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Surface Geometry of Charged Rotating Black Holes*

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Invariant measures of the surface geometry of a charged rotating (Kerr-Newman) black hole are examined. It is shown that as the rotation rate of the black hole increases, the equatorial circumference increases while the polar circumference decreases. This is analogous to effects in material rotating bodies. The number of parameters describing a charged Kerr black hole drops from three to two on its surface. It is found that a scale parameter η and a distortion parameter β describe this geometry very simply. There emerge two classes of Kerr metrics separated by $\beta = \frac{1}{2}$. For larger β the Gaussian curvature becomes negative on two polar-cap regions and the surface cannot be globally embedded in Euclidean 3-space. Possible physical effects are briefly discussed.

INTRODUCTION

There is considerable current interest in the physics of black holes.[1] Much of the insight gained has come from a detailed study of the Kerr-Newman[2,3] family of solutions to Einstein's gravitational field equation. These metrics represent a charged rotating black hole.

When a material body rotates, its surface deforms; there is the characteristic flattening of the poles and lengthening of the equatorial circumference. This paper shows in what way similar effects occur in black holes and in which way modifications arise due to the highly curved space-time.

I. THE SURFACE OF A BLACK HOLE

If there is one characteristic feature that defines a black hole, it is the existence of an event horizon.[4] That is, there exists a null hypersurface which is the boundary of the set of points that can be connected to null infinity by a causal curve. Without this "one-way membrane" there may occur a naked singularity with properties quite different from those of a black hole.[1]

When one takes a sequence of spacelike slices through this null hypersurface, one obtains a family of 2-geometries which, if closed, represents the surface of the black hole evolving in time. In the generic dynamical situation these surfaces will change their geometry depending on the particular slicing of the event horizon utilized.

In the special case of stationary one-black-hole metrics the charged Kerr metrics play an important role. The 2-geometry of the surface of such a black hole is independent of which spacelike slice of the event horizon one takes. Thus, for this class of black holes one can speak of *the* surface without having to specify the slicing.

Using the coordinate system of Kerr and Newman[3,5] the charged Kerr metric may be expressed as

$$\begin{aligned} ds^2 = &-[1-(2mr-e^2)\Sigma^{-1}]\,du^2 + 2dr\,du \\ &- 2a(2mr-e^2)\Sigma^{-1}\sin^2\theta\, d\phi\, du \\ &- 2a\sin^2\theta\, dr\, d\phi + \Sigma d\theta^2 \\ &+ [(r^2+a^2)^2 - \Delta a^2\sin^2\theta]\,\Sigma^{-1}\sin^2\theta\, d\phi^2, \end{aligned} \qquad (1)$$

$$\Sigma := r^2 + a^2\cos^2\theta, \quad \Delta := r^2 - 2mr + a^2 + e^2,$$

where the notation := indicates a definition.

The three parameters (m, a, e) represent, respectively, the total mass, angular momentum per unit mass, and the charge of the black hole. There are two symmetries, given by the Killing vectors:

$$T = \frac{\partial}{\partial u}, \quad \Phi = \frac{\partial}{\partial \phi}. \tag{2}$$

The surface on which the T Killing vector becomes null has been called the "stationary limit"[6]; the region between that and the event horizon is known as the "ergosphere".[7] The event horizon is a null hypersurface given by[8] $r = r_+$, where r_+ is a root of $\Delta = 0$:

$$r_+ := m + (m^2 - a^2 - e^2)^{1/2}. \tag{3}$$

Those intrinsic surfaces which occur inside r_+ will not be treated in this paper.[9]

II. KILLING HORIZONS

Carter[10] has shown that the one-way membrane for the charged Kerr metric is a *Killing* horizon. This means that the null hypersurface r_+ is invariant under the 2-dimensional Abelian group of isometries: (T translations) + (Φ rotations). Furthermore, the null generators of the hypersurface lie along trajectories of *one* Killing vector field of the group, namely,[11]

$$\xi := (r_+{}^2 + a^2)T + a\Phi. \tag{4}$$

Moreover, for $0 < a^2 + e^2 < m^2$ the event horizon r_+ is a *bifurcate* Killing horizon.[12] That is to say, the one-way membrane is composed of two Killing horizons which intersect in a compact, spacelike, totally geodesic, 2-surface S of fixed points of ξ. To see that this is the case, one shows that ξ is a zero vector in the tangent space at each point of S. Using a coordinate system which is well defined on S, such as the one used by Boyer and Lindquist,[8] one writes ξ as

$$\xi = (m^2 - a^2 - e^2)^{1/2}\left(v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\right). \tag{5}$$

Here S is defined by $u = v = 0$. Clearly, $\xi(p) = 0$ for any p on S.

Since there is now a geometrically selected 2-surface in r_+, one would like to know how it is related to any other spacelike 2-surface obtained by slicing the one-way membrane. The answer is: *All 2-dimensional spacelike slices through a 3-dimensional Killing horizon are isometric.*[13]

To see this, consider a Killing vector field ξ and a null hypersurface N such that ξ is a null generator on N. Choose coordinates x^1, x^2, x^3, x^4, with $\xi^a = \delta^a_4$. This restriction is invariant under the coordinate transformations $x^A \to f^A(x^1, x^2, x^3)$, where $A = 1, 2, 3$. By using one of these transformations, the equation for N may be made $x^3 = 0$.

Take x^1, x^2, x^4 as coordinates on N. Considering that $\partial/\partial x^4$ is null on N and orthogonal to N, the induced line element on N is given by ${}^{(3)}ds^2 = g_{\lambda\mu}dx^\lambda dx^\mu$ ($\lambda, \mu = 1, 2$). Here $g_{\lambda\mu}$ is independent of x^3 because we are on a hypersurface of constant x^3 and independent of x^4 because $\partial/\partial x^4$ is a Killing vector. Now, a 2-slice is defined by $x^4 = F(x^1, x^2)$. The line element on any such 2-slice is given by ${}^{(2)}ds^2 = g_{\lambda\mu}dx^\lambda dx^\mu$ which is manifestly independent of the choice of F, that is to say: All 2-slices are isometric.

Therefore, the geometry of the one-way membrane is determined by the Killing vector ξ and the 2-surface S. One obtains a coordinate representation of this geometry by using the maximal analytic extension[5] which explicitly covers S. Introducing an orthonormal dyad, one has

$${}^{(2)}ds^2 = (\omega_2)^2 + (\omega_3)^2, \tag{6}$$

$$\begin{aligned} \omega_2 &:= (r_+{}^2 + a^2\cos^2\theta)^{1/2} d\theta, \\ \omega_3 &:= (r_+{}^2 + a^2)(r_+{}^2 + a^2\cos^2\theta)^{-1/2}\sin\theta\, d^+\phi, \\ d^+\phi &:= d\phi - \omega dt, \\ \omega &:= a(r_+{}^2 + a^2)^{-1}. \end{aligned} \tag{7}$$

Henceforth, by a simple coordinate transformation, the + in $d^+\phi$ will be dropped.

Finally, one recalls that S is a marginally trapped surface,[10] i.e., a compact, spacelike, 2-surface such that the convergence of the ingoing and outgoing null geodesic normals vanishes. For S these null normals are parallel to the two repeated principal null directions of the Weyl tensor, which have vanishing expansion, shear, and twist on S.

III. THE SCALE AND DISTORTION PARAMETERS

It is convenient, in the investigation of the geometry intrinsic to the surface of a charged Kerr black hole, to introduce a pair of new parameters: *the scale parameter* η,

$$\eta := (r_+{}^2 + a^2)^{1/2}, \tag{8}$$

and the *distortion parameter* β,

$$\beta := a(r_+{}^2 + a^2)^{-1/2}. \tag{9}$$

The 2-metric [Eq. (6)] then takes the following simple form:

$$\begin{aligned} \omega_2 &:= \eta(1 - \beta^2\sin^2\theta)^{1/2} d\theta, \\ \omega_3 &:= \eta(1 - \beta^2\sin^2\theta)^{-1/2}\sin\theta\, d\phi. \end{aligned} \tag{10}$$

There is a degeneracy in the 3-parameter Kerr-Newman family on the event horizon, since the 2-metric there depends on only 2 parameters. That is, given a set of values for (m, a, e) one has a value for η and β from above. However, as a result of the degeneracy, a set of values for η and β determine only a uniquely while an algebraic expression relates m and e:

$$a=\eta\beta\,,\qquad m=\tfrac{1}{2}\eta(1-\beta^2)^{-1/2}(1+e^2/\eta^2)\,. \tag{11}$$

This can be understood by recalling what happens in the Reissner-Nordström solution ($\beta=0$). If η is held fixed then one can add charge to a Schwarzschild solution until $m^2=e^2$ *while maintaining the spherical symmetry of the surface of the black hole*. Likewise for any allowable β and η, one has a certain intrinsic geometry on the surface specified by the 2-metric in Eq. (10). A whole family of charged black holes *with this identical intrinsic surface geometry* exist and are given by Eqs. (11).

Once β and η are chosen, there is an upper limit on e fixed by the requirement that, if an event horizon is to exist, r_+ must be real:

$$m^2 \geq a^2+e^2\,. \tag{12}$$

Accordingly, one finds the range of charge and total mass energy for the family of black holes associated with the given β and η:

$$0\leq e\leq \eta(1-2\beta^2)^{1/2}\,,\qquad \tfrac{1}{2}\eta(1-\beta^2)^{-1/2}\leq m\leq \eta(1-\beta^2)^{1/2}\,. \tag{13}$$

The black holes which satisfy the *equality* in Eq. (12) will be called *extreme* charged Kerr black holes. The allowed families of charged Kerr metrics are illustrated in Fig. 1. One notices, incidentally, that the maximum value β can attain requires the charge to be zero;

$$\beta_{\max}=1/\sqrt{2}\simeq 0.707\,. \tag{14}$$

When this limit is exceeded [Eq. (12)] a naked singularity occurs.

IV. SURFACE AREA AND REVERSIBLE TRANSFORMATIONS

The surface area of a black hole is an invariant property of some interest.[4,14] Using Eq. (10) one calculates

$$A:=\int \omega^2\wedge\omega^3=4\pi\eta^2\,, \tag{15}$$

where $\wedge$ denotes the wedge product of differential forms. Notice that although the event horizon occurs at

$$r=r_+=\eta(1-\beta^2)^{1/2}\,, \tag{16}$$

the area of the 2-surfaces in the horizon depends only on η.

Now Hawking's general theorem[4] that the total surface area of black holes can never decrease may be applied to the case of one charged Kerr hole. From an investigation of the problem Christodoulou[14] has shown that there are three contributions to the total mass-energy of a charged rotating black hole. Two of these can be removed completely: the energy due to the charge e and that due to the angular momentum ma. The mass-energy of the Schwarzschild hole which is left is defined as the *irreducible mass* m_{ir}. As can be seen from Eq. (11) this means[14]

$$m_{ir}=\tfrac{1}{2}\eta\,,\qquad A=16\pi m_{\text{ir}}{}^2\,. \tag{17}$$

Christodoulou's statement of Hawking's theorem as applied to charged Kerr metrics is this: There is no physical process which can decrease the irreducible mass of a black hole. Those processes in which m_{ir} (or η), and hence the surface area A, remain unchanged are termed reversible transformations.[14]

Going back to Fig. 1 one understands the signifi-

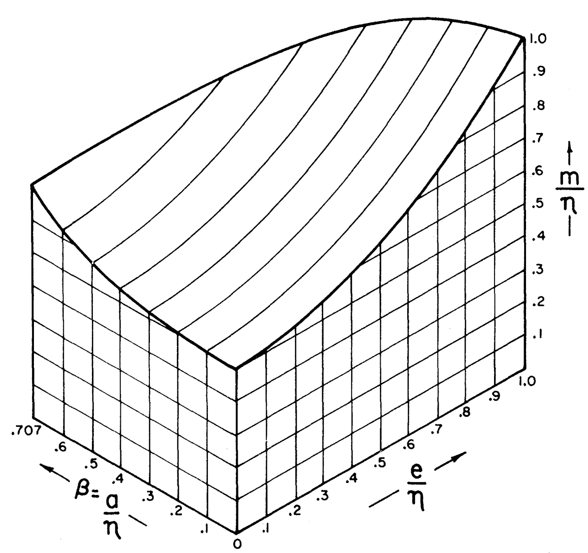

FIG. 1. This graph represents all those charged Kerr black holes which can be obtained from each other by a reversible transformation. Since this requires that η be fixed, we have a 2-surface in (m,a,e) space. The points in the boundary of this 2-surface defined by $\beta=0$ represent the Reissner-Nordström metrics; the points defined by $e=0$ represent uncharged Kerr metrics; the back boundary consists of points which represent extreme charged Kerr metrics. The charged black holes which lie along a line of constant β have identical intrinsic geometry on their horizons.

cance of the 2-surface in (m, a, e) space created by holding η fixed. All the black holes represented by points on this surface can be obtained from one another by reversible transformations.[15] Those black holes with common β but varying m and e have the same intrinsic geometry on their horizons as determined by Eq. (11). What will be considered in the following sections are ways of measuring what this geometry is.

V. CIRCUMFERENCES

To obtain a gross measure of the surface deformation one may compare the equatorial circumference c_e and the polar circumference c_p, these being defined as the proper length of the curves $\theta = \frac{1}{2}\pi$ and $\phi = 0$, respectively. Calculation yields

$$c_e := \int \omega_3 = 2\pi\eta(1-\beta^2)^{-1/2}, \tag{18}$$

$$c_p := \int \omega_2 = 4\eta E(\beta), \tag{19}$$

the last being a complete elliptical integral of the second kind. These circumferences are invariant since the curves are geodesics of the 2-metric.

In Fig. 2 there is a graph of c_e and c_p versus β in a reversible transformation. One notices that as angular momentum is added (β increases) the equatorial circumference increases and the polar circumference decreases. Another measure of this departure from spherical symmetry is plotted in Fig. 3, namely,

$$\delta = (c_e - c_p)/c_e. \tag{20}$$

These are some of the effects one would expect from the analogy with rotating material bodies.

The result for the equatorial circumference [Eq. (18)] recalls the identical result in Minkowski space when one is in a rotating reference system.[16] This suggests that one consider the distortion parameter β as being the product of the angular velocity of the black hole with its "effective" radius. In fact, using the definition of angular velocity of the surface of a black hole as proposed by Christodoulou[14] and Bardeen[17] [ω in Eq. (7)] one finds:

$$\beta = \omega\eta. \tag{21}$$

Defining the tangential velocity of the surface of a rotating black hole is a more delicate matter since there are no timelike world lines in the horizon for observers to travel on. Christodoulou[14] has proposed that the equatorial surface velocity of a black hole be taken as

$$v := \sqrt{g_{33}}\,\omega = \beta(1-\beta^2)^{-1/2}. \tag{22}$$

The definition has the attractive feature that as the extreme Kerr hole is approached ($\beta \to 1/\sqrt{2}$) the velocity tends to the velocity of light ($v \to 1$). From this viewpoint one can better understand why the surface disappears altogether for $\beta > \beta_{max}$.

By now, one can see that although the horizon occurs at $r = r_+ =$ constant, the surface is not metrically a 2-sphere. To gain more insight into the true intrinsic nature of the surface one must look locally.

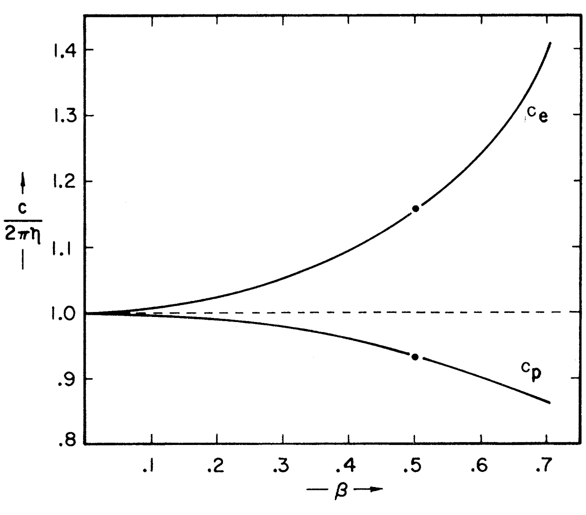

FIG. 2. The proper equatorial circumference c_e ($\theta = \frac{1}{2}\pi$) and the proper polar circumference c_p ($\phi = 0$) of a charged Kerr black hole are plotted here versus β in a reversible transformation. One sees the bulging of the equator and flattening of the poles caused by the rotation. Recall that the maximum value β may obtain is determined by the value of e. The dots on the curves represent the extreme Kerr hole when $e = \eta/\sqrt{2}$ ($\beta = \frac{1}{2}$).

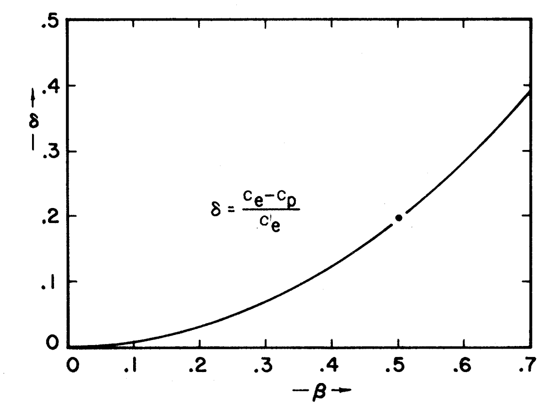

FIG. 3. Here is plotted another measure of the surface distortion on a black hole caused by rotation. The value of $\delta = (c_e - c_p)/c_e$ specifies how "oblate" the surface is becoming. The surface is not globally embeddible in Euclidean 3-space if $\beta > \frac{1}{2}$.

VI. GAUSSIAN CURVATURE

One would like an isometrically invariant local measure of the intrinsic deviation of the horizon from sphericity. Gauss's *theorema egregium*[18] provides such a measure – the Gaussian curvature K. This quantity measures the geometry intrinsic to the horizon itself and is independent of the embedding space. One recalls that a sphere of surface area $4\pi r^2$ has $K=1/r^2$, which is constant and positive. A plane has $K=0$, while a pseudosphere has $K=-1/r^2$.

In the Appendix the Gaussian curvature is calculated for the 2-surface of a charged Kerr black hole. One finds that

$$K(m,a,e,\theta)=(r_+{}^2+a^2)(r_+{}^2-3a^2\cos^2\theta)\times(r_+{}^2+a^2\cos^2\theta)^{-3}, \tag{23}$$

$$K(\eta,\beta,\theta)=\eta^{-2}[1-\beta^2(1+3\cos^2\theta)](1-\beta^2\sin^2\theta)^{-3}.$$

In particular, for the spherically symmetric Schwarzschild ($\beta=e=0$) and for the Reissner-Nordström ($\beta=0$) black holes one has, as expected for a sphere,

$$K=1/\eta^2=1/r_+{}^2. \tag{24}$$

However, K is a function of polar angle θ if the black hole is rotating ($\beta\neq 0$). When $\beta=\frac{1}{2}$ ($a=m_{\text{ir}}$), the Gaussian curvature becomes zero at the poles ($\theta=0$). This occurs in the uncharged case when $a=\sqrt{3}\,m/2$.[19] For $\frac{1}{2}<\beta\leq 1/\sqrt{2}$, there are two "polar caps" of negative Gaussian curvature on the surface.

Thus, there are two geometrically distinct classes of charged Kerr black holes depending on whether $\beta\lessgtr\frac{1}{2}$. The first class consists of black holes whose surfaces are like oblately deformed spheres which we are familiar with in Euclidean 3-space. They have everywhere positive Gaussian curvature. The second class is unlike any surface one can envision in our familiar 3-space. This is because there are regions of negative Gaussian curvature both *on* and around the axis of symmetry. The surface most resembles a hybrid sphere and pseudosphere.

The second class is nonempty only if the charge is small enough to allow $\beta>\frac{1}{2}$, i.e., only if $e<\eta/\sqrt{2}$ [Eq. (13)]. However, since Wald[20] has argued that $e\cong 0$ for black holes formed from collapse and Bardeen[21] has suggested that $a\cong m$ for black holes at the center of galaxies, the second class is likely to be of the most astrophysical interest.

One can use the general formula for the Gaussian curvature [Eq. (23)] to check the topology of the surface. If one integrates $K(\theta)$ and applies the Gauss-Bonnet theorem[22] one finds (see Appendix)

$$\int K(\eta,\beta,\theta)\omega^2\wedge\omega^3=4\pi=2\pi\chi, \tag{25}$$

where χ is the Euler characteristic of the surface. This tells us that the horizons are *topologically* 2-spheres.

VII. EMBEDDING

To visualize the intrinsic geometry of a black hole (as opposed to its appearance in space[23]), one may attempt to embed the surface isometrically[24] in our familiar Euclidean 3-space E^3. First, it is convenient to rewrite the metric by introducing a new coordinate μ and a metric function $f(\mu)$ defined by

$$\mu:=\cos\theta,$$

$$ds^2=\eta^2[f^{-1}(\mu)d\mu^2+f(\mu)d\phi^2], \tag{A1}$$

$$f(\mu):=(1-\mu^2)[1-\beta^2(1-\mu^2)]^{-1}, \tag{A2}$$

$$f':=\frac{df}{d\mu}.$$

From the Appendix, the isometric embedding map from $(\mu,\phi)\to(x,y,z)$ is given by

$$x=F(\mu)\cos\phi,\quad y=F(\mu)\sin\phi,\quad z=G(\mu), \tag{A8}$$

$$F(\mu)=\eta f^{1/2},\quad G(\mu)=\eta\int d\mu[f^{-1}(1-\tfrac{1}{4}f'^2)]^{1/2}. \tag{A11}$$

The condition for global embedding in E^3 is that the radicand in the integral for G be nonnegative definite. Since f is always nonnegative this implies

$$|f'|\leq 2. \tag{26}$$

But, $|f'|=2$ at the poles $\mu=\pm 1$. Therefore, the above condition for embedding is equivalent to

$$(f')'=f''(1)<0. \tag{27}$$

Using Eq. (A3), this means the Gaussian curvature K must be positive at the poles. Thus, *the surface of a charged Kerr black hole cannot be globally embedded in E^3 if $\beta>\frac{1}{2}$.*

By allowing z to become imaginary when $|f'|>2$, part of the surface (the polar caps) become embedded in a pseudo-Euclidean (PE^3) space with metric $ds^2=dx^2+dy^2-dz^2$ and part of the surface (centered on the equator) remains embedded in E^3, where the metric is $ds^2=dx^2+dy^2+dz^2$.

To obtain embedding diagrams one has to integrate numerically for G. The result is summarized in Fig. 4. The three black holes all are related by a reversible transformation. The sphere is a Schwarzschild hole ($\beta=0$). As β is increased

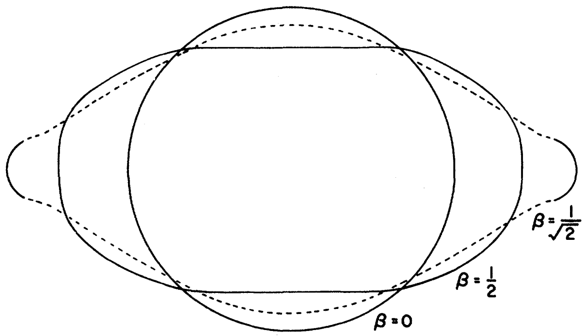

FIG. 4. By applying a series of reversible transformations, one obtains a sequence of embedding diagrams for the surface of the black hole. The sphere represents a Schwarzschild hole ($\beta = 0$). As β is increased, the embedded surface becomes more oblate. When $\beta = \frac{1}{2}$, the curvature at the poles becomes flat ($K = 0$). Beyond this, a global embedding in Euclidean 3-space is impossible. The polar caps are embedded in a pseudo-Euclidean 3-space (dotted lines). If the hole is uncharged, the extreme Kerr black hole occurs for $\beta = 1/\sqrt{2}$.

it distorts as expected for a rotating object until $\beta = \frac{1}{2}$. Thereafter, part of the surface must be embedded in a pseudoeuclidean space (dotted line) and part in Euclidean space (undotted line). If the black hole is uncharged, the last surface represents an extreme black hole with $\beta = 1/\sqrt{2}$.

One must remember that these embedding diagrams distort the "shape" of the black hole, i.e., the extrinsic curvature, while preserving the intrinsic geometry. Because the extrinsic curvature depends on the embedding space, a detailed analysis has been carried out only for the intrinsic geometry. An analysis of the former problem is now underway.

CONCLUSION

It is found that a set of new parameters, the scale parameter η and the distortion parameter β, enable one to see very clearly the effects of rotation on black holes. One striking feature which emerges is that Kerr metrics break into two classes separated by $\beta = \frac{1}{2}$. An open question is whether this geometric division has physical consequences. Does instability develop or could this be a bifurcation point for the Kerr solution[25]?

A clearer understanding of the surface geometry of the charged Kerr family of black holes will shed light on other problems in the physics of black holes. For instance, Bekenstein has introduced the concept of the surface tension of a black hole.[26] The models for black-hole vibration frequencies developed so far have assumed a spherical surface. How will the deviation from sphericity found here effect these results? Also, a better understanding of the intrinsic and extrinsic geometry of black-hole surfaces will lead to keener insight into the problem of finding an internal solution.

Finally, one will wish to study how many of the properties of the Kerr horizon carry over to nonstationary solutions. Is the analogy of a black-hole surface with a physical membrane a stable one? And if so, can one use the invariant quantities associated with such a surface to gain insight into astrophysical processes?

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APPENDIX

The calculation of the Gaussian curvature and the embedding formulas are given here. The metric is written in a standard form by the change of coordinates $\mu = \cos\theta$:

$$ds^2 = \eta^2[f^{-1}(\mu)d\mu^2 + f(\mu)d\phi^2], \tag{A1}$$

$$f(\mu) := (1-\mu^2)[1-\beta^2(1-\mu^2)]^{-1}. \tag{A2}$$

The Gaussian curvature for a metric of the above form is given by[27]

$$K(\mu) = -\tfrac{1}{2}f''(\mu)\eta^{-2}. \tag{A3}$$

The derivatives of $f(\mu)$ are

$$\begin{aligned} f'(\mu) &= -2\mu[1-\beta^2(1-\mu^2)]^{-2}, \\ f''(\mu) &= -2[1-\beta^2(1+3\mu^2)][1-\beta^2(1-\mu^2)]^{-3}. \end{aligned} \tag{A4}$$

Transforming back to the θ coordinate yields

$$K(\eta, \beta, \theta) = \eta^{-2}[1-\beta^2(1+3\cos^2\theta)][1-\beta^2\sin^2\theta]^{-3}. \tag{A5}$$

To check the topology of the surface one can apply the Gauss-Bonnet theorem which states[22] that

$$\int K\omega^2 \wedge \omega^3 = 2\pi\chi, \tag{A6}$$

where χ is the Euler characteristic of the surface. For the event horizon

$$\int K(\mu)\omega^2 \wedge \omega^3 = -\tfrac{1}{2}\int_0^{2\pi}\int_{-1}^{1} f''(\mu)d\mu \wedge d\phi$$

$$= \pi\,[f'(-1) - f'(1)] = 4\pi . \qquad (A7)$$

This yields $\chi = 2$; therefore the black-hole surface is topologically a 2-sphere. Hawking[4] has obtained a similar result in a more abstract setting.

The procedure for embedding an arbitrary surface of revolution in Euclidean 3-space (E^3) is standard.[28] One sets up a map from $(\mu, \phi) \to (x, y, z)$ by the formulas

$$x = F(\mu)\cos\phi , \quad y = F(\mu)\sin\phi , \quad z = G(\mu) \qquad (A8)$$

and equates the resulting 2-metric,

$$ds^2 = dx^2 + dy^2 + dz^2 = (F'^2 + G'^2)d\mu^2 + F^2 d\phi^2 , \qquad (A9)$$

to the 2-metric which is to be embedded [Eqs. (A1) and (A2)],

$$F'^2 + G'^2 = \eta^2 f^{-1} , \quad F = \eta^2 f . \qquad (A10)$$

Solving for F and G one obtains

$$F = \eta f^{1/2} , \quad G = \eta \int d\mu [f^{-1}(1 - \tfrac{1}{4} f'^2)]^{1/2} . \qquad (A11)$$

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