Lambda^\epsilon\phi^4$ in a power series in u_0 and ϵ, since this is a Feynman-graph calculation. The result, for any N, is a jumble of logarithms in k and Λ. There are arguments that, for any N, there is a unique choice $u_0 = u_0(\epsilon, N)$ for u_0 such that (1) the logarithms in Λ disappear (except for a wave-function renormalization factor; see below); (2) the logarithms in k exponentiate to a single power of k^2 as in Eq. (3.2), rather than many different powers of k^2 as in Eq. (3.3) (for $m = 0$ where m is the renormalized mass); and (3) the resulting theory is scale-invariant for $m = 0$. The arguments will be discussed briefly in Sec. IV; a more thorough discussion is given in Ref. 7.

Once one believes that there is such a function $u_0(\epsilon, N)$ there are several procedures for determining $u_0(\epsilon, N)$ and then calculating anomalous dimensions and other quantities of interest. The procedure used below is the procedure used earlier in a statistical-mechanical context.[6] The idea is to make use of the fact that there is a unique (scale-invariant) power of k^2 when $u_0 = u_0(\epsilon, N)$. This makes no reference to Λ, so for convenience we use units with $\Lambda = 1$. This choice of units is natural when one is doing statistical-mechanical calculations[6] but is somewhat strange to a field theorist. The reader may wish to redo the calculations with arbitrary Λ. One must then pay attention to wave-function renormalization. In order ϵ^2, for finite N, ϕ has an anomalus dimension. When this happens, amplitudes like H_B will have an over-all dependent factor which can be removed by wave-function renormalization. For $u_0 = u_0(\epsilon, N)$ there will be no other cutoff dependence.

The calculation of $u_0(\epsilon, N)$ and the anomalous dimension d_ϕ of ϕ will now be described. The quantities $u_0(\epsilon, N)$ and $d_\phi(\epsilon, N)$ will be determined simultaneously. Two amplitudes will be computed by Feynman graphs: the propagator $D(q^2)$ for zero mass and the four-point function $H_B(q_1, \ldots, q_4)$ for finite m and $q_1 = \cdots = q_4 = 0$ (this is one possible definition of the renormalized coupling constant). The idea is to make full use of the properties these amplitudes should have due to scale invariance when $u_0 = u_0(\epsilon, N)$. For example, in a scale-invariant theory, $D(q^2)$ behaves as $(q)^{(2d_\phi - d)}$. This follows directly (by dimensional analysis) from the definition of the propagator,

$$D(q) = \int_x e^{iq\cdot x} D(x)\,, \tag{3.5}$$

where

$$D(x) = \langle \Omega\,|\,T(\phi(x)\phi(0))\,|\,\Omega\rangle\,, \tag{3.6}$$

and the fact that $D(x)$ scales as

$$D(x) \propto (x)^{-(2d_\phi)} \tag{3.7}$$

when ϕ has dimension d_ϕ. The propagator will be studied for $\Lambda = 1$ and $m = 0$; the scale-invariant regime is $q \ll \Lambda$, i.e., $q \ll 1$. Hence one expects

$$D(q^2) \sim q^{(2d_\phi - d)} \tag{3.8}$$

for $q \ll 1$, when $u_0 = u_0(\epsilon, N)$. This formula will be compared later with an actual calculation of $D(q)$ by Feynman graphs.

The function H_B will be calculated for $m \ll \Lambda$, i.e., $m \ll 1$, but not $m = 0$. A prediction of the m dependence of $H_B(0,0,0,0)$ (to be denoted u_R) is needed. Consider first the propagator $D(q)$ for $m \neq 0$. For $m \ll q \ll \Lambda$, the propagator still behaves as in Eq. (3.8). [No wave-function renormalization will be performed when calculating either $D(q^2)$ or H_B. Otherwise, $D(q^2)$ might have m-dependent factors due to the wave-function renormalization. No wave-function renormalization is required: Since $\Lambda = 1$ is fixed, the propagator is finite without renormalization.] For $q \ll m$ all that happens is that m replaces q in Eq. (3.8); in particular

$$D(0) \propto m^{(2d_\phi - d)}\,. \tag{3.9}$$

It is evident for the free propagator $1/(q^2 - m^2)$ that q can be replaced by m for $q \ll m$; it can also be verified for individual Feynman graphs; a general argument is given in Ref. 7.

Consider now the function $H_B(q_1, q_2, q_3, q_4)$. The complete (improper) four-point function is a Fourier transform with respect to x, y, and z of $\langle \Omega\,|\,T(\phi(x)\phi(y)\phi(z)\phi(0))\,|\,\Omega\rangle$. If the momenta involved (q_1, q_2, q_3, say) are all of the same order of magnitude then dimensional analysis shows that this triple Fourier transform scales as $(q_1)^{(4d_\phi - 3d)}$ in the scaling region. The proper four-point function H_B is obtained by removing disconnected graphs (which does not change the scaling analysis) and then dividing by four propagators, one for each external line. Performing the division, using Eq. (3.8), one finds that the proper four-point function $H_B(q_1, q_2, q_3, q_4)$ scales as $(q_1)^{(d - 4d_\phi)}$ in the scaling region $m \ll q_1 \ll 1$. Of more interest is $u_R = H_B(0,0,0,0)$. Again one simply replaces q_1 by m, giving

$$u_R \propto m^{(d-4d_\phi)} . \tag{3.10}$$

The quantities that will actually be calculated by graphs are u_R and $D(0)$ for small but nonzero m, and $D(q^2)$ for $m=0$ and small q^2. To keep the calculation as simple as possible, only a partial mass renormalization will be carried out. Instead of making the customary two subtractions in self-energy diagrams on the mass shell, only a single subtraction at $q^2=0$ will be made. This means that the renormalized mass m is not the mass parameter in Feynman graphs; the latter will be denoted m_F. The single subtraction at $q^2=0$ means that

$$D(0)=1/m_F^{\ 2} \tag{3.11}$$

independently of u_0 or ϵ. It will be convenient to know how u_R scales with m_F. Combining Eqs. (3.9)–(3.11) one obtains

$$u_R \propto m_F^{2(4d_\phi-d)/(2d_\phi-d)} . \tag{3.12}$$

The idea is now to compare the predicted behavior of u_R and $D(q^2)$ with explicit graphical calculations. The result will be formulas for the two exponents $2(4d_\phi-d)/(2d_\phi-d)$ and $2d_\phi-d$ as functions of u_0 and $\epsilon=4-d$. For given ϵ (the independent variable) these two equations will determine the two unknowns d_ϕ and u_0.

The procedures for calculating graphs for non-

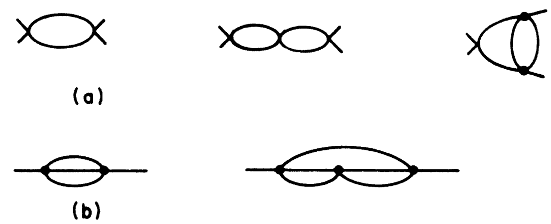

FIG. 2. (a) Diagrams determining u_R to order $u_0^{\ 3}$. (b) Diagrams determining $D(q)$ to order $u_0^{\ 2}$.

integer d have been explained in Sec. II and the Appendix. Apart from the noninteger d the graph calculations are standard; only the results will be quoted here. The calculations are identical except for notation to those reported previously[6] (in a statistical-mechanical context). All graphs were calculated for spacelike momenta in a Euclidean metric. The cutoff was introduced by using a cutoff propagator $[q^2(1+q^2)^2+m^2]^{-1}$ ($q^2=q_0^{\ 2}+\vec{q}^{\,2}$ because of the Euclidean metric) for all internal lines. Changing the form of the cutoff propagator changes u_0 but not d_ϕ.

The diagrams for u_R and $D(q^2)$ were both calculated through order $u_0^{\ 3}$. The diagrams calculated are shown in Fig. 2. The constant and u_0 terms are known exactly in ϵ; the $u_0^{\ 2}$ term was calculated to order ϵ, and the $u_0^{\ 3}$ term was calculated only for $\epsilon=0$ (one calculates only terms that will ultimately be of order ϵ^3 or less once u_0 is known as a function of ϵ). Only logarithmic terms were needed for the $u_0^{\ 3}$ and $u_0^{\ 2}\epsilon$ terms.

The results are as follows[6]:

$$u_R=u_0+(N+8)(u_0^{\ 2}/4\pi^2)(2\ln m_F+\tfrac{17}{6})-(N+8)(u_0^{\ 2}/4\pi^2)\epsilon[\ln^2 m_F+(1+2K')\ln m_F]$$
$$+(u_0^{\ 3}/4\pi^4)[(N+8)^2(\ln^2 m_F+\tfrac{17}{6}\ln m_F)-(10N+44)\ln m_F] . \tag{3.13}$$

The constant K' will affect u_0 but not d_ϕ or any other anomalous dimension. Its definition is as follows. If $K(d)$ is

$$K(d)=2^{-(d-1)}\pi^{-d/2}[\Gamma(\tfrac{1}{2}d)]^{-1} , \tag{3.14}$$

where Γ is the usual gamma function, then

$$K'=\left.\frac{d\ln K(x)}{dx}\right|_{x=4} . \tag{3.15}$$

For $D(q^2)$ only the logarithmic terms are computed:

$$q^2D(q^2)=1+(N+2)(u_0^{\ 2}/16\pi^4)\ln q^2-(N+2)(u_0^{\ 2}/16\pi^4)\epsilon[\tfrac{1}{2}(\ln q^2)^2+(2K'-\tfrac{9}{4})\ln q^2]$$
$$+(N+2)(N+8)(u_0^{\ 3}/16\pi^6)[\tfrac{1}{4}(\ln q^2)^2-\tfrac{1}{3}\ln q^2] . \tag{3.16}$$

Consider now the requirement (3.12) on u_R. This means that the logarithms of m_F must exponentiate. To order $u_0^{\ 2}$ there is no problem: One can write

$$u_R=u_0[1+\tfrac{17}{6}(N+8)(u_0/4\pi^2)]\exp[(N+8)(u_0/2\pi^2)\ln m_F] . \tag{3.17}$$

Hence to order u_0 one must have

$$\frac{2(d-4d_\phi)}{d-2d_\phi}=(N+8)\,\frac{u_0}{2\pi^2}. \tag{3.18}$$

Similarly, to order u_0^2 the requirement (3.8) gives

$$d_\phi-\tfrac{1}{2}d=-1+(N+2)(u_0^2/16\pi^4). \tag{3.19}$$

Through first order in u_0 the second equation gives canonical dimensions for ϕ: $d_\phi=\frac{1}{2}(d-2)+O(u_0^2)$. Substituting this in Eq. (3.18) one gets

$$4-d=\epsilon=(N+8)u_0/2\pi^2. \tag{3.20}$$

Hence to order ϵ

$$u_0(\epsilon,N)=2\pi^2\epsilon/(N+8). \tag{3.21}$$

Substituting in Eq. (3.19) gives

$$d_\phi=\tfrac{1}{2}(d-2)+\frac{N+2}{4(N+8)^2}\,\epsilon^2 \tag{3.22}$$

to order ϵ^2.

With the information available in Eqs. (3.13) and (3.16) one can calculate u_0 to order ϵ^2, and d_ϕ to order ϵ^3. There are also consistency conditions that must be satisfied. For example, the requirement that the logarithms exponentiate means the coefficient of $\ln^2 m_R$ in Eq. (3.13) for u_R must be $(N+8)^2(u_0^3/8\pi^4)=\pi^2\epsilon^3/(N+8)$ to order ϵ^3. This consistency condition is satisfied. For $D(q^2)$ the consistency condition is that the $\ln^2 q^2$ terms vanish in order ϵ^3. They do.

The result for d_ϕ to order ϵ^3 is

$$d_\phi=\tfrac{1}{2}(d-2)+\frac{N+2}{4(N+8)^2}\,\epsilon^2+\frac{N+2}{4(N+8)^2}\left[\frac{6(3N+14)}{(N+8)^2}-\frac{1}{4}\right]\epsilon^3. \tag{3.23}$$

The remarkable feature of this result is how small the anomalous part of d_ϕ is. The variable ϵ has a built-in scale: The only sensible values of ϵ are 1 and 2. Negative ϵ is not permitted because it corresponds to a negative value of u_0, which is meaningless for a ϕ^4 theory (see Sec. II). In the other direction, $\epsilon=3$ corresponds to a space-time without any space. The maximum for any N of $(N+2)/[4(N+8)^2]$ is $\frac{1}{96}$ (for $N=4$). The coefficient of ϵ^3 is never much larger than $\frac{1}{96}$. So d_ϕ, to order ϵ^3, differs from its canonical value by 0.1 at most. Higher orders could change this result, but there is independent evidence from statistical mechanics that the anomalous part of d_ϕ is indeed small: about 0.03 for $\epsilon=1$ and exactly 0.125 for $\epsilon=2$, both for $N=1$. (The value 0.125 comes from the Yang-Onsager calculation of the spin-spin correlation function for the two-dimensional Ising model.) See Ref. 7 for further discussion and references.

The smallness of the anomaly in d_ϕ does not mean all anomalies are small. For example the anomaly in the dimension of ϕ^2, for $N=\infty$, is $2-(d-2)=\epsilon$. For $\epsilon=2$ this anomaly is 2. [Note that from Eq. (3.23) the anomaly for d_ϕ vanishes in the limit $N\to\infty$, as expected from Sec. II.]

Anomalous dimensions have also been calculated for the composite operators ϕ^2 and $\phi\nabla_{\alpha_1}\cdots\nabla_{\alpha_n}\phi$ for any even n. See Ref. 7 for details of the calculation. For $N>1$ there are two forms of each of these operators. For example, for ϕ^2 one can consider either $\sum_i\phi_i^2$ or the set of operators $\phi_i\phi_j$ with $i\neq j$. The latter operators form a tensor with respect to the internal symmetry [O(N)]. In the case $N=3$, $\sum_i\phi_i^2$ has isospin 0, and $\phi_i\phi_j$ ($j\neq i$) has $I=2$. For convenience these operators will continue to be labeled $I=0$ and $I=2$ for any N. The anomalous dimensions of these operators, to order ϵ^2, are as follows:

$$d_A(\phi^2,I=0)=d-2+\frac{N+2}{N+8}\,\epsilon+\frac{N+2}{2(N+8)^3}\,\epsilon^2(13N+44), \tag{3.24}$$

$$d_A(\phi^2,I=2)=d-2+\frac{2\epsilon}{N+8}-\frac{\epsilon^2}{2(N+8)^3}\,(N^2-18N-88), \tag{3.25}$$

$$d_A(\phi\nabla^n\phi,I=0)=d-2+n+\frac{N+2}{2(N+8)}\,\epsilon^2\left[1-\frac{6}{n(n+1)}\right]\quad(n\text{ even}), \tag{3.26}$$

$$d_A(\phi\nabla^n\phi,I=2)=d-2+n+\frac{N+2}{2(N+8)^2}\,\epsilon^2\left[1-\frac{2(N+6)}{N+2}\,\frac{1}{n(n+1)}\right]\quad(n\text{ even}), \tag{3.27}$$

where $\phi\nabla^n\phi$ is the nth-rank tensor operator $\phi\nabla_{\alpha_1}\cdots\nabla_{\alpha_n}\phi$. [Only the traceless tensor part was considered, and pieces behaving like gradients such as $\nabla(\phi\nabla^{n-1}\phi)$ were removed. The resulting operator is guaranteed to have a unique dimension. Without these extractions the operator could be a sum of terms with different anomalous dimensions.] The terms $d-2$ for ϕ^2 and $d-2+n$ for $\phi\nabla^n\phi$ are the canonical dimension; the ϵ and ϵ^2 terms are the anomalies.

All the anomalies given above are small for any N except for ϕ^2 with $I=0$. The anomalies for the tensors are especially small, being at most twice the anomaly for d_ϕ (which occurs for $n\to\infty$). The anomaly for $\phi\nabla_\alpha\nabla_\beta\phi$, $I=0$, is 0. This is as expected, for the traceless tensor part of $\phi\nabla_\alpha\nabla_\beta\phi$ should be the traceless tensor part of the stress-energy tensor. The stress-energy tensor is required to have canonical dimensions, from symmetry arguments. The significance of these results was explained in the Introduction.

IV. EXISTENCE OF THE FUNCTION $u_0(\epsilon, N)$

The purpose of this section is to make plausible the basic assumptions of the ϵ expansion, namely the existence of a scale-invariant ϕ^4 theory with small coupling u_0 and the existence of the function $u_0(\epsilon, N)$. Two incomplete and unrigorous explanations of these assumptions will be offered, the first using the field equation and the other using the renormalization group. For further discussion see Refs. 1 and 7.

Consider the field equation of the ϕ^4 theory (for $N=1$) in the zero-mass limit,

$$\nabla_\mu\nabla^\mu\phi=-4\lambda_0\phi^3\,. \tag{4.1}$$

Consider first dimension 4. In the free-field zero-mass case $\nabla_\mu\nabla^\mu\phi$ and ϕ^3 have the same dimensions, and naively one would expect scale invariance to hold in the presence of interaction. However, the interaction changes the scaling properties of the operator ϕ^3: To order λ_0 the anomalous dimension of ϕ^3 is $\frac{3}{2}(d-2)+\frac{2}{9}\pi^2\lambda_0$. The dimension of $\nabla_\mu\nabla^\mu\phi$ does not change to order λ_0. Hence the dimensions of both sides of Eq. (4.1) do not match and scaling is broken. See Refs. 16 and 8.

One can restore the scale invariance by reducing d. Then, to first order in λ_0, the condition for matching anomalous dimensions of $\nabla_\mu\nabla^\mu\phi$ and ϕ^3 is

$$\tfrac{1}{2}(d+2)=\tfrac{3}{2}(d-2)+\tfrac{2}{9}\pi^2\lambda_0\,, \tag{4.2}$$

giving

$$d=4-\tfrac{2}{9}\pi^2\lambda_0 \tag{4.3}$$

or

$$\epsilon=\tfrac{2}{9}\pi^2\lambda_0\,. \tag{4.4}$$

So one might expect a scale-invariant solution with a specific coupling constant λ_0 of order ϵ.

In order that ϕ^3 have an anomalous dimension for $d<4$ one must give λ_0 some cutoff dependence, e.g., $\lambda_0=u_0\Lambda^\epsilon$, as discussed in Sec. II. Otherwise the field theory is super-renormalizable and ϕ^3 has canonical dimensions. The reason for this is simple. In the free-field theory for $d<4$, ϕ^3 has a smaller dimension than $\nabla_\mu\nabla^\mu\phi$. This means that short-distance matrix elements of $\phi^3(x)$ are much smaller than short-distance matrix elements of $\nabla_\mu\nabla^\mu\phi$ (not summed over μ, otherwise $\nabla_\mu\nabla^\mu\phi$ is zero). This means that adding $\lambda_0\phi^3$ to the field equation has little effect at short distances. If one instead adds $u_0\Lambda^\epsilon\phi^3$ to the field equation then the interaction is effective to distances of order Λ^{-1}; in the limit $\Lambda\to\infty$ the short-distance behavior is changed by the interaction and ϕ^3 has the anomalous dimension cited above (to order u_0).

Now the renormalization-group argument will be presented. It will be illustrated with specific formulas for N large, but should apply for any N. Suppose one can set up a renormalization-group equation for the ϕ^4 theory in $4-\epsilon$ dimensions analogous to the renormalization-group equation for electrodynamics in 4 dimensions.[3,4] This means one defines a renormalized coupling constant λ_k depending on a reference momentum k. For $k\sim\Lambda$, λ_k is essentially the bare constant λ_0; for $k\to 0$ λ_k becomes the renormalized coupling constant. The renormalization-group equation is a differential equation for λ_k. When $m\ll k\ll\Lambda$ (m is the renormalized mass) and $d=4$, the renormalization-group equation is

$$\frac{d\lambda_k}{d\ln k}=\psi(\lambda_k)\,. \tag{4.5}$$

For small λ_k,

$$\psi(\lambda_k)=+c\lambda_k{}^2\,, \tag{4.6}$$

where c is a positive constant. In particular ψ is independent of both Λ and m and depends on k only through λ_k. For $d=4-\epsilon$ with ϵ small the argument that ψ is mass- and cutoff-independent hopefully still applies. However, ψ cannot be independent of k, because λ_k itself has dimensions, the same as λ_0. The dimensions of λ_0 are (mass)$^\epsilon$; hence $k^{-\epsilon}\lambda_k$ is dimensionless. (In renormalized perturbation theory dimensions in the sense of dimensional analysis do not become anomalous; it is only the space-time dependence of operators which may show anomalous behavior.[4]) Hence to preserve dimensions the renormalization-group equation should read

$$\frac{d\lambda_k}{d\ln k}=k^\epsilon\psi(k^{-\epsilon}\lambda_k)\,. \tag{4.7}$$

To a first approximation $\psi(x)$ should still be cx^2 for small x and ϵ.

This result can be illustrated using the explicit solution of the theory for $N\to\infty$. One may define λ_k to be the four-point function $H_B(q_1,\ldots,q_4)$ for $(q+q_1)^2=-k^2$ ($q+q_1$ spacelike). Then

$$\lambda_k=\frac{8\lambda_0}{1+4N\lambda_0 v_B(-k^2)}\,. \tag{4.8}$$

From this one finds

$$\frac{d\lambda_k}{d\ln k^2} = \frac{-32N\lambda_0^{\,2}}{[1+4N\lambda_0 v_B(-k^2)]^2}\,\frac{dv_B(-k^2)}{d\ln k^2}$$

$$= -\lambda_k^{\,2}\tfrac{1}{2}N\,\frac{dv_B(-k^2)}{d\ln k^2}\,. \tag{4.9}$$

For mass zero one obtains

$$\frac{dv_B(-k^2)}{d\ln k^2} = -\tfrac{1}{2}\epsilon C_B(\epsilon)k^{-\epsilon}$$

$$\simeq \frac{-k^{-\epsilon}}{16\pi^2} \tag{4.10}$$

for small ϵ. Hence one may write

$$\frac{d\lambda_k}{d\ln k^2} = k^\epsilon \psi(k^{-\epsilon}\lambda_k) \tag{4.11}$$

for small ϵ, with

$$\psi(x) = Nx^2/32\pi^2\,. \tag{4.12}$$

Returning to the general renormalization-group analysis (arbitrary N), it is convenient to define a dimensionless coupling constant u_k by

$$u_k = k^{-\epsilon}\lambda_k\,. \tag{4.13}$$

The differential equation for u_k derived from Eq. (4.7) is

$$\frac{du_k}{d\ln k^2} = -\tfrac{1}{2}\epsilon u_k + \psi(u_k)\,. \tag{4.14}$$

For small u_k and ϵ this becomes

$$\frac{du_k}{d\ln k^2} = -\tfrac{1}{2}\epsilon u_k + c(N)u_k^{\,2}\,. \tag{4.15}$$

The equation for u_k has a special fixed-point solution for u_k, namely

$$u_k = u^* = \epsilon/2c(N)\,. \tag{4.16}$$

For small ϵ, u^* is also small, so it is legitimate to use the approximate equation (4.15) to calculate u^*. In four dimensions, a fixed-point solution defines a scale-invariant theory. This is also true in $4-\epsilon$ dimensions, as can be seen from the $N\to\infty$ case. For $N\to\infty$ we obtained a scale-invariant theory by letting $\lambda_0 = u_0\Lambda^\epsilon$ and then letting $\Lambda\to\infty$. In this limit (for small ϵ and $m=0$)

$$\lambda_k = \frac{16\pi^2\epsilon}{N}\,k^\epsilon\,, \tag{4.17}$$

which gives $u_k = u^*$ independent of k. So the scaling theory corresponds to the fixed-point solution for u_k.

Thus the renormalization-group argument predicts there will be a scale-invariant theory associated with a small dimensionless effective coupling constant $u_k = u^* \sim \epsilon$.

It remains to discuss the existence of the unique value $u_0 = u_0(\epsilon, N)$. Suppose one solves the approximate renormalization-group equation with $u_\Lambda = u_0$, with u_0 small but otherwise arbitrary. The solution is

$$u_k = \frac{u_0 k^{-\epsilon}}{\Lambda^{-\epsilon} + 2cu_0(k^{-\epsilon} - \Lambda^{-\epsilon})/\epsilon}\,. \tag{4.18}$$

In the limit $\Lambda\to\infty$, holding u_0 fixed, this gives $u_k = u^*$, so if one does an exact calculation the value of u_0 is immaterial. However, a sum of the perturbation expansion in u_0 of u_k will contain all the powers $k^{-\epsilon}$, $k^{-2\epsilon}$, $k^{-3\epsilon}$, etc., causing much confusion, unless

$$2cu_0\epsilon^{-1} = 1\,, \tag{4.19}$$

i.e., $u_0 = u^*$. At $u_0 = u^*$, u_k is the constant u^* also. So one chooses $u_0 = u^*$ to avoid the appearance of powers $k^{-\epsilon}$, $k^{-2\epsilon}$, etc. not connected with scale invariance.

In the actual calculation of Sec. III u_0 is not precisely u^*. The reason is that $\psi(u_k)$ has a cutoff dependence when $k\sim\Lambda$ and the true boundary condition is $u_k\to u_0$ for $k\to\infty$; hence the dependence of u_k for $k\ll\Lambda$ on u_0 is more complicated than described here. However, the function $u_0(\epsilon, N)$ still exists. See Ref. 7.

V. CORRECTIONS TO THE LARGE-N APPROXIMATION

In Sec. II it was shown that the anomalous dimension d_ϕ in $u_0\Lambda^\epsilon\phi^4$ theory is canonical in the limit $N\to\infty$. In this section the term of order $1/N$ in d_ϕ will be computed for any dimension d in the range $2<d<4$ $(0<\epsilon<\tau)$. The purpose of this section is to show that d_ϕ is anomalous in order $1/N$, and more generally to show how corrections to the $N\to\infty$ limit are calculated.[17]

First, one must determine the diagrams which contribute to the propagator in order $1/N$. The simplest procedure is to define an effective four-point vertex consisting of the sum of all bubbles as shown in Fig. 3(a), and then construct diagrams using the effective vertex in all graphs. Graphs using effective vertices are not permitted to have any explicit bubbles. Nor are one-line loops permitted [see Fig. 3(b)], since they are removed by mass renormalization. (Here, as in Sec. III, the mass m_F appearing in Feynman graphs is defined so that $D(0) = 1/m_F^{\,2}$.) Hence explicit loops must involve at least three lines. "Loops" in this context means loops involving a sum over an internal index. It is then easy to see that the only graph contributing in order $1/N$ to the propagator is the graph of Fig. 3(c), which has no loops. Graphs with three-line loops are of order $1/N^2$ or smaller.

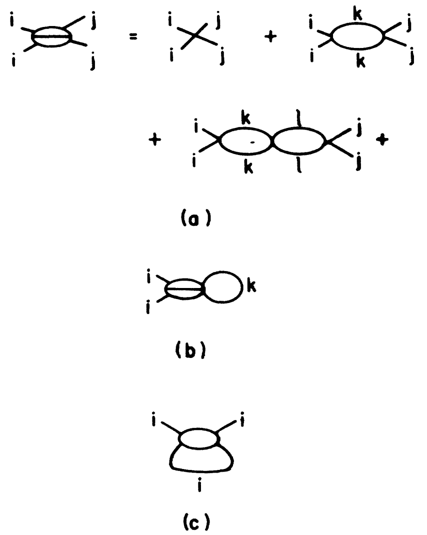

FIG. 3. (a) Definition of effective vertex (left-hand side) as sum of bubbles. The indices k and l are summed over. (b) Example of a one-line loop (k is summed over). (c) Graph determining $1/N$ correction to propagator. This graph has no loops.

The graph of Fig. 3(c) will be calculated in the Euclidean metric. The full propagator to order $1/N$ is found to be

$$D(q)=\frac{1}{q^2}\left\{1-\frac{1}{q^2}\int_k \frac{8u_0\Lambda^\epsilon}{1+4u_0\Lambda^\epsilon N v_B(k^2)}\times\left[\frac{1}{(k+q)^2+m^2}-\frac{1}{k^2+m^2}\right]\right\} \tag{5.1}$$

(remember that u_0 is assumed to be of order $1/N$). The subtraction $1/(k^2+m^2)$ is the required single subtraction at $q=0$. Let $m=0$ and let $q\ll\Lambda$. A straightforward analysis (for $0<\epsilon<2$) shows that the dominant part of the integral is a logarithmic integration coming from the region $q\ll k\ll\Lambda$. Considering only this range of k, the integral simplifies; one gets

$$D(q)\simeq\frac{1}{q^2}\left\{1-\frac{2}{q^2}\int_k \frac{k^\epsilon}{NC_B(\epsilon)}\times\left[-\frac{2k\cdot q}{k^4}+\frac{4(k\cdot q)^2}{k^6}-\frac{q^2}{k^6}\right]\right\}, \tag{5.2}$$

with the integral restricted to $q<k<\Lambda$. Since the radial part of $\int_k$ involves $k^{3-\epsilon}dk$, one sees that the q^2 terms inside the integral indeed involve logarithmic integrations (the term linear in q vanishes due to rotational symmetry).

Explicit calculation now gives

$$D(q)=\frac{1}{q^2}\left[1+\eta\ln\left(\frac{q}{\Lambda}\right)\right], \tag{5.3}$$

with

$$\eta=\frac{\epsilon^2}{4N}\,\frac{\sin\frac{1}{2}\epsilon\pi}{\frac{1}{2}\epsilon\pi}\,\frac{(4-\epsilon)\Gamma(3-\epsilon)}{\Gamma(3-\frac{1}{2}\epsilon)\Gamma(3-\frac{1}{2}\epsilon)}. \tag{5.4}$$

One can rewrite $D(q)$, to order $1/N$, as

$$D(q)=\frac{q^\eta\Lambda^{-\eta}}{q^2}, \tag{5.5}$$

which corresponds to an anomalous dimension

$$d_\phi=\tfrac{1}{2}(d-2+\eta). \tag{5.6}$$

To renormalize $D(q)$ it is now necessary to perform a wave-function renormalization.

It is easily verified that the large-N behavior of Eq. (3.23) agrees, to order $1/N$ and ϵ^3, with Eqs. (5.4) and (5.6).

VI. CONCLUSION

The importance of the ϵ expansion and $N\to\infty$ limit is that it provides a large class of models whose true short-distance behavior is both calculable and nontrivial. Previously this was true only of the Thirring model. In four-dimensional perturbation theory for renormalizable theories like ϕ^4 theory or quantum electrodynamics one cannot as yet calculate the true short-distance behavior because too many graphs are involved.

It is unfortunate that the models introduced here become trivial in four dimensions. However, it seems likely that a thorough study of these models in less than four dimensions will generate new ideas about the nature of field theory that do not depend on dimensionality and may apply to four-dimensional theories as well. It should be instructive to study the behavior of high-energy scattering, deep-inelastic scattering, bound states, etc. in these models.

Note added in proof. Included in the models treated in this paper are some true field theories in 2 space and 1 time dimension, namely, the ϕ^4 and $(\bar\psi\psi)^2$ theories for large N. The discussion of theories with noninteger d was emphasized because it is so instructive. See also recent papers by Schroer, Mitter, and Brezin, LeGuillou, and Zinn-Justin.[18]

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APPENDIX: SPACE WITH NONINTEGRAL DIMENSION d

In this Appendix a Euclidean space of nonintegral dimension d will be discussed. A Lorentz space-time of dimension d is equivalent to one time dimension (as usual) combined with a Euclidean space of dimension $d-1$.

First, integration will be defined for nonintegral d.[5,6] Second, γ matrices will be discussed for nonintegral d.[5] Finally, the space itself will be considered.

Three simple principles plus a normalization condition *uniquely* define the integral $\int d^d p$ for any d, real or complex. The first principle is linearity:

$$\int d^d p[af_1(p)+bf_2(p)] = a\int d^d p f_1(p) + b\int d^d p f_2(p), \tag{A1}$$

where f_1 and f_2 are aribtrary functions and a and b are arbitrary constants. The second principle is invariance to translations:

$$\int d^d p f(p+q) = \int d^d p f(p), \tag{A2}$$

where f is an arbitrary function and q an arbitrary fixed vector. The third principle is a scaling law:

$$\int d^d p f(sp) = s^{-d}\int d^d p f(p), \tag{A3}$$

where f is again arbitrary, and s is an arbitrary positive constant. In other words, volumes in momentum space scale as (momentum)^d.

There will be functions f which are not integrable in d dimensions, due to bad behavior as $|p|\to\infty$, for example. The problem of delineating the class of functions which are integrable and therefore actually obey these principles will be left to the experts in such matters.

To see that the three principles uniquely define $\int d^d p$ except for a normalization constant, one examines the integral of a generating function. Namely, consider the integral

$$I(s,q) = \int d^d p\, e^{-sp^2+p\cdot q}. \tag{A4}$$

A translation gives

$$I(s,q) = \int d^d p e^{-sp^2+q^2/4s}. \tag{A5}$$

A scale transformation gives

$$I(s,q) = s^{-d/2}e^{q^2/4s}\int d^d p\, e^{-p^2} \tag{A6}$$

The remaining integral is independent of s and q. One can generate any function of the form $f(p^2, p\cdot q_1, \ldots, p\cdot q_n)$ depending on any n vectors q_i besides p by using differentiation and summation with respect to s and q of the generating function $e^{-sp^2+p\cdot q}$. For example: Differentiating m times with respect to s gives the function $(p^2)^m\, e^{-sp^2+p\cdot q}$; summing over m, one can produce any function of p^2 times $e^{-sp^2+p\cdot q}$. Replacing q by $s_1q_1+\cdots+s_nq_n$, where the s_i are scalars, and differentiating with respect to $s_1, \ldots, s_n$ allows one to build up polynomials in $p\cdot q_1, \ldots, p\cdot q_n$. These can then be summed to give arbitrary functions. Convergence problems are left for the experts to contemplate. To set the normalization, note that

$$\int d^d p\, e^{-p^2} = \pi^{d/2} \tag{A7}$$

for any integer d; it is natural (but also arbitrary) to use this normalization for nonintegral d. In all the applications of this paper, changing this normalization condition can always be compensated for by a renormalization of the coupling constant, and does not affect results such as the anomalous dimensions.

Thus

$$I(s,q^2) = s^{-d/2}e^{q^2/4s}\pi^{d/2}. \tag{A8}$$

This is an analytic function of d, so one expects the integrals of all reasonable functions generated from I to be analytic in d also; expanding in powers of $\epsilon = 4-d$ poses no difficulties.

An arbitrary graph involving only scalar-particle propagators can be reduced to Gaussian momentum integrals by Schwinger's trick (see Sec. II). There remains the problem of spin. This has been discussed in detail by 't Hooft and Veltman.[5] The problem that arises is that the ϵ tensor $\epsilon_{\mu\nu\pi\sigma}$ has no known continuation to nonintegral d, and likewise for γ_5. Continuation is possible if one considers a theory without γ_5 in the interaction (e.g., electrodynamics or a scalar Yukawa interaction), provided one also discusses only scalar invariant functions. In this case γ matrices occur only inside traces, and no γ_5's or ϵ tensors arise.

Consider a trace of a product of γ matrices not involving γ_5. The following rules completely determine any such trace[5]:

(1) The trace of an odd number of γ matrixes is 0.

(2) The trace operation is cyclic, e.g., $\mathrm{Tr}ABC = \mathrm{Tr}CAB$.

(3) The anticommutation rule $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ holds.

(4) $\mathrm{Tr}I = 2^{d/2}$.

These rules are satisfied by the irreducible representation of the γ matrix algebra for even d. For odd d, rules (1) and (4) are incorrect for the irreducible representation[5]: For odd d the trace of $d\,\gamma$ matrices gives the ϵ tensor and TrI is $2^{d/2-1}$.

The definition (4) of TrI when d is nonintegral is arbitrary. Different choices of TrI, for given d, will in general define different theories. The reason for this is that a single trace over an internal loop can involve two, three, four, or more coupling constants, so one cannot compensate a redefinition of the trace by a rescaling of the coupling constant. However, only a two-line loop occurs in Sec. II, so the value of TrI is irrelevant in Sec. II.

This completes the definitions of nonintegral d needed for the text. Now the basic questions must be discussed. What is a space of noninteger d? What is an integral in this space?

First, it will be useful to have $\int d^d p$ expressed directly as an integral. First consider an integrand $f(p^2)$. One obtains

$$\frac{1}{(2\pi)^d}\int d^d p\, f(p^2)=K_d\int_0^\infty p^{d-1}f(p^2)dp\,, \tag{A9}$$

where

$$K_d=\frac{2}{(4\pi)^{d/2}\Gamma(\frac{1}{2}d)} \tag{A10}$$

[as in Eq. (3.14)]. The proof of this is that

$$(2\pi)^d K_d\int_0^\infty p^{d-1}e^{-sp^2}dp=\pi^{d/2}s^{-d/2}\,, \tag{A11}$$

in agreement with the generating function [see Eq. (A8)]. For Re$d\le 0$ the formula (A9) is still basically correct, except one must introduce subtractions to make the integral converge. The subtracted formula is

$$\frac{1}{(2\pi)^d}\int d^d p\, f(p^2)=K_d\int_0^\infty p^{d-1}\Big[f(p^2)-f(0)-p^2f'(0)-\cdots-\frac{p^{2l}}{l!}f^{(l)}(0)\Big]dp\,,$$

where l is the largest integer smaller than $\frac{1}{2}(-\text{Re}d)$. This formula is well defined provided Re$d\neq 0, -2, -4$, etc., and provided $f(p^2)$ is differentiable $l+1$ times at $p^2=0$. This formula, with $f(p^2)=e^{-sp^2}$, still agrees with the generating functional. The case Re$d=-2l$, Im$d\neq 0$ is best handled by a limiting procedure; it is unimportant for applications in this paper. The case $d=0, -2, -4$, etc. is interesting. In this case the integral (A12) collapses to the form

$$\int d^{-2l}p\, f(p^2)=\pi^{-l}f^{(l)}(0)\,. \tag{A13}$$

The cause of the collapse is that as $d\to -2l$, from above, the integral becomes divergent at $p=0$ but $K_d\to 0$.

Now consider the more general integrand $f(p^2, p\cdot q_1,\ldots,p\cdot q_n)$. One proceeds as follows. Introduce n coordinates such that the n coordinate axes span the space generated by $q_1,\ldots,q_n$ (even if $n>d$; see later for further discussion). Then separate p into the n components $p_1,\ldots,p_n$ parallel to these axes and $p_\perp$ consisting of the remainder of p. Then $p\cdot q_1,\ldots,p\cdot q_n$ all depend only on $p_1,\ldots,p_n$, while p^2 is

$$p^2=p_1{}^2+\cdots+p_n{}^2+p_\perp{}^2\,. \tag{A14}$$

Then

$$\int d^d p\, f(p^2,p\cdot q_1,\ldots,p\cdot q_n)=\int_{-\infty}^\infty dp_1\cdots\int_{-\infty}^\infty dp_n\int d^{d-n}p_\perp f(p^2,p\cdot q_1,\ldots,p\cdot q_n)\,, \tag{A15}$$

where $\int d^{d-n}p_\perp$ is as defined previously for a function of $p_\perp{}^2$ only. It is easily verified that this formula agrees with the generating functional.

If Re$(d-n)>0$ it is straightforward to convert this integral to spherical coordinates. If Re$(d-n)<0$ the subtractions in the definition of $\int d^{d-n}p_\perp$ might cause troubles in such a conversion. For $n=1$, the integral in polar coordinates is

$$\int d^d p\, f(p^2,p\cdot q)\,(2\pi)^{-d}=K_{d-1}\int_0^\infty p^{d-1}dp\int_{-1}^1(\sin\theta)^{d-2}d\theta\, f(p^2,pq\cos\theta)\,, \tag{A16}$$

valid for Re$d>1$.

What is a d-dimensional space? A vector p in this space has an infinite number of components. Only for integral d can one restrict the number of components to be d. To see this restriction, let d be a positive integer; suppose one is given d linearly independent vectors q_l. Now the integral $\int d^d p$ is replaced by $\int dp_1\cdots\int dp_d\int d^0 p_\perp$. The explicit formula for $\int d^0p_\perp$ (A13) means that $p_\perp=0$, so p also has four components. This is the way the number of components of all vectors is restricted to be d, for integer d. This mechanism obviously fails for nonintegral d.

So what one has is an infinite-dimensional space with an integral $\int d^d p$ defined for functions of scalar products in this space.The integrals imitates the scaling properties of a d-dimensional space because of the requirement (A3). There are no convergence requirements in the space in the sense that one never explicitly specifies the complete set of components $\{p_i\}$ of p, and one does not demand that $\sum_i p_i^2$ converge. One can only separate p into a finite set of components $p_1, \ldots, p_n$ and a perpendicular vector $p_\perp$.

Note that $\int d^d p$ is not a positive operation, due to the subtractions in $\int d^{d-n} p_\perp$ when $n > d$.

There are other ways in which the space imitates a d-dimensional space. For example,

$$\int d^d p (p^2 e^{-p^2}) = d \int d^d p \, p_1{}^2 e^{-p^2}, \tag{A17}$$

where p_1 is a component of p.

A more difficult question is whether one can define field operators $\phi(x)$ and $\psi(x)$ themselves for noninteger d. I do not know. In statistical mechanics another question arises: Can one define crystal lattices in a d-dimensional space? I do not know, although some lattice sums can be continued to noninteger d.[19]

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