Perturbative Calculations of Symmetry Breaking*

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The one-loop contributions to fermion and pseudo-Goldstone masses are calculated for the general class of renormalizable guage theories. It is shown explicitly that when the masses are subject to any type of zeroth-order symmetry relation for all values of the parameters in the Lagrangian, the divergences in the one-loop corrections to these symmetry relations cancel. The finite parts of these corrections are evaluated and discussed. Other topics considered include the connection of this work with that of Coleman and E. Weinberg, the constraints obeyed by scalar coupling constants, and the path-integral derivation of the Feynman rules for general renormalizable gauge theories.

I. INTRODUCTION AND SUMMARY

The development of unified renormalizable gauge theories of the weak and electromagnetic interactions[1] has yielded, as an unexpected by-product, a new view of the origins of approximate symmetry.[2-6] Gauge theories are so constrained by the requirement of renormalizability that the physical parameters of the theory, calculated in zeroth-order perturbation theory after spontaneous symmetry breaking, will often be subject to certain symmetry relations for all values of the parameters in the Lagrangian. (Here and below, "all values" means "all values in at least a finite range.") Those symmetry relations that merely reflect some unbroken subgroup of the underlying gauge group remain valid to all orders, and do not concern us here. However, there are other kinds of zeroth-order symmetry relations which do receive corrections from higher-order effects, and which can therefore account for observed approximate symmetries. For instance there are three separate types of fermion mass relations which may arise in zeroth order for all values of the parameters in the theory and yet may not remain exact in higher orders:

(1) Symmetry relations which arise because not all of the scalar-field types which could enter in the Yukawa interactions actually appear in the theory.[2]

(2) Symmetry relations which arise because the scalar-field vacuum expectation values are invariant under some group of symmetries which, although not symmetries of the whole Lagrangian, are symmetries of *any* quartic gauge-invariant polynomial in the scalar field.[4]

(3) Symmetry relations which arise because the scalar-field vacuum expectation values are subject to constraints other than those of type 2.[3]

Renormalizability assures us that when a zeroth-order symmetry relation arises for all values of the parameters in the Lagrangian, the higher-order corrections are finite and calculable.[7] It is this feature, of course, which makes this new approach so attractive.[8] The specific mechanisms by which divergences cancel in these calculations have been previously outlined[2] in general terms only for fermion mass relations of type 1; detailed calculations of the proton-neutron mass difference[9,10] and the electron-muon mass ratio[5] have been carried out in various illustrative models of this type.

The purpose of this paper is to describe in detail how to carry out one-loop calculations of symmetry-breaking effects in general renormalizable gauge theories. The emphasis throughout will be on generality, because we do not wish to lose sight of the universal features of these calculations in a hotchpotch of specific models.

Section II sets the stage. Using a previously developed formalism,[11] we write down a general renormalizable Lagrangian, invariant under an arbitrary gauge group G. The Lagrangian is constructed from general multiplets of spin-0 and spin-$\frac{1}{2}$ fields and a set of spin-1 gauge fields. The symmetry is broken in zeroth order by allowing the scalar fields ϕ_i to develop nonvanishing vacuum expectation values λ_i, determined by the condition that the scalar-field polynomial $P(\phi)$ appearing in the Lagrangian be stationary at $\phi_i = \lambda_i$. The Feynman rules for the theory with spontaneous symmetry breaking are then given in a general "ξ gauge."[12] These rules are manifestly renormalizable for $\xi \neq 0$ and reduce for $\xi \to 0$ to the rules derived by canonical quantization[11] in the "unitarity" gauge. All physical quantities are calculated in a general ξ gauge, and are explicitly shown to be ξ-independent. (The calculations could in fact be carried through perfectly well in the unitarity gauge, as shown previously[2] for fermion mass relations of type 1.)

Section III deals with the tadpoles: graphs with a single external scalar line. These are of no direct physical significance, but they describe the shift in the vacuum expectation value of the scalar fields due to higher-order effects, and therefore play a fundamental role in calculations of physical effects.

Section IV treats the corrections to fermion mass relations of all types, 1, 2, and 3. The corrections to these mass relations are explicitly found to be finite, and are given by Eqs. (4.25)–(4.30). In theories in which all gauge meson masses except the photon's are roughly equal and much greater than all fermion masses, the corrections to fermion mass relations of type 1 turn out to be roughly the same as would be produced by electromagnetism alone, but with the ultraviolet cutoff replaced by a typical vector-boson mass. For this reason, it seems unlikely that the neutron-proton mass difference can be naturally made to come out with the observed sign if isospin is an approximate symmetry of type 1, a difficulty that is in fact encountered in detailed calculations [9] using a specific model. However, in corrections to mass relations of types 2 or 3 there are additional tadpole contributions, which can be larger than the usual electromagnetic term if the scalar-boson masses are smaller than the vector-boson masses, and which can have any sign.

Sections V and VI deal with theories in which the polynomial $P(\phi)$ is forced by gauge invariance and renormalizability to be invariant under some group $\overline{G}$ larger than the symmetry group G of the Lagrangian.[4] (This includes all theories with fermion mass relations of type 2.) The feature which makes these theories so interesting is that in addition to the Goldstone bosons corresponding to broken-symmetry generators of G which are eliminated by the Higgs mechanism,[13] there are also "pseudo-Goldstone bosons"[4] corresponding to those broken symmetry generators of $\overline{G}$ which are not in the algebra of G. These pseudo-Goldstone bosons are not eliminated by the Higgs mechanism, and although their mass vanishes in zeroth order it receives finite contributions from higher-order effects. The hope is that the pion and its relatives may eventually be identified as pseudo-Goldstone bosons in a suitable theory of the strong, weak, and electromagnetic interactions. Another special feature of these theories is that there is a continuous infinity of physically inequivalent values of the "vector" λ_i at which the polynomial $P(\phi)$ is stationary.[14] In Sec. V it is shown that this ambiguity is to be removed by the condition that the tadpole graphs must vanish if the single external scalar line corresponds to a pseudo-Goldstone boson. Using the tadpole results calculated in Sec. III, this condition may also be expressed as the requirement that at the physical value of λ, the "one-loop potential" $V_1(\phi)$ is $\overline{G}$-invariant, where

$$V_1(\lambda)=\frac{1}{64\pi^2}\,\mathrm{Tr}(M^4\ln M^2)-\frac{1}{16\pi^2}\,\mathrm{Tr}(m^4\ln m^2)+\frac{3}{64\pi^2}\,\mathrm{Tr}(\mu^4\ln\mu^2)\,, \tag{1.1}$$

with M, m, and μ the zeroth-order mass matrices of the spin-0, $-\frac{1}{2}$, and -1 fields, respectively. In Sec. VI the masses of the pseudo-Goldstone bosons are calculated; the mass matrix is found to be

$$\mathfrak{M}^2{}_{AB}=-[\overline{\mu}^{-1}\Pi\overline{\mu}^{-1}]_{AB}\,, \tag{1.2}$$

where $\overline{\mu}$ and Π are matrices defined by

$$\overline{\mu}^2{}_{AB}=-(\theta_A\lambda\,,\,\theta_B\lambda)\,,$$

$$\Pi_{AB}=-\left(\theta_A\lambda\,,\frac{\partial}{\partial\lambda}\right)\left(\theta_B\lambda\,,\frac{\partial}{\partial\lambda}\right)V_1(\lambda)\,,$$

with θ_A the Ath generator of the group $\overline{G}$ of symmetries of the polynomial $P(\phi)$. It follows that if all components of λ_i are of the same order of magnitude λ, and if all gauge coupling constants are of the same order of magnitude e, then all pseudo-Goldstone boson masses are of order

$$\mathfrak{M}\sim e^2\lambda\sim e\mu\,, \tag{1.3}$$

where μ is a typical intermediate-vector-boson mass. This estimate applies in the case of an "unlocking" symmetry [4] $\overline{G}$ even when one of the unlocked scalar multiplets has a much smaller vacuum expectation value than the other.

Section VII describes the connection between the present work and an interesting recent paper by Coleman and E. Weinberg.[15] These authors employ a functional formalism, in which the sum of all connected proper Feynman diagrams with n external scalar lines is given by the nth derivative with respect to λ of a potential $V(\lambda)$, which in the one-loop approximation is just the quantity

$$V(\lambda)=P(\lambda)+V_1(\lambda)\,, \tag{1.4}$$

with $V_1(\lambda)$ the same as the function (1.1). In this formalism the true vacuum expectation value of ϕ_i is located at any point where $V(\phi)$ is stationary. Their results turn out to be in agreement with the values of tadpole graphs and pseudo-Goldstone boson masses derived here, except that for all gauges other than Landau gauge the tadpole graphs calculated in Sec. V contain an extra term, which is not the derivative of any potential. This extra term is due to the presence of λ-dependent gauge-determining and scalar-ghost terms in the effective interaction Hamiltonian. However, since the present results agree with those of Ref. 15 in the Landau gauge, which was the gauge used there, and since all physical quantities such as masses are gauge-independent, there is no real discrepancy between our results and those of Ref. 15.

It should also be noted that Coleman and Weinberg were chiefly concerned with the implications of a proposed mass-renormalization condition, that the second derivatives of $V(\lambda)$ with respect to λ should vanish at $\lambda=0$. This condition allows a solution in which the zeroth-order vacuum expectation value of ϕ_i vanishes, and yet the symmetries are spontaneously broken by higher-order effects which shift the minimum of $V(\phi)$ away from $\phi=0$. This circumstance leads to resemblances between their work and that presented here in Secs. V and VI, in that the qualitative character of the symmetry breaking can only be learned after carrying through a one-loop calculation. However, in our case (and also in one section of Ref. 15) this is due to an invariance group $\overline{G}$ of the polynomial $P(\phi)$ which is forced on us by G invariance and renormalizability, rather than to a more-or-less arbitrary mass renormalization condition.

There are five appendixes, some of which present material that may be independently useful. In particular, Appendix A outlines the derivation of the general ξ-gauge Feynman rules[16] in the path-integral formalism,[17] and Appendix B derives constraints on the trilinear and quadrilinear scalar-meson coupling constants. These constraints are responsible for the fortunate circumstance that the formulas for the fermion and pseudo-Goldstone boson masses derived here can be used without knowing all the details of the Lagrangian.

So far, no one model has emerged which accounts for the observed weak and electromagnetic interactions of leptons and hadrons with sufficient elegance to win our allegiance. It is hoped that this paper, by serving as a general primer for perturbative calculations of symmetry breaking, will aid in the search for such a model. Of course, we will not be able to rely on "one-loop" calculations in any realistic treatment of the approximate symmetries of hadrons. However, it turns out that the perturbative calculations presented here stand in a very close correspondence with current-algebra calculations valid to all orders in the strong interactions. The current-algebra treatment of approximate symmetries will be described in a forthcoming paper.

II. REVIEW OF THE GENERAL THEORY

We shall work with a completely general renormalizable Lagrangian possessing gauge invariance with respect to some compact semisimple Lie group G. Such a Lagrangian is formed from a set of Hermitian gauge fields $A_{\alpha\mu}$, a multiplet (perhaps reducible) of spin-$\frac{1}{2}$ fields $\psi_n(x)$, and a multiplet (also perhaps reducible, and hence without loss of generality real) of spin-0 fields $\phi_i(x)$. Following Ref. 11, the form of the Lagrangian is

$$\mathcal{L}=-\tfrac{1}{4}F_{\alpha\mu\nu}F_{\alpha}^{\mu\nu}-\tfrac{1}{2}(D_\mu\phi)_i(D^\mu\phi)_i-\overline{\psi}\gamma^\mu D_\mu\psi-\overline{\psi}m_0\psi-P(\phi)-\overline{\psi}\Gamma_i\psi\phi_i\,, \tag{2.1}$$

where

(a) The gauge-covariant curl is

$$F_{\alpha\mu\nu}=\partial_\mu A_{\alpha\nu}-\partial_\nu A_{\alpha\mu}-C_{\alpha\beta\gamma}A_{\beta\mu}A_{\gamma\nu}\,, \tag{2.2}$$

where $C_{\alpha\beta\gamma}$ are a set of real totally antisymmetric structure constants, proportional to one or more gauge coupling constants.

(b) The gauge-covariant derivative of the scalar field $\phi_i(x)$ is

$$(D_\mu\phi)_i\equiv\partial_\mu\phi_i+i(\theta_\alpha)_{ij}\phi_jA_{\alpha\mu}\,, \tag{2.3}$$

where $(\theta_\alpha)_{ij}$ is the matrix representing the αth generator of G on the scalar-field multiplet. The matrix θ_α is proportional to the gauge coupling constants, and satisfies the antisymmetry, Hermiticity, and commutation relations

$$(\theta_\alpha)_{ij}=(\theta_\alpha)^*_{ji}=-(\theta_\alpha)_{ji}\,, \tag{2.4}$$

$$[\theta_\alpha,\theta_\beta]=iC_{\alpha\beta\gamma}\theta_\gamma\,. \tag{2.5}$$

(c) The gauge-covariant derivative of the spin-$\frac{1}{2}$ field $\psi_n(x)$ is

$$(D_\mu\psi)_n=\partial_\mu\psi_n+i(t_\alpha)_{nm}\psi_mA_{\alpha\mu}\,, \tag{2.6}$$

where $(t_\alpha)_{nm}$ is the matrix representing the αth generator of G on the spin-$\frac{1}{2}$ multiplet. The matrix t_α is proportional to the gauge coupling constants, and satisfies the Hermiticity and commutation relations

$$t_\alpha^\dagger=t_\alpha\,, \tag{2.7}$$

$$[t_\alpha,t_\beta]=iC_{\alpha\beta\gamma}t_\gamma\,. \tag{2.8}$$

We are using a metric $\eta_{\mu\nu}$ with nonzero elements +1, +1, +1, −1, for $\mu=\nu=1,2,3,0$, and Dirac matrices with

$$\begin{gathered}\{\gamma_\mu,\gamma_\nu\}=2\eta_{\mu\nu}\,,\\ \gamma_4=-i\gamma_0,\quad\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4,\\ \vec{\gamma}^\dagger=\vec{\gamma},\quad\gamma_4^\dagger=\gamma_4,\quad\gamma_5^\dagger=\gamma_5,\\ \overline{\psi}\equiv\psi^\dagger\gamma_4\,.\end{gathered} \tag{2.9}$$

The matrix t_α may include terms proportional to γ_5 as well as 1.

(d) The bare-mass matrix m_0 is Hermitian and G-invariant in the sense that

$$m_0^\dagger=\gamma_4m_0\gamma_4\,, \tag{2.10}$$

$$[t_\alpha, \gamma_4 m_0] = 0. \tag{2.11}$$

(e) The function $P(\phi)$ is a real 4th-order polynomial in ϕ, and is G-invariant in the sense that

$$\frac{\partial P(\phi)}{\partial \phi_i}(\theta_\alpha)_{ij}\phi_j = 0\,. \tag{2.12}$$

(f) The Yukawa coupling matrices Γ_i contain any Yukawa coupling constants, and may contain terms proportional to γ_5. They are Hermitian and G-covariant, in the sense that

$$\Gamma_i^\dagger = \gamma_4 \Gamma_i \gamma_4\,, \tag{2.13}$$

$$[t_\alpha, \gamma_4\Gamma_i] = -(\theta_\alpha)_{ij}\gamma_4\Gamma_j\,. \tag{2.14}$$

The G invariance of the Lagrangian is broken by allowing the scalar fields ϕ_i spontaneously to develop a nonvanishing vacuum expectation value. In zeroth order, this vacuum expectation value is a quantity λ_i, defined as some solution of the symmetry-breaking condition

$$\left.\frac{\partial P(\phi)}{\partial \phi_i}\right|_{\phi=\lambda} = 0\,. \tag{2.15}$$

The field ϕ_i may conveniently be replaced in (2.1) with a shifted field ϕ_i', with

$$\phi_i = \phi_i' + \lambda_i\,. \tag{2.16}$$

As a result of the symmetry breaking, the spin-1 particles pick up a zeroth-order mass matrix[13] μ, with

$$\begin{aligned}(\mu^2)_{\alpha\beta} &= -(\theta_\alpha\lambda)_i(\theta_\beta\lambda)_i \\ &= \lambda_i(\theta_\alpha\theta_\beta\lambda)_i\,,\end{aligned} \tag{2.17}$$

while the formal zeroth-order mass matrices M and m of the spin-0 and $-\frac{1}{2}$ fields become

$$M^2{}_{ij} = \left.\frac{\partial^2 P(\phi)}{\partial\phi_i\partial\phi_j}\right|_{\phi=\lambda}\,, \tag{2.18}$$

$$m = m_0 + \Gamma_i\lambda_i\,. \tag{2.19}$$

We can and will define the fermion fields so that m contains no terms proportional to γ_5, and therefore

$$\gamma_4 m \gamma_4 = m^\dagger = m\,.$$

By differentiating (2.12) with respect to ϕ_k, setting $\phi = \lambda$, and using (2.15), we see that M^2 has eigenvectors $\theta_\alpha\lambda$ with eigenvalue zero:

$$M^2\theta_\alpha\lambda = 0\,. \tag{2.20}$$

The corresponding fields $(\theta_\alpha\lambda)_i\phi_i$ describe the Goldstone bosons of this theory.

The derivation of the Feynman rules in a convenient "ξ gauge"[12] is sketched in Appendix A.[16] The effective interaction Lagrangian $\mathcal{L}'$ is obtained by replacing ϕ_i in Eq. (2.1) with $\lambda_i + \phi_i'$, setting aside all terms quadratic in the fields ϕ', A, and ψ, and adding certain terms which describe the interaction of a complex spinless fermion "ghost" field ω_α:

$$\begin{aligned}\mathcal{L}' = {} & \tfrac{1}{2}(\partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu})C_{\alpha\beta\gamma}A_\beta{}^\mu A_\gamma{}^\nu \\ & -\tfrac{1}{4}C_{\alpha\beta\gamma}C_{\alpha\delta\epsilon}A_{\beta\mu}A_{\gamma\nu}A_\delta{}^\mu A_\epsilon{}^\nu \\ & -i\partial_\mu\phi_i'(\theta_\alpha)_{ij}\phi_j' A_\alpha{}^\mu - (\theta_\beta\theta_\alpha\lambda)_i\phi_i' A_\alpha{}^\mu A_{\beta\mu} \\ & -\tfrac{1}{2}(\theta_\beta\theta_\alpha)_{ij}\phi_i'\phi_j' A_\alpha{}^\mu A_{\beta\mu} - i\bar\psi\gamma^\mu t_\alpha\psi A_{\alpha\mu} \\ & -\tfrac{1}{6}f_{ijk}\phi_i'\phi_j'\phi_k' - \tfrac{1}{24}f_{ijkl}\phi_i'\phi_j'\phi_k'\phi_l' \\ & -\bar\psi\Gamma_i\psi\phi_i' - \partial_\mu\omega_\alpha^* C_{\alpha\beta\gamma}\omega_\beta A_\gamma{}^\mu \\ & -\xi^{-1}\omega_\alpha^*\omega_\beta(\theta_\beta\theta_\alpha\lambda)_i\phi_i'\,,\end{aligned} \tag{2.21}$$

with

$$f_{ijk} \equiv \left.\frac{\partial^3 P(\phi)}{\partial\phi_i\partial\phi_j\partial\phi_k}\right|_{\phi=\lambda}\,, \tag{2.22}$$

$$f_{ijkl} \equiv \left.\frac{\partial^4 P(\phi)}{\partial\phi_i\partial\phi_j\partial\phi_k\partial\phi_l}\right|_{\phi=\lambda}\,. \tag{2.23}$$

(Although the coupling constants f_{ijk} and f_{ijkl} are not known, they are shown in Appendix B to be subject to important constraints, which in fact will turn out to provide sufficient information about the f's for our calculations to go through.) The propagators of the A_α^μ, ϕ_i', ψ_n, and ω_α fields are given by

$$\begin{aligned}\Delta^A_{\alpha\mu,\beta\nu}(k) = {} & \eta_{\mu\nu}(k^2+\mu^2)^{-1}{}_{\alpha\beta} \\ & +(1-\xi)k_\mu k_\nu[(k^2+\mu^2)^{-1}(\xi k^2+\mu^2)^{-1}]_{\alpha\beta}\,,\end{aligned} \tag{2.24}$$

$$\begin{aligned}\Delta^\phi_{ij}(k) = {} & (k^2+M^2)^{-1}{}_{ij} \\ & +(\theta_\alpha\lambda)_i(\theta_\beta\lambda)_j(k^2)^{-1}(\xi k^2+\mu^2)^{-1}{}_{\alpha\beta}\,,\end{aligned} \tag{2.25}$$

$$\Delta^\psi_{nm}(k) = (i\gamma^\mu k_\mu + m)^{-1}{}_{nm}\,, \tag{2.26}$$

$$\Delta^\omega_{\alpha\beta}(k) = \xi(\xi k^2+\mu^2)^{-1}{}_{\alpha\beta}\,. \tag{2.27}$$

The propagators of field derivatives such as $\partial_\mu A_{\alpha\mu} - \partial_\nu A_{\alpha\mu}$ or $\partial_\mu\phi_i$ are simply given by the corresponding derivatives of these propagators, and there are no propagators here which mix the fields of particles of different spin. In addition to the internal line propagators (2.24)–(2.27) and the vertex factors dictated by the interaction (2.21), we must as usual insert extra factors: $i(2\pi)^4$ for each vertex, $-i(2\pi)^{-4}$ for each internal line, and -1 for each spin-$\frac{1}{2}$ or ghost loop, as well as momentum-space wave functions for each external A, ϕ, or ψ line.

The above Feynman rules reduce in the limit $\xi \to 0$ to the Feynman rules derived[11] by canonical quantization in the "unitarity" gauge. This is a gauge in which all Goldstone-boson fields are absent, so that

$$(\theta_A\lambda)_i\phi_i = 0\,.$$

The propagator of the ϕ fields in this gauge may be taken as

$$\Delta^{\phi}_{ij}(k) = [(k^2+M^2)^{-1}\Pi]_{ij} ,$$

where Π is the projection matrix[2] on the subspace perpendicular to all $\theta_\alpha\lambda$:

$$\Pi_{ij} = \delta_{ij} + (\theta_\alpha\lambda)_i(\mu^{-2})_{\alpha\beta}(\theta_\beta\lambda)_j .$$

Recalling that $M^2\theta_\alpha\lambda$ vanishes, we see that this propagator is the same as (2.25) in the limit $\xi\to 0$. Also, Eqs. (2.21) and (2.27) show that for $\xi\to 0$, the whole effect of the ghost fields is to generate an effective interaction Lagrangian

$$-\sum_{N=1}^{\infty}\frac{\delta^4(0)}{N}[i(2\pi)^4]^{N-1}\left(\frac{-i}{(2\pi)^4}\right)^N(-)^N\operatorname{Tr}(\mu^{-2}\Phi')^N = -i\delta^4(0)\operatorname{Tr}\ln(1+\mu^{-2}\Phi'),$$

where

$$\Phi'_{\beta\alpha} \equiv \phi'_i(\theta_\beta\theta_\alpha\phi')_i .$$

We recognize this as the extra term generated by "summing the springs" in the unitarity gauge.[11]

In what follows, we are going to show explicitly that the sum of one-loop diagrams is ξ-independent, and the results obtained could just as well be derived in the unitarity gauge. However, the calculations below will be worked out in the general ξ gauge, because the explicit verification of ξ independence provides a useful check of our calculations, and because the milder degree of divergence encountered for $\xi\neq 0$ helps to calm our fears regarding manipulation of divergent integrals.

III. TADPOLES

The first diagrams to be calculated here will be those with a single external ϕ'_i line and no other external lines. These "tadpole" diagrams play an important role in corrections to various mass relations, and as we shall see in Sec. V, they are of fundamental importance in removing the ambiguities in λ which arise when $P(\phi)$ is invariant under a group $\overline{G}$ of symmetries larger than the gauge group G.

Before beginning our calculation of the tadpole diagrams, it is useful to consider their physical effects. Generally, the effect of the one-loop tadpoles is just the same as that of shifting the zeroth-order vacuum expectation value λ_i by an amount

$$\delta_T\lambda_i \equiv -i(2\pi)^{-4}\Delta^{\phi}_{ij}(0)T_j \quad , \tag{3.1}$$

where T_j is the sum of all proper one-loop tadpole diagrams. For instance, Eqs. (2.17)–(2.19) show that M^2, m, and μ^2 may be regarded as functions of λ, with derivatives

$$\frac{\partial M^2{}_{ij}}{\partial\lambda_k} = f_{ijk} , \tag{3.2}$$

$$\frac{\partial m_{nm}}{\partial\lambda_k} = (\Gamma_k)_{nm} , \tag{3.3}$$

$$\frac{\partial\mu^2{}_{\alpha\beta}}{\partial\lambda_k} = (\{\theta_\alpha,\theta_\beta\}\lambda)_k , \tag{3.4}$$

while inspection of the interaction (2.21) shows that the one-loop tadpole contributions to the scalar, spinor, and vector-meson mass matrices are

$$\delta M^2{}_{ij} = f_{ijk}\delta_T\lambda_k = \frac{\partial M^2{}_{ij}}{\partial\lambda_k}\delta_T\lambda_k , \tag{3.5}$$

$$\delta m = \Gamma_k\delta_T\lambda_k = \frac{\partial m}{\partial\lambda_k}\delta_T\lambda_k , \tag{3.6}$$

$$\delta\mu^2{}_{\alpha\beta} = (\{\theta_\alpha,\theta_\beta\}\lambda)_k\delta_T\lambda_k = \frac{\partial\mu^2{}_{\alpha\beta}}{\partial\lambda_k}\delta_T\lambda_k . \tag{3.7}$$

The same conclusion can be reached on more general grounds. Recall that the shifted field ϕ'_i is defined by Eqs. (2.15) and (2.16) in such a way that its *zeroth-order* vacuum expectation value vanishes, so there are no zeroth-order tadpole graphs. However, this definition of ϕ'_i does not forbid higher-order tadpole graphs, which in the one-loop approximation generate a vacuum expectation value, which is just given by the quantity (3.1):

$$\langle\phi'_i\rangle_0 = \delta_T\lambda_i . \tag{3.8}$$

If we had defined the shifted field as $\phi_i - \lambda_i - \delta_T\lambda_i$, its vacuum expectation value would vanish in the one-loop approximation, so to this order there would be no tadpoles. The whole effect of the tadpoles is therefore just the same as a shift in λ_i by the amount (3.1). A more detailed argument along these lines is given in Appendix C.

Equation (3.1) makes no sense unless $\Delta^{\phi}(k)T$ is finite in the limit $k\to 0$, which requires that T have no components along directions u for which M^2u vanishes:

$$u_iT_i = 0 \ \text{ if } \ M^2{}_{ij}u_j = 0 . \tag{3.9}$$

The same result is derived in Appendix C from the requirement that one-loop corrections should not produce a singular change in the vacuum expectation values of the ϕ_i. This condition will be discussed in detail at the end of this section and in Sec. V. For the present, we merely note that

(3.9) [with (2.20) and (2.25)] allows the effective change in λ_i given by (3.1) to be written

$$\delta_T\lambda_i = -i(2\pi)^{-4}M^{-2}{}_{ij}T_j\ , \tag{3.10}$$

with an obvious interpretation of M^{-2}.

Now, to the calculation. In the one-loop approximation, there are four tadpole diagrams, shown in Fig. 1. The scalar, spinor, vector, and ghost loops yield the respective contributions

$$T_i^{(\phi)} = -\tfrac{1}{2}\int d^4k\, f_{ijk}\Delta^{\phi}_{jk}(k)\,, \tag{3.11}$$

$$T_i^{(\psi)} = +\int d^4k\, \mathrm{Tr}\{\Gamma_i\Delta^{\psi}(k)\}\ , \tag{3.12}$$

$$T_i^{(A)} = -(\theta_\beta\theta_\alpha\lambda)_i\eta^{\mu\nu}\int d^4k\,\Delta^A_{\alpha\mu,\beta\nu}(k)\,, \tag{3.13}$$

$$T_i^{(\omega)} = +\xi^{-1}(\theta_\beta\theta_\alpha\lambda)_i\int d^4k\,\Delta^{\omega}_{\alpha\beta}(k)\,, \tag{3.14}$$

where Δ^ϕ, Δ^ψ, Δ^A, and Δ^ω are the propagators given by Eqs. (2.24)–(2.27). The ξ-dependent part of (3.11) can be calculated using the constraint derived in Appendix B:

$$f_{ijk}(\theta_\alpha\lambda)_j(\theta_\beta\lambda)_k = -(M^2\theta_\alpha\theta_\beta\lambda)_i\ . \tag{3.15}$$

[See Eq. (B8).] We find

$$T_i^{(\phi)} = -\tfrac{1}{2}\int d^4k\, f_{ijk}(k^2+M^2)^{-1}{}_{jk} + \tfrac{1}{2}(M^2\theta_\alpha\theta_\beta\lambda)_i\int d^4k(k^2)^{-1}(\xi k^2+\mu^2)^{-1}{}_{\alpha\beta}\,.$$

Also, (2.24) gives

$$\eta^{\mu\nu}\Delta^A_{\alpha\mu,\beta\nu}(k) = \left(\frac{3}{k^2+\mu^2}+\frac{1}{\xi k^2+\mu^2}\right)_{\alpha\beta}\ ,$$

the second term of which cancels the ghost-loop term (3.14). Putting this together, we have for the total tadpole

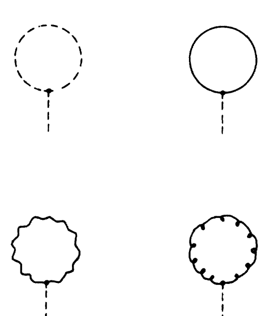

FIG. 1. Feynman diagrams for the tadpole T. (Here dashed lines refer to scalar fields, solid lines refer to spinor fields, wavy lines refer to gauge fields, and looped lines refer to ghost fields.)

$$\begin{aligned}T_i &= T_i^{(\phi)}+T_i^{(\psi)}+T_i^{(A)}+T_i^{(\omega)}\\ &= -\tfrac{1}{2}\int d^4k\, f_{ijk}(k^2+M^2)^{-1}{}_{jk}\\ &\quad+\int d^4k\,\mathrm{Tr}\{\Gamma_i(i\gamma^\mu k_\mu+m)^{-1}\}\\ &\quad-3(\theta_\beta\theta_\alpha\lambda)_i\int d^4k(k^2+\mu^2)^{-1}{}_{\alpha\beta}\\ &\quad+\tfrac{1}{2}(M^2\theta_\alpha\theta_\beta\lambda)_i\int d^4k(k^2)^{-1}(\xi k^2+\mu^2)^{-1}\,.\end{aligned} \tag{3.16}$$

It proves very convenient to use Eqs. (3.2)–(3.4) to express the first three terms in (3.16) as derivatives with respect to λ_i:

$$\int d^4k\, f_{ijk}(k^2+M^2)^{-1}{}_{jk} = \frac{\partial}{\partial\lambda_i}\int d^4k\,\mathrm{Tr}\ln(k^2+M^2)\,, \tag{3.17}$$

$$\begin{aligned}\int d^4k\,\mathrm{Tr}\{\Gamma_i(i\gamma^\mu k_\mu+m)^{-1}\} &= \frac{\partial}{\partial\lambda_i}\int d^4k\,\mathrm{Tr}\ln(i\gamma_\mu k^\mu+m)\\ &= \frac{1}{2}\frac{\partial}{\partial\lambda_i}\int d^4k[\mathrm{Tr}\ln(i\gamma_\mu k^\mu+m)\\ &\qquad+\mathrm{Tr}\ln(-i\gamma_\mu k^\mu+m)]\\ &= 2\frac{\partial}{\partial\lambda_i}\int d^4k\,\mathrm{Tr}\ln(k^2+m^2)\,,\end{aligned} \tag{3.18}$$

$$(\theta_\alpha\theta_\beta\lambda)_i\int d^4k(k^2+\mu^2)^{-1}{}_{\alpha\beta} = \frac{1}{2}\frac{\partial}{\partial\lambda_i}\int d^4k\,\mathrm{Tr}\ln(k^2+\mu^2)\,. \tag{3.19}$$

[Note that $\mathrm{Tr}\ln(i\gamma_\mu k^\mu+m)$ is understood to include a sum over Dirac indices as well as particle types, while $\mathrm{Tr}\ln(k^2+m^2)$ is understood to include only a sum over particle types. This difference accounts for an extra factor of 4 in (3.18).] Equation (3.16) may thus be written

$$T_i = -i(2\pi)^4\frac{\partial V_1}{\partial\lambda_i} + \tfrac{1}{2}(M^2\theta_\alpha\theta_\beta\lambda)_i\int d^4k(k^2)^{-1}(\xi k^2+\mu^2)^{-1}{}_{\alpha\beta}\,, \tag{3.20}$$

where V_1 is the one-loop "potential"

$$V_1 = -\frac{i}{2(2\pi)^4}\int d^4k\,\mathrm{Tr}\ln(k^2+M^2) + \frac{2i}{(2\pi)^4}\int d^4k\,\mathrm{Tr}\ln(k^2+m^2) - \frac{3i}{2(2\pi)^4}\int d^4k\,\mathrm{Tr}\ln(k^2+\mu^2)\,. \tag{3.21}$$

The second term in Eq. (3.20) is both ξ-dependent and divergent, but it has an M^2 factor which will cancel the M^{-2} factor in Eq. (3.10), so that the ξ dependence and divergence of this term can be canceled by the ξ dependence and divergences in nontadpole diagrams. As we shall see in Secs. IV and VI, this is just what happens. On the other hand, the $\partial V_1/\partial\lambda$ term in (3.20) does not have an M^2 factor to cancel the M^{-2} factor in (3.10), so we cannot expect nontadpole graphs, which do not have M^{-2} factors, to be of any help in dealing with this term. Fortunately, (3.21) shows that V_1 is ξ-independent, so gauge invariance is no problem here. However, (3.21) also shows that V_1 is quartically divergent, so we face a problem in eliminating the infinities in the potential.

The clue to this problem is provided by the observation that if we added to $P(\phi)$ a small correction polynomial $\Delta P(\phi)$, then λ would be shifted by an amount $\Delta\lambda$ given by the condition

$$0=\left(\frac{\partial}{\partial\phi_i}[P(\phi)+\Delta P(\phi)]\right)_{\phi=\lambda+\Delta\lambda}$$
$$=\frac{\partial^2 P(\lambda)}{\partial\lambda_i\partial\lambda_j}\Delta\lambda_j+\frac{\partial\Delta P(\lambda)}{\partial\lambda_i}\,,$$

or, using (2.18),

$$\Delta\lambda_i=-M^{-2}{}_{ij}\frac{\partial\Delta P(\lambda)}{\partial\lambda_j}\,. \qquad (3.22)$$

As we have seen, the physical effect of the tadpole contribution produced by any polynomial term $\Delta V(\lambda)$ in the potential $V_1(\lambda)$ is the same as would be produced by shifting λ_i an amount $\Delta\lambda_i$, given by (3.10) and (3.20) as

$$\Delta\lambda_i=-M^{-2}{}_{ij}\frac{\partial}{\partial\lambda_j}\Delta V(\lambda)\,,$$

and, according to (3.22), this shift in λ_i could be produced by adding to $P(\phi)$ a correction polynomial $\Delta V(\lambda)$. [As an alternative way of reaching the same conclusions, we may note that if we added a correction polynomial $\Delta V(\lambda)$ to $P(\phi)$, but continued to quantize using the old shifted field $\phi_i-\lambda_i$, then ΔV would produce a new tadpole contribution, given by the part of $\Delta V(\lambda+\phi')$ linear in ϕ':

$$\Delta T_i=-i(2\pi)^4\frac{\partial\Delta V(\lambda+\phi')}{\partial\phi'_i}\bigg|_{\phi'=0}$$
$$=-i(2\pi)^4\frac{\partial}{\partial\lambda_i}\Delta V(\lambda)\,,$$

which is just the tadpole contribution associated with $\Delta V(\lambda)$.] However, $P(\phi)$ is *any* G-invariant quartic polynomial in ϕ, so any polynomial terms in $V_1(\lambda)$, which are G-invariant and not greater than 4th order in λ, produce tadpole contributions to physical quantities which are the same as would be produced by appropriate changes in the constants appearing in the polynomial $P(\phi)$. This implies that such terms can be absorbed into a renormalization of the constants in $P(\phi)$, a point we will not pursue here. More importantly for our present purposes, this means that if the G invariance and quartic character of the polynomial $P(\phi)$ imposes constraints on λ_i that do not depend on the values of the constants appearing in $P(\phi)$, and if these constraints lead to zeroth-order relations among masses or other physical parameters, then *such zeroth-order relations will be unaffected by any terms in $V(\phi)$ which are G-invariant and of not more than 4th order in λ.*

Now we can calculate V_1 using the formula

$$\int d^4k\,\ln(k^2+x)=\tfrac{1}{2}i\pi^2x^2\ln x+q(x)\,, \qquad (3.23)$$

where $q(x)$ is a quadratic polynomial in x with divergent coefficients:

$$q(x)=\int \ln k^2 d^4k+x\int (k^2)^{-1}d^4k$$
$$-\tfrac{1}{2}x^2\left[\int (k^2)^{-2}d^4k+\tfrac{1}{2}i\pi^2(\ln 0+\tfrac{3}{2})\right]. \qquad (3.24)$$

It follows that

$$V_1(\lambda)=\frac{1}{64\pi^2}\mathrm{Tr}(M^4\ln M^2)-\frac{1}{16\pi^2}\mathrm{Tr}(m^4\ln m^2)$$
$$+\frac{3}{64\pi^2}\mathrm{Tr}(\mu^4\ln\mu^2)+V_{1\infty}(\lambda)\,, \qquad (3.25)$$

where

$$V_{1\infty}(\lambda)=\frac{i}{(2\pi)^4}[-\tfrac{1}{2}\mathrm{Tr}q(M^2)$$
$$+2\,\mathrm{Tr}q(m^2)-\tfrac{3}{2}\mathrm{Tr}q(\mu^2)]. \qquad (3.26)$$

However, M^2, m^2, and μ^2 are quadratic polynomials in λ, so $V_{1\infty}$ is a quartic polynomial in λ. Further, $V_{1\infty}$ is a G-invariant function of λ in the sense that

$$\frac{\partial V_{1\infty}}{\partial\lambda_i}(\theta_\alpha)_{ij}\lambda_j=0\,,$$

because $\mathrm{Tr}M^2$, $\mathrm{Tr}M^4$, $\mathrm{Tr}m^2$, $\mathrm{Tr}m^4$, $\mathrm{Tr}\mu^2$, and $\mathrm{Tr}\mu^4$ are all G-invariant functions of λ. [See Appendix D.] The above argument then shows that the term $V_{1\infty}(\lambda)$ in (3.25) cannot produce corrections to any constraints in physical parameters which may arise in zeroth order, as long as these constraints are independent of the parameters appearing in the polynomial $P(\phi)$. Hence, for the purpose of calculating corrections to such constraints, we can drop the term $V_{1\infty}$ in the potential (3.25), and write the "effective one-loop potential" as

$$V_{1,\mathrm{eff}}(\lambda)=\frac{1}{64\pi^2}\mathrm{Tr}(M^4\ln M^2)-\frac{1}{16\pi^2}\mathrm{Tr}(m^4\ln m^2)+\frac{3}{64\pi^2}\mathrm{Tr}(\mu^4\ln\mu^2)\,. \tag{3.27}$$

Note that a change of mass units used to calculate the logarithms would produce correction terms in $V_{1,\mathrm{eff}}$ proportional to $\mathrm{Tr}M^4$, $\mathrm{Tr}m^4$, or $\mathrm{Tr}\mu^4$, but such terms are again G-invariant quartic polynomials in λ, and hence cannot produce corrections to the zeroth-order constraints which concern us here.

The gradient of this potential may be evaluated using Eqs. (3.2)–(3.4):

$$\frac{\partial V_{1,\mathrm{eff}}}{\partial\lambda_i}=\frac{1}{64\pi^2}f_{jki}[2M^2\ln M^2+M^2]_{jk}-\frac{1}{16\pi^2}(\Gamma_i)_{nm}[8m^3\ln m+2m^3]_{nm}+\frac{3}{64\pi^2}(\{\theta_\alpha,\theta_\beta\}\lambda)_i[2\mu^2\ln\mu^2+\mu^2]_{\alpha\beta}\,. \tag{3.28}$$

The nonlogarithm terms here may be written as derivatives with respect to λ_i of a linear combination of the quantities $\mathrm{Tr}M^4$, $\mathrm{Tr}m^4$, and $\mathrm{Tr}\mu^4$, all of which are G-invariant and quartic in λ; hence, by the same reasoning that allowed us to drop the term $V_{1\infty}$, we may simplify (3.28) to read

$$\left(\frac{\partial V_1(\lambda)}{\partial\lambda_i}\right)_{\mathrm{eff}}=\frac{1}{32\pi^2}f_{jki}(M^2\ln M^2)_{jk}-\frac{1}{2\pi^2}(\Gamma_i)_{nm}(m^3\ln m)_{nm}+\frac{3}{32\pi^2}(\{\theta_\alpha,\theta_\beta\}\lambda)_i(\mu^2\ln\mu^2)_{\alpha\beta}\,. \tag{3.29}$$

For the same reason, it makes no difference what mass units we use to calculate the logarithms in (3.29).

The total effective tadpole is now just given by using (3.29) as the first term in Eq. (3.20):

$$T_i=-i(2\pi)^4\left(\frac{\partial V_1(\lambda)}{\partial\lambda_i}\right)_{\mathrm{eff}}+\tfrac{1}{2}(M^2\theta_\alpha\theta_\beta\lambda)_i\int d^4k\,(k^2)^{-1}(\xi k^2+\mu^2)^{-1}{}_{\alpha\beta}\,. \tag{3.30}$$

We return now to the condition (3.9) which was encountered in the course of discussing the physical effect of tadpoles. The second term in our formula (3.20) for T_i obviously satisfies this condition, so (3.9) may be written

$$\frac{\partial V_1(\lambda)}{\partial\lambda_i}u_i=0\quad\text{if}\quad M^2u=0\,. \tag{3.31}$$

One set of eigenvectors of M^2 with eigenvalue zero is provided by the "Goldstone" vectors $\theta_\alpha\lambda$. [See Eq. (2.20).] Thus, (3.31) always requires at least that

$$\frac{\partial V_1(\lambda)}{\partial\lambda_i}(\theta_\alpha\lambda)_i=0\,. \tag{3.32}$$

This just means that $V_1(\lambda)$ must be G-invariant. However, Eq. (3.21) and the theorem proved in Appendix D show immediately that $V_1(\lambda)$ *is* G-invariant, so that (3.32) is satisfied for all λ, and tells us nothing new. Condition (3.9) or (3.31) is of physical interest only when there are vectors u, other than the vectors $\theta_\alpha\lambda$, for which M^2u vanishes. This possibility is explored in Sec. V.

IV. FERMION MASS CORRECTIONS

Let us now apply this formalism to calculate the one-loop corrections to zeroth-order fermion mass relations. There are six one-loop diagrams shown in Fig. 2. The vector-boson exchange contributes to the self-energy insertion $\Sigma(p)$ an amount

$$\Sigma^{(A)}(p)=\frac{i}{(2\pi)^4}\int d^4k\,\gamma^\mu t_\beta\Delta^\psi(p-k)\gamma^\nu t_\alpha\Delta^A_{\mu\beta,\nu\alpha}(k)\,, \tag{4.1}$$

the scalar-boson-exchange contribution is

$$\Sigma^{(\phi)}(p)=-\frac{i}{(2\pi)^4}\int d^4k\,\Gamma_i\Delta^\psi(p-k)\Gamma_j\Delta^\phi_{ij}(k)\,, \tag{4.2}$$

and the total tadpole contribution is

$$\Sigma^{(T)}(p)=\frac{i}{(2\pi)^4}\Gamma_iM^{-2}{}_{ij}T_j\,. \tag{4.3}$$

The fermion masses are at the poles of the corrected propagator

$$\Delta^{\psi\prime}(p)=[i\gamma_\mu p^\mu+m-\Sigma(p)]^{-1}\,,$$

where m is the zeroth-order mass matrix (2.19).

In calculating the masses, it is useful to keep

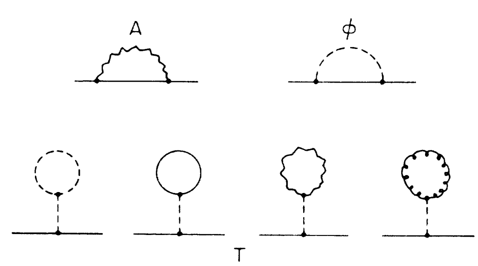

FIG. 2. Feynman diagrams for the fermion self-energy. (Conventions same as in Fig. 1.)

in mind that a term in $\Sigma(p)$ with a factor $(i\gamma_\mu p^\mu + m)$ on the extreme right or left can affect the residues of these poles, but not their positions. That is, if $\Sigma(p)$ has the form

$$\Sigma(p) = (i\gamma_\mu p^\mu + m)F(p) + G(p)(i\gamma_\mu p^\mu + m) + \Sigma_{\rm eff}(p),$$

with F, G, and $\Sigma_{\rm eff}$ matrix functions of p, then to first order in F, G, and $\Sigma_{\rm eff}$, the corrected propagator may be written

$$\Delta^{\psi\prime}(p) = [1 - F(p)]^{-1}[i\gamma_\mu p^\mu + m - \Sigma_{\rm eff}(p)]^{-1} \times [1 - G(p)]^{-1},$$

so that the positions of the poles are determined by $\Sigma_{\rm eff}$, not F or G. Thus we may freely replace any factor $\gamma_\mu p^\mu$ with im whenever it appears on the extreme left or right in any term in $\Sigma(p)$.

We begin by decomposing (4.1)–(4.3) further into various separate contributions to $\Sigma(p)$. The vector-boson-exchange term $\Sigma^{(A)}(p)$ contains a term arising from the $g_{\mu\nu}$ part of Δ^A,

$$\Sigma^{(A1)}(p) = \frac{i}{(2\pi)^4}\int d^4k\, \gamma^\mu t_\alpha \Delta^\psi(p-k)\gamma_\mu t_\beta \times (k^2+\mu^2)^{-1}{}_{\alpha\beta}, \tag{4.4}$$

and a term arising from the $k_\mu k_\nu$ part of Δ^A. This latter term may in turn be decomposed into several terms by making the substitution

$$\begin{aligned}
k_\mu\gamma^\mu t_\beta\Delta^\psi(p-k)k_\nu\gamma^\nu t_\alpha &\to i[i(p_\mu - k_\mu)\gamma^\mu + m]t_\beta\Delta^\psi(p-k)k_\nu\gamma^\nu t_\alpha\\
&= i\gamma_4[i(p_\mu-k_\mu)\gamma_4\gamma^\mu + \gamma_4 m]t_\beta\Delta^\psi(p-k)k_\nu\gamma^\nu t_\alpha\\
&= i\gamma_4 t_\beta\gamma_4 k_\nu\gamma^\nu t_\alpha + i\gamma_4[\gamma_4 m, t_\beta]\Delta^\psi(p-k)k_\nu\gamma^\nu t_\alpha\\
&\to i\gamma_4 t_\beta\gamma_4 k_\nu\gamma^\nu t_\alpha - \gamma_4[\gamma_4 m, t_\beta]\Delta^\psi(p-k)\gamma_4 t_\alpha\gamma_4[i(p_\mu-k_\mu)\gamma^\mu + m]\\
&= i\gamma_4 t_\beta\gamma_4 k_\nu\gamma^\nu t_\alpha - \gamma_4[\gamma_4 m, t_\beta]\Delta^\psi(p-k)\gamma_4[t_\alpha, \gamma_4 m] - \gamma_4[\gamma_4 m, t_\beta]t_\alpha .
\end{aligned}$$

(An arrow indicates the replacement of $p_\mu\gamma^\mu$ with im on the left or right. Note that t_α may contain terms proportional to γ_5, so it does not necessarily commute with γ_μ, but t_α does commute with $\gamma_4\gamma_\mu$. The liberal use of γ_4 matrices here saves us from having to decompose t_α explicitly into right- and left-hand parts.) The first term here vanishes upon symmetric integration. The second and third terms yield contributions to $\Sigma^{(A)}(p)$:

$$\Sigma^{(A2)}(p) = -\frac{i}{(2\pi)^4}(1-\xi)\int d^4k\big((k^2+\mu^2)^{-1}(\xi k^2+\mu^2)^{-1}\big)_{\alpha\beta}\gamma_4[\gamma_4 m, t_\beta]\Delta^\psi(p-k)\gamma_4[t_\alpha, \gamma_4 m], \tag{4.5}$$

$$\Sigma^{(A3)}(p) = -\frac{i}{(2\pi)^4}(1-\xi)\int d^4k\big((k^2+\mu^2)^{-1}(\xi k^2+\mu^2)^{-1}\big)_{\alpha\beta}\gamma_4[\gamma_4 m, t_\beta]t_\alpha . \tag{4.6}$$

Similarly, the ϕ exchange term can be broken into two parts, arising from the first and second terms in (2.25):

$$\Sigma^{(\phi 1)}(p) = -\frac{i}{(2\pi)^4}\int d^4k\, \Gamma_i\Delta^\psi(p-k)\Gamma_j \times (k^2+M^2)^{-1}{}_{ij}, \tag{4.7}$$

$$\Sigma^{(\phi 2)}(p) = -\frac{i}{(2\pi)^4}(\theta_\alpha\lambda)_i(\theta_\beta\lambda)_j \times \int d^4k\,\Gamma_i\Delta^\psi(p-k) \times \Gamma_j(k^2)^{-1}(\xi k^2+\mu^2)^{-1}{}_{\alpha\beta}. \tag{4.8}$$

Finally, the tadpole term may be broken into two parts, arising from the first and second terms of Eq. (3.20):

$$\Sigma^{(T1)}(p) = \Gamma_i M^{-2}{}_{ij}\frac{\partial V_1}{\partial\lambda_j}, \tag{4.9}$$

$$\Sigma^{(T2)}(p) = \frac{i}{2(2\pi)^4}\Gamma_i(\theta_\alpha\theta_\beta\lambda)_i \times \int d^4k\,(k^2)^{-1}(\xi k^2+\mu^2)^{-1}{}_{\alpha\beta}. \tag{4.10}$$

At this point, we can begin to take note of the interesting cancellations among the various terms, and put the result in a ξ-independent form. Note first that

$$\begin{aligned}
(\theta_\alpha\lambda)_i\Gamma_i &= -\lambda_j(\theta_\alpha)_{ji}\Gamma_i\\
&= -\gamma_4\lambda_j[t_\alpha, \gamma_4\Gamma_j]\\
&= -\gamma_4[t_\alpha, \gamma_4 m]
\end{aligned}$$

and

$$\begin{aligned}
(\theta_\alpha\theta_\beta\lambda)_i\Gamma_i &= \lambda_j(\theta_\beta)_{jk}(\theta_\alpha)_{ki}\Gamma_i\\
&= \gamma_4\lambda_j[t_\beta, [t_\alpha, \gamma_4\Gamma_j]]\\
&= \gamma_4[t_\beta, [t_\alpha, \gamma_4 m]].
\end{aligned}$$

[See Eq. (2.14).] Also, the matrix in (4.6) may be written

$$\begin{aligned}\gamma_4[\gamma_4 m, t_\beta]t_\alpha+\gamma_4[\gamma_4 m, t_\alpha]t_\beta &= \gamma_4[t_\beta,[t_\alpha,\gamma_4 m]]+\gamma_4[\gamma_4 m, t_\beta t_\alpha]\\ &\to \gamma_4[t_\beta,[t_\alpha,\gamma_4 m]]+\gamma_4[-i\gamma_4\gamma_\mu p^\mu, t_\beta t_\alpha]\\ &= \gamma_4[t_\beta,[t_\alpha,\gamma_4 m]].\end{aligned}$$

These relations allow us to combine denominators using the identity

$$\left[\frac{1}{k^2}-\frac{(1-\xi)}{k^2+\mu^2}\right]\frac{1}{\xi k^2+\mu^2}=\frac{1}{k^2(k^2+\mu^2)} .$$

In this way, we find that the $A2$ and $\phi 2$ terms and the $A3$ and $T2$ terms combine to give the ξ-independent contributions

$$\begin{aligned}\Sigma^{(A\phi)}(p) &= \Sigma^{(A2)}(p)+\Sigma^{(\phi 2)}(p)\\ &= \frac{i}{(2\pi)^4}\int d^4k\,\gamma_4[\gamma_4 m, t_\beta]\Delta^\psi(p-k)\\ &\quad\times\gamma_4[\gamma_4 m, t_\alpha](k^2)^{-1}(k^2+\mu^2)^{-1}{}_{\alpha\beta},\end{aligned} \tag{4.11}$$

$$\begin{aligned}\Sigma^{(AT)}(p) &= \Sigma^{(A3)}+\Sigma^{(T2)}\\ &= \frac{i}{2(2\pi)^4}\gamma_4[t_\beta,[t_\alpha,[\gamma_4 m]]]\\ &\quad\times\int d^4k\,(k^2)^{-1}(k^2+\mu^2)^{-1}{}_{\alpha\beta}.\end{aligned} \tag{4.12}$$

Thus $\Sigma(p)$ may be written as the sum of terms

$$\begin{aligned}\Sigma(p) &= \Sigma^{(A1)}(p)+\Sigma^{(\phi 1)}(p)\\ &\quad+\Sigma^{(A\phi)}(p)+\Sigma^{(AT)}+\Sigma^{(T1)},\end{aligned} \tag{4.13}$$

every one of which is explicitly ξ-independent. As promised, the ξ dependence in the tadpole cancels against the ξ dependence in nontadpole graphs.

Although ξ-independent, $\Sigma(p)$ is still divergent. The divergence in the potential V_1 term has already been discussed in Sec. III, where it was shown that $\partial V_1/\partial\lambda_i$ may be replaced with the finite quantity (3.28). This leaves us with a linear divergence in the $A1$ and $\phi 1$ terms and a logarithmic divergence in the AT term. Rather than isolate these divergent terms to check explicitly that they do not affect zeroth-order fermion mass relations, as already done in Ref. 2, we will proceed directly with our calculation using a cutoff Λ to evaluate divergent integrals; we will then check that the physically relevant parts of $\Sigma(p)$ are cutoff-independent for $\Lambda\to\infty$.

The details of the calculation of the $A1$, $\phi 1$, $A\phi$, and AT terms are presented in Appendix E. It turns out to be extremely convenient here to leave the zeroth-order fermion mass matrix as a general, possibly nondiagonal, matrix, but to diagonalize the vector-meson mass matrix μ, writing[11]

$$\mu^2{}_{\alpha\beta}C_{\beta N}=\mu_N{}^2 C_{\alpha N}, \tag{4.14}$$

with the eigenvectors C_N forming a complete orthonormal set

$$\sum_N C_{\alpha N}C_{\beta N}=\delta_{\alpha\beta},$$

$$C_{\alpha N}C_{\alpha M}=\delta_{NM}.$$

We also diagonalize the scalar-meson mass matrix,

$$M^2{}_{ij}u_{jK}=M_K{}^2 u_{iK}, \tag{4.15}$$

with the eigenvectors u_K also forming a complete orthonormal set

$$\sum_K u_{iK}u_{jK}=\delta_{ij},$$

$$u_{iK}u_{iL}=\delta_{KL}.$$

The matrix generators associated with the vector mesons of definite mass are

$$\bar t_N\equiv C_{\alpha N}t_\alpha,\quad \bar\theta_N\equiv C_{\alpha N}\theta_\alpha, \tag{4.16}$$

while the Yukawa coupling matrices associated with scalar mesons of definite mass are

$$\bar\Gamma_K\equiv u_{iK}\Gamma_i . \tag{4.17}$$

Inspection of Eqs. (E1)–(E4) shows that the divergent parts of $\Sigma(p)$ are of the form

$$\begin{aligned}\Sigma_\infty &= \sum_N(am\bar t_N\bar t_N+b\gamma_4\bar t_N\gamma_4 m\bar t_N+c\gamma_4\bar t_N\bar t_N\gamma_4 m)\\ &\quad+\sum_K(dm\gamma_4\bar\Gamma_K\gamma_4\bar\Gamma_K+e\bar\Gamma_K m\bar\Gamma_K),\end{aligned} \tag{4.18}$$

where a through e are divergent constants. Specifically, we have

$$a=\frac{\ln\Lambda^2}{32\pi^2},\quad b=-\frac{3\ln\Lambda^2}{16\pi^2},\quad c=-\frac{\ln\Lambda^2}{32\pi^2},$$

$$d=-\frac{\ln\Lambda^2}{32\pi^2},\quad e=\frac{\ln\Lambda^2}{16\pi^2}$$

(but our discussion will not depend on these particular values). Because a, b, and c are N-independent and d and e are K-independent, we can rewrite this as

$$\begin{aligned}\Sigma_\infty &= amt_\alpha t_\alpha+b\gamma_4 t_\alpha\gamma_4 m t_\alpha\\ &\quad+c\gamma_4 t_\alpha t_\alpha\gamma_4 m+dm\gamma_4\Gamma_i\gamma_4\Gamma_i+e\Gamma_i m\Gamma_i .\end{aligned} \tag{4.19}$$

Using Eq. (2.19) to express m in terms of the matrices m_0 and Γ_i, we see that the change $-\Sigma_\infty$ in the fermion mass matrix produced by Σ_∞ is precisely the same as would be produced by adding a correction to the original Lagrangian

$$\Delta\mathcal{L} = -\bar{\psi}\Delta\Gamma_i\psi\phi_i - \bar{\psi}\Delta m_0\psi\,, \tag{4.20}$$

with

$$\Delta\Gamma_i = -a\Gamma_i t_\alpha t_\alpha - b\gamma_4 t_\alpha\gamma_4\Gamma_i t_\alpha - c\gamma_4 t_\alpha t_\alpha\gamma_4\Gamma_i - d\Gamma_i\gamma_4\Gamma_j\gamma_4\Gamma_j - e\Gamma_j\Gamma_i\Gamma_j\,, \tag{4.21}$$

$$\Delta m_0 = -am_0 t_\alpha t_\alpha - b\gamma_4 t_\alpha\gamma_4 m_0 t_\alpha - c\gamma_4 t_\alpha t_\alpha\gamma_4 m_0 - dm_0\gamma_4\Gamma_i\gamma_4\Gamma_j - e\Gamma_j m_0\Gamma_j\,. \tag{4.22}$$

However, by using (2.14), (2.11), and (2.8), and recalling the antisymmetry of the structure constants and θ matrices, we can easily check that $\Delta\Gamma_i$ and Δm_0 have the same G-transformation properties as Γ_i and m_0, i.e.,

$$[t_\alpha, \gamma_4\Delta\Gamma_i] = -(\theta_\alpha)_{ij}\gamma_4\Delta\Gamma_j\,, \tag{4.23}$$

$$[t_\alpha, \gamma_4\Delta m_0] = 0\,, \tag{4.24}$$

so that $\Delta\mathcal{L}$ is G-invariant. Also, $\Delta\mathcal{L}$ is Lorentz-invariant because the γ_4 factors in (4.21) just have the effect of changing the sign of all γ_5 terms in t_α and Γ_i. Finally, $\Delta\mathcal{L}$ is obviously renormalizable. But our original Lagrangian $\mathcal{L}$ was supposed to be the most general possible G-invariant Lorentz-invariant renormalizable Lagrangian that could be constructed from ϕ_i, ψ, and $A_{\alpha\mu}$, so adding $\Delta\mathcal{L}$ to $\mathcal{L}$ just amounts to a change in the parameters in $\mathcal{L}$. This allows us to absorb the infinite self-mass correction $-\Sigma_\infty$ into a renormalization of the parameters in the Lagrangian. Of greater immediate interest, we can now conclude that if the zeroth-order mass matrix m is subject to constraints which do not depend on the parameters in the Lagrangian, then such constraints will be unaffected by the divergent term Σ_∞, so that this term may be dropped.[18] The same reasoning allows us also to drop any other terms in $\Sigma(p)$ which are of the form (4.18). With this understanding, the individual terms in $\Sigma(p)$ are now

$$\Sigma_{\rm eff}^{(A1)} = \frac{1}{16\pi^2}\sum_N\int_0^1 dx[-2m\bar{t}_N(1-x) + 4\gamma_4\bar{t}_N\gamma_4 m]\ln\left(\mu_N^2 + \frac{m^2x^2}{1-x}\right)\bar{t}_N\,, \tag{4.25}$$

$$\Sigma_{\rm eff}^{(A\phi)} = \frac{1}{16\pi^2}\sum_N\frac{1}{\mu_N^2}\int_0^1 dx\{(1-x)m[\gamma_4 m, \bar{t}_N]\gamma_4 + \gamma_4[\gamma_4 m, \bar{t}_N]m\}\left\{\ln\left(\frac{m^2x^2}{1-x}\right) - \ln\left(\mu_N^2 + \frac{m^2x^2}{1-x}\right)\right\}\gamma_4[\gamma_4 m, \bar{t}_N]\,, \tag{4.26}$$

$$\Sigma_{\rm eff}^{(AT)} = \frac{1}{32\pi^2}\sum_N\gamma_4[\bar{t}_N,[\bar{t}_N, \gamma_4 m]]\ln\mu_N^2\,, \tag{4.27}$$

$$\Sigma_{\rm eff}^{(\phi 1)} = -\frac{1}{16\pi^2}\sum_K\int_0^1 dx[-(1-x)m\gamma_4\bar{\Gamma}_K\gamma_4 + \bar{\Gamma}_K m]\ln[m^2x^2 + M_K^2(1-x)]\bar{\Gamma}_K\,, \tag{4.28}$$

and using (3.29) in (4.9),

$$\Sigma_{\rm eff}^{(T1)} = \frac{1}{32\pi^2}\Gamma_i M^{-2}{}_{ij}\Big[f_{klj}(M^2\ln M^2)_{kl} - 16{\rm Tr}(m^3\ln m\Gamma_j) + 6\sum_N(\bar{\theta}_N{}^2\lambda)_j\mu_N{}^2\ln\mu_N{}^2\Big]\,. \tag{4.29}$$

(Here m is still a matrix which in general does not commute with $\bar{t}_N$.) It should be noted that the corrections to zeroth-order fermion mass relations produced by each term here are separately independent of how we choose our mass units in evaluating logarithms; this has already been shown for (4.29) in Sec. III, while for (4.25)–(4.28) the change in Σ produced by replacing m^2, M^2, and μ^2 in logarithms with λm^2, λM^2, and $\lambda\mu^2$ is of the form (4.18), and hence cannot affect zeroth-order mass relations.

The self-energy insertions are now all p-independent. They therefore yield a correction to the fermion mass matrix, given simply by

$$\delta m = -\Sigma_{\rm eff}^{(A1)} - \Sigma_{\rm eff}^{(A\phi)} - \Sigma_{\rm eff}^{(AT)} - \Sigma_{\rm eff}^{(\phi 1)} - \Sigma_{\rm eff}^{(T1)}\,. \tag{4.30}$$

It is useful to look at how these results simplify when we compute corrections to various specific kinds of zeroth-order mass relation

Type 1. Mass relations of type 1 arise solely because of the representation content of the scalar fields which enter in the Yukawa coupling,[2] rather than from constraints on λ_i. We note that both the AT and $T1$ terms are linear combinations of the Yukawa coupling matrices Γ_i. [For (4.28) this is obvious. For (4.27), use (2.14) twice.] Thus the AT and $T1$ terms cannot enter in corrections to zeroth-order mass relations of type 1, and the corrections are given by

$$\delta m = -\Sigma_{\rm eff}^{(A1)} - \Sigma_{\rm eff}^{(A\phi)} - \Sigma_{\rm eff}^{(\phi 1)}\,. \tag{4.31}$$

If the zeroth-order mass relation arises because the part of λ contributed by the scalar fields with Yukawa couplings is invariant under a subgroup S of G, then only the gauge couplings break S, and the $\phi 1$ term may be dropped.

Type 2. Mass relations of type 2 arise[4] because G invariance and renormalizability forces both the Yukawa interaction and the polynomial $P(\phi)$ to be invariant under some group $\overline{G}$ larger than G, while the solution λ_i of Eq. (2.15) happens to be invariant under some subgroup $\overline{S}$ of $\overline{G}$, so that m is also invariant under $\overline{S}$. In this case it is only the gauge couplings that break $\overline{S}$, so (4.29) simplifies to

$$\Sigma_{\text{eff}}^{(T1)}=\frac{3}{16\pi^2}\Gamma_i M^{-2}{}_{ij}\sum_N(\overline{\theta}_N{}^2\lambda)_j\mu_N{}^2\ln\mu_N{}^2, \quad (4.32)$$

and the $\phi 1$ term may be dropped. However, the AT and $T1$ terms as well as the $A1$ and $A\phi$ terms may contribute to the corrections to the $\overline{S}$ invariance of m.

Type 3. Mass relations of type 3 arise[3] because the solution λ_i of Eq. (2.15) is subject to some constraints for all values of the parameters in $P(\phi)$, other than constraints corresponding to the invariance of λ_i under some subgroup of a group of symmetries of $P(\phi)$. In this case, all terms in Eqs. (4.25)–(4.29) are potentially capable of contributing to the higher-order corrections to the zeroth-order mass relation.

It is also useful to see how the contributions to δm simplify in the likely (though by no means certain[2]) event that all vector particles except the photon are much heavier than all fermions. In this case, the m^2 term in the logarithm in (4.25) can be dropped in all terms except the photon term $N=\gamma$. Setting $\overline{t}_\gamma$ equal to the fermion charge e, we then have

$$\Sigma_{\text{eff}}^{(A1)}\simeq\frac{1}{16\pi^2}\int_0^1 dx\left\{\sum_{N\neq\gamma}\left[-2m\overline{t}_N{}^2(1-x)+4\gamma_4\overline{t}_N\gamma_4 m\overline{t}_N\right]\ln\mu_N{}^2+2e^2m(1+x)\ln\left(\frac{m^2x^2}{1-x}\right)\right\}.$$

The x integral is now very simple, and we find

$$\Sigma_{\text{eff}}^{(A1)}\simeq\frac{1}{16\pi^2}\sum_{N\neq\gamma}(-m\overline{t}_N{}^2+4\gamma_4\overline{t}_N\gamma_4 m\overline{t}_N)\ln\mu_N{}^2+\frac{3}{16\pi^2}e^2m\left(-\tfrac{1}{2}+\ln m^2\right). \quad (4.33)$$

With the same approximation of large $\mu_N{}^2$ for $N\neq\gamma$, we see that the $A\phi$ term (4.26) and the $\text{Tr}(m^3\ln m\Gamma_j)$ term in (4.29) are both negligible. Also, the ratio of the Yukawa to the gauge couplings is of the same order as the ratio of the fermion to the vector masses, so the $\phi 1$ term is also negligible. The AT term and the other $T1$ terms do contribute to this order, but these terms do not receive any contributions from the photon.

As an example, suppose for a moment that all vector bosons except the photon are degenerate, with common mass μ. As already remarked, we can evaluate the logarithms using any mass units we like, so let us choose to measure all masses in units of μ. The $\ln\mu_N{}^2$ terms in (4.33), (4.27), and (4.29) can then be dropped. For zeroth-order mass relations of types 1 and 2, the $M^2\ln M^2$ term in (4.29) may also be dropped, so for $m^2\ll\mu^2$, we have

$$\delta m\simeq-\Sigma_{\text{eff}}^{(A1)}$$

$$\simeq-\frac{3e^2}{16\pi^2}m\left[\tfrac{1}{2}+\ln\left(\frac{\mu^2}{m^2}\right)\right]. \quad (4.34)$$

This is just the old Weisskopf result[19] but with the ultraviolet cutoff replaced with a finite quantity, the vector-boson mass μ.

Although (4.34) applies only for the case of degenerate massive vector bosons, it should provide a pretty fair approximation to the corrections to type 1 mass relations, even when the massive vector-boson masses are somewhat different. This is why calculations[9] of the neutron-proton mass difference, in theories in which isospin is a type 1 approximate symmetry, persistently give the same wrong sign as in the Weisskopf formula. However, for mass relations of type 2 and 3, there is a possibility that the tadpole terms can be much larger than the $A1$ terms. That is, if the massive vector bosons are not strictly degenerate, and if one or more of the Higgs bosons (*not* the Goldstone or pseudo-Goldstone bosons) are much lighter than all the massive vector bosons, then the largest correction to the zeroth-order mass relations comes from the vector-boson tadpole term in (4.29), and so

$$\delta m\simeq-\frac{3}{16\pi^2}\Gamma_i M^{-2}{}_{ij}\sum_N(\overline{\theta}_N{}^2\lambda)_j\mu_N{}^2\ln\mu_N{}^2. \quad (4.35)$$

This can have any sign, suggesting that perhaps isospin may be an approximate symmetry of type 2 or 3 rather than type 1.

V. PSEUDOSYMMETRY BREAKING

In this section and the next, we will consider the possibility[4] that the G invariance and quartic character of the polynomial $P(\phi)$ require it to be invariant under some group $\overline{G}$ larger than G. As already remarked, those "pseudosymmetries" which are in $\overline{G}$ but not G, and which are not broken in zeroth order, generate approximate symmetries of "type 2" which are broken by higher-order effects. Our concern here is more with the pseudosymmetries which are already broken in zeroth order.

The existence of such pseudosymmetries renders the symmetry-breaking condition (2.15) for λ_i somewhat ambiguous.[14] Given any solution λ, we can find other solutions $D(g)\lambda$, where $D(g)$ is the matrix corresponding to any arbitrary element g of $\overline{G}$. If g is a member of the true symmetry group G, then λ and $D(g)\lambda$ are physically equivalent solutions, and this ambiguity is harmless. However, if g is in $\overline{G}$ but not in G, then it is only a symmetry of the polynomial $P(\phi)$ but not of the whole Lagrangian, so λ and $D(g)\lambda$ are physically inequivalent, and we need extra conditions to determine which is the physically relevant solution of λ.

Of course, even without pseudosymmetries, it is common to find a discrete set of possible solutions λ of Eq. (2.15), each corresponding to a different physical theory. The new element introduced by the pseudosymmetries is that we can have a *continuous* set of physically inequivalent solutions $D(g)\lambda$ of Eq. (2.15). To be more specific, if θ_A is any generator of the connected part of $\overline{G}$, then the $\overline{G}$ invariance of the polynomial $P(\phi)$ requires that, for all ϕ,

$$\frac{\partial P(\phi)}{\partial \phi_i}(\theta_A\phi)_i=0, \tag{5.1}$$

so the solutions to Eq. (2.15), which requires $\partial P/\partial\phi_i$ to vanish at $\phi=\lambda_i$, have $\overline{N}$ degrees of freedom, where $\overline{N}$ is the number of independent vectors of the form $\theta_A\lambda$. However, since λ and $\lambda+i\epsilon_\alpha\theta_\alpha\lambda$ are physically equivalent solutions of (2.15), the physically relevant ambiguity is only $\overline{N}-N$ dimensional, where N is the number of independent vectors of form $\theta_\alpha\lambda$. We need $\overline{N}-N$ extra conditions on λ to remove the ambiguity.

These conditions are supplied by the requirement (3.31):

$$\frac{\partial V_1(\lambda)}{\partial \lambda_i}u_i=0 \quad \text{if} \quad M^2{}_{ij}u_j=0, \tag{5.2}$$

which was derived in Sec. III from the requirement that scalar particles with vanishing zeroth-order mass should not have tadpoles. (This condition is also derived in Appendix C from the requirement that $\langle\phi_i'\rangle_0$ have a well-behaved perturbative expansion.) Each nonzero vector $\theta_A\lambda$ defines an eigenvector of M^2 with eigenvalue zero,

$$M^2\theta_A\lambda$$

[see Eq. (B5)], so (5.2) requires that

$$\frac{\partial V_1(\lambda)}{\partial\lambda_i}(\theta_A\lambda)_i=0\,. \tag{5.3}$$

The potential $V_1(\lambda)$ is invariant with respect to G, not $\overline{G}$, so this condition is automatically satisfied only for the generators θ_α of G, not for the other generators of $\overline{G}$. Thus the number of independent constraints here equals the number $\overline{N}$ of independent vectors $\theta_A\lambda$ minus the number N of independent vectors $\theta_\alpha\lambda$, which is just what we need.

In using (5.3), we may obviously discard any terms in $V_1(\lambda)$ which are invariant for all λ under the symmetry generated by θ_A. In particular, since the polynomial $P(\phi)$ is supposed to be invariant under the group $\overline{G}$ for all values of the parameters in $P(\phi)$, *all* quartic G-invariant polynomials in λ are invariant under $\overline{G}$, so we may drop the infinite part (3.26) of $V_1(\lambda)$, which was shown to be G-invariant and quartic in Sec. III. Also, since $\mathrm{Tr}(M^4\ln M^2)$ is invariant under any of the symmetries of the polynomial $P(\phi)$ (see Appendix D) we can drop this term in (3.25), so that (5.3) becomes simply

$$0=(\theta_A\lambda)_i\frac{\partial}{\partial\lambda_i}\left[3\,\mathrm{Tr}(\mu^4\ln\mu^2)-4\,\mathrm{Tr}(m^4\ln m^2)\right]. \tag{5.4}$$

Finally, in the most interesting cases, the symmetries of the polynomial $P(\phi)$ are also symmetries of the Yukawa term, so that $\mathrm{Tr}(m^4\ln m^2)$ is invariant under all these symmetries [see Appendix D again] and hence may be dropped from (5.4), leaving as our condition on λ just

$$0=(\theta_A\lambda)_i\frac{\partial}{\partial\lambda_i}\mathrm{Tr}(\mu^4\ln\mu^2), \tag{5.5}$$

or, using (3.4),

$$0=(\theta_A\lambda)_i(\{\theta_\alpha,\theta_\beta\}\lambda)_i(\mu^2\ln\mu^2)_{\alpha\beta}\,. \tag{5.6}$$

This condition plays an important role in calculations of the masses of the "pseudo-Goldstone" bosons carried out in Sec. VI. It also has an interesting variational interpretation. Consider the "orbit" in the representation space of the scalar fields, formed by solutions of Eq. (2.15) of the form

$$\lambda=D(g)\lambda_0, \tag{5.7}$$

where λ_0 is any particular solution of (2.15), and g sweeps over the group $\overline{G}$ of symmetries of $P(\phi)$.

Any infinitesimal change in g produces in λ a change of form

$$\delta\lambda = i\epsilon_A\theta_A\lambda\,,$$

with ϵ_A a set of arbitrary infinitesimal parameters. Hence Eq. (5.3) just tells us that the true physical λ is that particular solution of (2.15) which minimizes $V_1(\lambda)$ on the orbit (5.7) (or at least extremizes it.) Since $\overline{G}$ is a compact group and V_1 is a continuous function of λ, it must indeed have such a minimum on the orbit (5.7); hence, given any particular solution λ_0 of Eq. (2.15), we can be sure that there is another solution λ on the same orbit which also satisfies (5.3).

VI. PSEUDO-GOLDSTONE BOSON MASSES

There is an eigenvector $\theta_A\lambda$ of the scalar mass matrix M^2 with eigenvalue zero for each broken symmetry of the polynomial $P(\phi)$. Those eigenvectors associated with generators θ_α of true symmetries define the true Goldstone bosons,[20] which are eliminated by the Higgs mechanism.[13] However, if the polynomial $P(\phi)$ is also invariant under pseudosymmetries which are not symmetries of the whole Lagrangian, then the corresponding vectors $\theta_A\lambda$ define scalar fields with vanishing zeroth-order mass, which are not killed by the Higgs mechanism, but which pick up a finite mass from higher-order corrections. We shall now calculate the mass matrix of these "pseudo-Goldstone" bosons[4] in the one-loop approximation.

There are eleven diagrams here, shown in Fig. 3. The contributions of these diagrams at zero-boson four-momentum are as follows:

$$\Pi^{(AA)}_{ij}(0) = -\frac{2i}{(2\pi)^4}(\theta_\alpha\theta_\beta\lambda)_i(\theta_\gamma\theta_\delta\lambda)_j\int d^4k\,\Delta^A_{\alpha\mu,\gamma\nu}(k)\Delta^A_{\beta\lambda,\delta\rho}(k)\eta^{\mu\lambda}\eta^{\nu\rho}\,, \tag{6.1}$$

$$\Pi^{(\phi\phi)}_{ij}(0) = -\frac{i}{(2\pi)^4}f_{ikl}f_{jpq}\int d^4k\,\Delta^\phi_{kp}(k)\Delta^\phi_{lq}(k)\,, \tag{6.2}$$

$$\Pi^{(\phi A)}_{ij}(0) = \frac{i}{(2\pi)^4}(\theta_\alpha)_{ki}(\theta_\beta)_{lj}\int d^4k\,k^\mu k^\nu\Delta^A_{\mu\alpha,\nu\beta}(k)\Delta^\phi_{kl}(k)\,, \tag{6.3}$$

$$\Pi^{(\psi\psi)}_{ij}(0) = \frac{i}{(2\pi)^4}\int d^4k\,\mathrm{Tr}\{\Gamma_i\Delta^\psi(k)\Gamma_j\Delta^\psi(k)\}\,, \tag{6.4}$$

$$\Pi^{(\omega\omega)}_{ij}(0) = \frac{i}{(2\pi)^4\xi^2}(\theta_\gamma\theta_\delta\lambda)_i(\theta_\beta\theta_\alpha\lambda)_j\int d^4k\Delta^\omega_{\alpha\gamma}(k)\Delta^\omega_{\delta\beta}(k)\,, \tag{6.5}$$

$$\Pi^{(A)}_{ij}(0) = \frac{i}{(2\pi)^4}(\theta_\beta\theta_\alpha)_{ij}\int d^4k\,\Delta^A_{\alpha\mu,\beta\nu}(k)\eta^{\mu\nu}\,, \tag{6.6}$$

$$\Pi^{(\phi)}_{ij}(0) = \frac{i}{2(2\pi)^4}f_{ijkl}\int d^4k\,\Delta^\phi_{kl}(k)\,, \tag{6.7}$$

$$\Pi^{(T)}_{ij}(0) = \frac{i}{(2\pi)^4}f_{ijk}M^{-2}{}_{kl}T_l\,, \tag{6.8}$$

with T_l the total tadpole given by Eq. (3.16). After a lengthy calculation, using the constraints on f_{ijk} and f_{ijkl} derived in Appendix B, we find that

$$\begin{aligned}
\Pi_{ij}(0) = &-\frac{3i}{2(2\pi)^4}(\{\theta_\alpha,\theta_\beta\}\lambda)_i(\{\theta_\gamma,\theta_\delta\}\lambda)_j\int d^4k\,(k^2+\mu^2)^{-1}{}_{\alpha\gamma}(k^2+\mu^2)^{-1}{}_{\beta\delta}\\
&-\frac{i}{2(2\pi)^4}f_{ikl}f_{jpq}\int d^4k\,(k^2+M^2)^{-1}{}_{kp}(k^2+M^2)^{-1}{}_{lq}\\
&-\frac{i}{(2\pi)^4}\int d^4k\,(k^2)^{-1}(\xi k^2+\mu^2)^{-1}{}_{\alpha\beta}[-M^2\theta_\alpha(k^2+M^2)^{-1}\theta_\beta M^2+M^2\theta_\alpha(k^2+M^2)^{-1}M^2\theta_\beta+\theta_\alpha M^2(k^2+M^2)^{-1}\theta_\beta M^2]_{ij}\\
&+\frac{i}{(2\pi)^4}\int d^4k\,\mathrm{Tr}\{\Gamma_i(i\gamma_\lambda k^\lambda+m)^{-1}\Gamma_j(i\gamma_\lambda k^\lambda+m)^{-1}\}\\
&-\frac{i}{2(2\pi)^4}(M^2\theta_\gamma\theta_\alpha\lambda)_i(M^2\theta_\beta\theta_\delta\lambda)_j\int d^4k\,(k^2)^{-1}(\xi k^2+\mu^2)^{-1}{}_{\alpha\beta}(\xi k^2+\mu^2)^{-1}{}_{\gamma\delta}\\
&+\frac{3i}{2(2\pi)^4}(\{\theta_\beta,\theta_\alpha\})_{ij}\int d^4k\,(k^2+\mu^2)^{-1}{}_{\alpha\beta}+\frac{i}{2(2\pi)^4}f_{ijkl}\int d^4k\,(k^2+M^2)^{-1}{}_{kl}
\end{aligned}$$

$$-\frac{i}{2(2\pi)^4} f_{ijk}M^{-2}{}_{kl} f_{lpq}\int d^4k\,(k^2+M^2)^{-1}{}_{pq}+\frac{i}{(2\pi)^4} f_{ijk}M^{-2}{}_{kl}\int d^4k\,\mathrm{Tr}\{\Gamma_l(i\gamma_\lambda k^\lambda+m)^{-1}\}$$

$$-\frac{3i}{2(2\pi)^4} f_{ijk}M^{-2}{}_{kl}(\{\theta_\alpha,\theta_\beta\}\lambda)_l\int d^4k\,(k^2+\mu^2)^{-1}{}_{\alpha\beta}\,. \tag{6.9}$$

This is not yet ξ-independent, nor should we expect it to be, for we are not on the mass shell for general i and j. In order to specialize (6.9) to the Goldstone and pseudo-Goldstone directions, for which $q=0$ *is* on the zeroth-order mass shell, we must contract with $(\theta_A\lambda)_i(\theta_B\lambda)_j$. Again, the constraints proved in Appendix B come to our aid, and we find a great many cancellations. Specifically, the third and fifth terms drop out immediately, while the second, seventh, and eighth terms cancel. The remaining terms yield

$$\Pi_{AB}=-\frac{3i}{2(2\pi)^4}(\theta_A\lambda)_i(\{\theta_\alpha,\theta_\beta\}\lambda)_i(\theta_B\lambda)_j(\{\theta_\gamma,\theta_\delta\}\lambda)_j\int d^4k\,(k^2+\mu^2)^{-1}{}_{\alpha\gamma}(k^2+\mu^2)^{-1}{}_{\beta\delta}$$

$$+\frac{3i}{2(2\pi)^4}(\theta_A\lambda)_i(\{\theta_\beta,\theta_\alpha\}\theta_B\lambda)_i\int d^4k\,(k^2+\mu^2)^{-1}{}_{\alpha\beta}$$

$$+\frac{3i}{4(2\pi)^4}(\{\theta_A,\theta_B\}\lambda)_i(\{\theta_\alpha,\theta_\beta\}\lambda)_i\int d^4k\,(k^2+\mu^2)^{-1}{}_{\alpha\beta}$$

$$+\frac{i}{(2\pi)^4}(\theta_A\lambda)_i(\theta_B\lambda)_j\int d^4k\,\mathrm{Tr}\{\Gamma_i(i\gamma_\lambda k^\lambda+m)^{-1}\Gamma_j(i\gamma_\lambda k^\lambda+m)^{-1}\}$$

$$-\frac{i}{2(2\pi)^4}(\{\theta_A,\theta_B\}\lambda)_l\int d^4k\,\mathrm{Tr}\{\Gamma_l(i\gamma_\lambda k^\lambda+m)^{-1}\}\,, \tag{6.10}$$

where

$$\Pi_{AB}\equiv(\theta_A\lambda)_i(\theta_B\lambda)_j\Pi_{ij}(0)\,. \tag{6.11}$$

This is now ξ-independent, as expected. However, we still must worry about the presence of ultraviolet divergences.

It proves extremely convenient at this point to use Eqs. (3.18) and (3.19) to rewrite (6.10) in the economical form

$$\Pi_{AB}=-\tfrac{1}{2}\{L_A,L_B\}U\,, \tag{6.12}$$

where L_A is the Lie derivative

$$L_A\equiv(\theta_A\lambda)_i\frac{\partial}{\partial\lambda_i} \tag{6.13}$$

and U is a function of λ given by

$$U=-\frac{3i}{2(2\pi)^4}\int d^4k\,\mathrm{Tr}\ln(\mu^2+k^2)$$

$$+\frac{2i}{(2\pi)^4}\int d^4k\,\mathrm{Tr}\ln(m^2+k^2)\,. \tag{6.14}$$

We note that U is the same as the "potential" V_1 defined by Eq. (3.21), except that the scalar-meson term in V_1 does not appear in U. In fact, such a term could not contribute in (6.12) anyway, because $\mathrm{Tr}\ln(k^2+M^2)$ is necessarily invariant under the whole group $\overline{G}$ which leaves the polynomial $P(\phi)$ invariant (see Appendix D) and therefore L_AU vanishes for all λ and A. For this reason, (6.12) could just as well be written

$$\Pi_{AB}=-\tfrac{1}{2}\{L_A,L_B\}V_1\,. \tag{6.15}$$

Also, we should note here that the order of Lie derivatives acting on V_1 does not matter, because

$$[L_A,L_B]=[\theta_B,\theta_A]_{ij}\lambda_j\frac{\partial}{\partial\lambda_i}$$

$$=-iC_{ABC}L_C\,, \tag{6.16}$$

and, as shown in Sec. V, L_CV_1 vanishes for the physical value of λ. [See Eq. (5.3).] Hence (6.15) could just as well be written

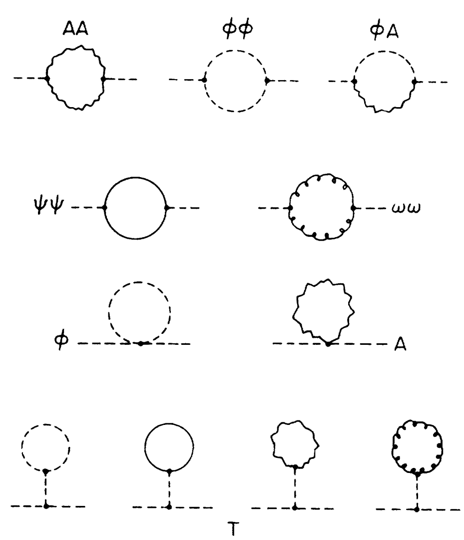

FIG. 3. Feynman diagrams for the scalar self-energy. (Conventions same as in Fig. 1.)

$$\Pi_{AB} = -L_A L_B V_1 = -L_B L_A V_1 . \quad (6.17)$$

We can now pause to verify Goldstone's theorem[20] in the one-loop approximation. If either θ_A or θ_B is one of the generators θ_α of true symmetries of the whole Lagrangian, then (6.17) shows that Π_{AB} vanishes, because $L_\alpha V_1$ vanishes for all λ. (See Appendix D.) It is only when both θ_A and θ_B are symmetries of the polynomial $P(\phi)$ but *not* of the whole Lagrangian that Π_{AB} can be nonzero.

The disposition of the infinities is now perfectly straightforward. It has already been shown in Sec. III that the infinite part of V_1 is a quartic G-invariant polynomial in λ. However, the pseudosymmetries are supposed to arise for *any* quartic G-invariant polynomial, so the infinite part of V_1 must be invariant under the transformation generated by θ_A and θ_B, and so cannot contribute in (6.17). Thus the potential V_1 in (6.17) can be replaced with the finite effective potential (3.27).

As already mentioned, the scalar-meson term in V_1 does not contribute to Π_{AB} because it is invariant under all the symmetries of the polynomial $P(\phi)$. In the most interesting cases, the Yukawa interaction is invariant under all pseudosymmetries,[4] so the spinor term in V_1 is also invariant under all the symmetries of the polynomial $P(\phi)$, and does contribute in (6.17). We have then, simply,

$$\Pi_{AB} = -\frac{3}{64\pi^2} L_A L_B \operatorname{Tr}\{\mu^4 \ln \mu^2\} . \quad (6.18)$$

To calculate the pseudo-Goldstone boson masses, we recall that the mass eigenvalues are defined by the poles of the corrected propagator

$$\Delta^{\phi\prime}_{ij}(k) = [k^2 + M^2 - \Pi(k^2)]^{-1}{}_{ij} ,$$

so for each mass value $\mathfrak{M}$, there is a vector u with

$$[-\mathfrak{M}^2 1 + M^2 - \Pi(-\mathfrak{M}^2)]_{ij} u_j = 0 . \quad (6.19)$$

For the pseudo-Goldstone bosons, u is close to an eigenvector $c_A \theta_A \lambda$ of M^2 with eigenvalue zero,

$$u = c_A \theta_A \lambda + \delta u ,$$

and the mass $\mathfrak{M}$ is close to zero. To first order in $\mathfrak{M}^2$, Π, and δu, Eq. (6.19) then gives

$$-\mathfrak{M}^2 c_A \theta_A \lambda + M^2 \delta u - \Pi(0) c_A \theta_A \lambda \simeq 0 . \quad (6.20)$$

Taking the scalar product of this equation with $\theta_B \lambda$ eliminates the unknown δu term, and gives

$$\mathfrak{M}^2 \bar{\mu}^2{}_{BA} c_A = -\Pi_{BA} c_A , \quad (6.21)$$

where $\bar{\mu}$ is an extension of the previously defined vector-boson mass matrix

$$\bar{\mu}^2{}_{BA} \equiv -(\theta_B \lambda)_i (\theta_A \lambda)_i , \quad (6.22)$$

and Π_{BA} is the quantity defined by Eq. (6.11). That is, the pseudo-Goldstone boson masses (squared) are the eigenvalues of the Hermitian matrix

$$\mathfrak{M}^2{}_{AB} \equiv -[\bar{\mu}^{-1} \Pi \bar{\mu}^{-1}]_{AB} . \quad (6.23)$$

Rather than continue this discussion in general terms, let us now turn to one particularly promising kind of pseudosymmetry, that which arises from the unlocking[4] of different scalar-field multiplets. Suppose that the scalar-field multiplet ϕ can be decomposed into two multiplets, χ and η, and that the most general quartic G-invariant polynomial in ϕ is invariant under separate G-transformations on χ or η. The generators θ_A of the symmetries of the polynomial $P(\phi)$ may then be labeled $\theta_{\alpha\chi}$ and $\theta_{\alpha\eta}$, with $\theta_{\alpha\chi}$ acting only on the χ multiplet and $\theta_{\alpha\eta}$ acting only on the η multiplet. The generators of the true symmetries of the whole Lagrangian are then just

$$\theta_\alpha = \theta_{\alpha\chi} + \theta_{\alpha\eta} .$$

Similarly, the vector-boson mass matrix may be expressed as

$$(\mu^2)_{\beta\alpha} = (\mu_\chi{}^2)_{\beta\alpha} + (\mu_\eta{}^2)_{\beta\alpha} ,$$

where $\mu_\chi{}^2$ and $\mu_\eta{}^2$ are the matrices defined by (6.22):

$$(\mu_\chi{}^2)_{\beta\alpha} = \bar{\mu}^2{}_{\beta\chi,\alpha\chi} = -(\theta_{\beta\chi}\lambda)_i (\theta_{\alpha\chi}\lambda)_i ,$$

$$(\mu_\eta{}^2)_{\beta\alpha} = \bar{\mu}^2_{\beta\eta,\alpha\eta} = -(\theta_{\beta\eta}\lambda)_i (\theta_{\alpha\eta}\lambda)_i .$$

Also, the mixed components of (6.22) vanish:

$$\bar{\mu}^2_{\beta\chi,\alpha\eta} = \bar{\mu}^2_{\beta\eta,\alpha\chi} = 0 .$$

Finally, the self-energy matrix Π_{AB} vanishes when θ_A or θ_B is one of the true symmetry generators θ_α, so that

$$\Pi_{\beta\chi,\alpha\chi} = -\Pi_{\beta\eta,\alpha\chi} = -\Pi_{\beta\chi,\alpha\eta} = +\Pi_{\beta\eta,\alpha\eta} \equiv \pi_{\beta\alpha} .$$

The eigenvalue equations (6.21) then become

$$\mathfrak{M}^2 \mu_\chi{}^2 u = -\pi u + \pi v ,$$

$$\mathfrak{M}^2 \mu_\eta{}^2 v = \pi v - \pi u ,$$

where u and v are the χ and η parts of the eigenvector

$$u_\alpha \equiv c_{\alpha\chi} , \quad v_\alpha \equiv c_{\alpha\eta} .$$

The solution of this eigenvalue problem may be written

$$u = -\mu_\chi{}^{-2} w , \quad v = +\mu_\eta{}^{-2} w ,$$

and the eigenvalue condition on $\mathfrak{M}^2$ now reads

$$\mathfrak{M}^2 w = \pi(\mu_\chi^{-2} + \mu_\eta^{-2}) w \,.$$

Although still very general, these results allow us to answer an interesting physical question. In general, (6.21) leads us to expect pseudo-Goldstone boson masses to be of order

$$\mathfrak{M} \sim e^2 \lambda \sim e\mu \,,$$

where e, λ, and μ are typical values of θ_α, λ_i, and μ_N. However, suppose that the vacuum expectation values of the η fields are much larger than those of the χ fields.[4] What is then the order of magnitude of the pseudo-Goldstone boson masses? Since $\mu_\eta^2 \gg \mu_\chi^2$, we can write (6.17) as

$$\pi_{\beta\alpha} = -\frac{3}{64\pi^2} L_{\beta\chi} L_{\alpha\chi} \mathrm{Tr}\{\mu^4 \ln \mu^2\}$$

$$\simeq -\frac{3}{32\pi^2} L_{\beta\chi} L_{\alpha\chi} \mathrm{Tr}\{\mu_\eta^2 (\ln\mu_\eta^2 - \tfrac{1}{2}) \mu_\chi^2\} \,.$$

The term $-\frac{1}{2}$ may be dropped, because $\mathrm{Tr}\{\mu_\eta^2 \mu_\chi^2\}$ is a quartic G-invariant polynomial in λ, and hence, by virtue of the assumed unlocking, must be invariant under $\theta_{\alpha\chi}$ and $\theta_{\alpha\eta}$ transformations as well as θ_α transformations. (By the same reasoning, the mass units used to calculate the logarithm do not matter.) A simple calculation then gives

$$\pi_{\beta\alpha} \simeq -\frac{3}{32\pi^2} \mathrm{Tr}\{\mu_\eta^2 \ln \mu_\eta^2 [\mathcal{T}_\beta, [\mathcal{T}_\alpha, \mu_\chi^2]]\} \,,$$

where $\mathcal{T}_\alpha$ is the matrix

$$(\mathcal{T}_\alpha)_{\beta\gamma} \equiv -i C_{\alpha\beta\gamma} \,.$$

We see that π is of order $e^2 \mu_\eta^2 \mu_\chi^2$, where e is a typical gauge coupling constant. The matrix $\pi(\mu_\chi^{-2} + \mu_\eta^{-2})$ is then of order $e^2 \mu_\eta^2$, so the pseudo-Goldstone boson masses are of order $e\mu_\eta$, and hence of order $e\mu$, where μ is a typical vector-boson mass. This result is mildly discouraging if we think that μ is of order 50 GeV and want to consider the pion to be a pseudo-Goldstone boson. However, in this case the "one-loop" approximation is quite inappropriate, because the Yukawa couplings are not of order e but rather of order 1.

VII. COMPARISON WITH THE VARIATIONAL APPROACH

The results obtained in the foregoing sections have a variational interpretation which is interesting both for its own sake and as a point of contact with the work of Coleman and Weinberg.[15]

Coleman and Weinberg define a "potential" by summing up all connected single-particle irreducible graphs having any number of external zero-momentum boson lines, each associated with a c-number factor ϕ_i, using for this purpose the original "stage one" Yang-Mills theory with unbroken symmetry in Landau gauge. Their result for this potential, keeping only graphs with not more than one loop, is just

$$V(\phi) = P(\phi) + V_1(\phi) \,, \tag{7.1}$$

with V_1 given by Eq. (3.21). They then calculate the vacuum expectation value of ϕ from the condition that $V(\phi)$ be stationary:

$$\frac{\partial V(\phi)}{\partial \phi_i} = 0 \quad \text{at} \quad \phi = \langle\phi\rangle_0 \,. \tag{7.2}$$

In zeroth order, this gives $\langle\phi_i\rangle_0$ equal to the vector λ_i defined by Eq. (2.15). Setting

$$\langle\phi_i\rangle_0 = \lambda_i + \delta\lambda_i \,, \tag{7.3}$$

the terms of the first order in $\delta\lambda_i$ and V_1 give

$$\left.\frac{\partial V_1(\phi)}{\partial \phi_i}\right|_{\phi=\lambda} + M^2{}_{ij} \delta\lambda_j = 0 \,. \tag{7.4}$$

This result may be compared with the result for $\delta\lambda$ derived here in the broken-symmetry formalism. According to Eq. (C4), we have

$$-i(2\pi)^4 M^2{}_{ij} \delta\lambda_j + T_i = 0 \,. \tag{7.5}$$

Equations (7.4) and (7.5) are the same if the one-loop tadpole is given by

$$T_i = -i(2\pi)^4 \frac{\partial V_1}{\partial \lambda_i} \,. \tag{7.6}$$

However, Eq. (3.20) shows that (7.6) is *not* true for general ξ, but only in Landau gauge, where $\xi \to \infty$.

In a sense it should not be surprising that (7.4) and (7.5) are inconsistent for gauges other than Landau gauge, because Coleman and Weinberg[15] used Landau gauge to calculate $V(\phi)$. However, we can easily see that for general ξ, the tadpole given by Eq. (3.20) is not the derivative of *any* potential, because

$$\frac{\partial T_i}{\partial \lambda_j} \neq \frac{\partial T_j}{\partial \lambda_i} \,.$$

This failure of the functional approach can be traced back to Eq. (A8).[21] The Lagrangian contains λ_i, not only in the boson field $\phi_i = \phi_i' + \lambda_i$, but also in the gauge-determining term,

$$-\tfrac{1}{2}\xi[\partial_\mu A_\alpha^\mu + i\xi^{-1}(\theta_\alpha\lambda)_i \phi_i]^2 \,, \tag{7.7}$$

and in the scalar-ghost interaction,

$$-\xi^{-1} \omega_\alpha^* \omega_\beta (\theta_\beta \theta_\alpha \lambda)_i \phi_i \,. \tag{7.8}$$

Hence in general, differentiation with respect to λ_i is not the same as adding an extra zero-momentum ϕ_i' line. It is only in Landau gauge, where $\xi \to \infty$, that (7.7) and (7.8) become λ-independent, so that the functional formalism can be used without modification.

Coleman and Weinberg[15] also calculate the scalar-boson self-energy from the second derivatives of the potential at its minimum:

$$M^2{}_{ij} - \Pi_{ij}(0) = \frac{\partial^2 V(\phi)}{\partial\phi_i \partial\phi_j}\bigg|_{\phi=\lambda+\delta\lambda} .$$

Expanding to first order in V_1 and $\delta\lambda$, this gives

$$\Pi_{ij}(0) = -\frac{\partial^2 V_1(\phi)}{\partial\phi_i\partial\phi_j}\bigg|_{\phi=\lambda} - f_{ijk}\delta\lambda_k .$$

Restricting our attention to the Goldstone and pseudo-Goldstone directions this yields the formula

$$\begin{aligned}\Pi_{AB} &= (\theta_A\lambda)_i(\theta_B\lambda)_j\Pi_{ij}(0)\\ &= -(\theta_A\lambda)_i(\theta_B\lambda)_j\frac{\partial^2 V_1(\lambda)}{\partial\lambda_i\partial\lambda_j} + (M^2\theta_A\theta_B\lambda)_k\delta\lambda_k .\end{aligned}$$

[See Eq. (B8).] But Eq. (7.4) allows us to rewrite this in the form

$$\begin{aligned}\Pi_{AB} &= -(\theta_A\lambda)_i(\theta_B\lambda)_j\frac{\partial^2 V_1(\lambda)}{\partial\lambda_i\partial\lambda_j} - (\theta_A\theta_B\lambda)_j\frac{\partial V_1(\lambda)}{\partial\lambda_j}\\ &= -\left[(\theta_B\lambda)_j\frac{\partial}{\partial\lambda_j}\right]\left[(\theta_A\lambda)_i\frac{\partial}{\partial\lambda_i}\right]V_1(\lambda),\end{aligned}$$

in agreement with Eq. (6.17). We see that where the functional approach works, in Landau gauge, the pseudo-Goldstone boson mass matrix is indeed given by the curvature of the potential $V(\lambda)$ near its minimum, as expected.

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APPENDIX A: DERIVATION OF THE EFFECTIVE LAGRANGIAN[16]

According to Faddeev and Popov,[17] whenever the action functional $I[a]$ of a set of fields $q_p(x)$ is invariant under some group of local transformations, it may be replaced with an effective action

$$I_{\text{eff}}[q] = I[q] - \tfrac{1}{2}\int f_\alpha(x)f_\alpha(x)d^4x - i\ln\operatorname{Det}M, \tag{A1}$$

where $f_\alpha(x)$ is an arbitrary function of $q_p(x)$ and its derivatives, which determines our choice of gauge, and M is a "matrix" defined by the gauge transformation property of f_α: Under an infinitesimal gauge transformation $1+i\epsilon_\alpha T_\alpha$, f undergoes the change

$$\delta f_\alpha(x) = \int M_{\alpha x,\beta y} f_\beta(y)d^4y . \tag{A2}$$

By introducing a set of complex spinless fermion "ghost" fields $\omega_\alpha(x)$, the effective action may also be written

$$I_{\text{eff}}[q,\omega] = I[q] - \tfrac{1}{2}\int f_\alpha(x)f_\alpha(x)d^4x - \int \omega^*_\alpha(x)M_{\alpha x,\beta y}\omega_\beta(y)d^4x\,d^4y . \tag{A3}$$

The propagators of the fields $q_p(x)$ and $\omega_\alpha(x)$ are simply the reciprocals of the "matrices" appearing in the terms of I_{eff} quadratic in q or ω, while the interaction vertices are determined in the usual way by the nonquadratic terms in I_{eff}.

In our case, the action I is given by the integral of the Lagrangian (2.1):

$$I[A,\phi,\psi] = \int d^4x\,\mathcal{L}. \tag{A4}$$

When we replace $\phi_i(x)$ in $\mathcal{L}$ with $\lambda_i + \phi'_i(x)$, we find a quadratic term which mixes the scalar and gauge fields:

$$-i(\theta_\alpha\lambda)_i\partial_\mu\phi'_i A_\alpha{}^\mu .$$

In order eventually to cancel this term, we choose the gauge-determining function f_α as[12]

$$f_\alpha = \sqrt{\xi}\,[\partial_\mu A_\alpha{}^\mu + i\xi^{-1}(\theta_\alpha\lambda)_i\phi'_i] , \tag{A5}$$

with ξ a free gauge-determining parameter. Under an infinitesimal gauge transformation $1+i\epsilon_\alpha T_\alpha$, the fields $A_\alpha{}^\mu$ and ϕ' undergo the changes

$$\begin{aligned}\delta A_\alpha{}^\mu &= C_{\alpha\beta\gamma}A_\beta{}^\mu\epsilon_\gamma - \partial_\mu\epsilon_\alpha,\\ \delta\phi'_i &= i\epsilon_\alpha(\theta_\alpha)_{ij}\phi_j ,\end{aligned}$$

and therefore

$$\delta f_\alpha = \sqrt{\xi}\,[-C_{\alpha\beta\gamma}\partial_\mu(A_\gamma{}^\mu\epsilon_\beta) - \Box^2\epsilon_\alpha - \xi^{-1}(\theta_\alpha\lambda)_i(\theta_\beta\phi)_i\epsilon_\beta] .$$

This may be put in the form (A2), with

$$M_{\alpha x,\beta y} = \sqrt{\xi}\,[-C_{\alpha\beta\gamma}A_\gamma{}^\mu(x)\partial_\mu\delta^4(x-y) - \delta_{\alpha\beta}\Box^2\delta^4(x-y) + \xi^{-1}(\theta_\beta\theta_\alpha\lambda)_i\phi_i(x)\delta^4(x-y)] . \tag{A6}$$

Using (A5) and (A6) in (A3) shows that the effective action may be expressed in terms of an effective Lagrangian

$$I_{\rm eff}=\int d^4x\,\mathcal{L}_{\rm eff}(x)\,, \tag{A7}$$

with

$$\mathcal{L}_{\rm eff}=-\tfrac{1}{4}F_{\alpha\mu\nu}F_\alpha{}^{\mu\nu}-\tfrac{1}{2}(D_\mu\phi)_i(D^\mu\phi)_i-\bar\psi\gamma^\mu D_\mu\psi-\bar\psi m_0\psi-P(\phi)-\bar\psi\Gamma_i\psi\phi_i-\tfrac{1}{2}\xi[\partial_\mu A_\alpha{}^\mu+i\xi^{-1}(\theta_\alpha\lambda)_i\phi_i']^2$$
$$-(\partial_\mu\omega_\alpha^*)C_{\alpha\beta\gamma}\omega_\beta A_\gamma{}^\mu-\partial_\mu\omega_\alpha^*\partial^\mu\omega_\alpha-\xi^{-1}\omega_\alpha^*\omega_\beta(\theta_\beta\theta_\alpha\lambda)_i\phi_i\ . \tag{A8}$$

(A factor $\sqrt{\xi}$ has been absorbed into the normalization of the ghost fields.)

We may immediately read off from (A8) the part of the effective Lagrangian quadratic in the fields ϕ', ψ, ω, and A. As promised, the mixed terms drop out, and we find

$$\mathcal{L}_{\rm eff}^{\rm quad}=-\tfrac{1}{4}(\partial_\mu A_{\alpha\nu}-\partial_\nu A_{\alpha\mu})(\partial^\mu A_\alpha{}^\nu-\partial^\nu A_\alpha{}^\mu)-\tfrac{1}{2}(\partial_\mu\phi_i')(\partial^\mu\phi_i')-\tfrac{1}{2}\mu^2{}_{\alpha\beta}A_{\alpha\mu}A_\beta{}^\mu-\bar\psi\gamma^\mu\partial_\mu\psi$$
$$-\bar\psi m\psi-\tfrac{1}{2}M^2{}_{ij}\phi_i'\phi_j'-\tfrac{1}{2}\xi(\partial_\mu A_\alpha{}^\mu)(\partial_\nu A_\alpha{}^\nu)+\frac{1}{2\xi}(\theta_\alpha\lambda)_i(\theta_\alpha\lambda)_j\phi_i'\phi_j'-\partial_\mu\omega_\alpha^*\partial^\mu\omega_\alpha-\xi^{-1}\mu^2{}_{\alpha\beta}\omega_\alpha^*\omega_\beta\,,$$

or, discarding various gradient terms,

$$\mathcal{L}_{\rm eff}^{\rm quad}=\tfrac{1}{2}A_\alpha{}^\nu[\delta_{\alpha\beta}\eta_{\mu\nu}\Box^2-(1-\xi)\delta_{\alpha\beta}\partial_\mu\partial_\nu-\mu^2{}_{\alpha\beta}\eta_{\mu\nu}]A_\beta{}^\mu$$
$$+\tfrac{1}{2}\phi_i'[\delta_{ij}\Box^2-M^2{}_{ij}+\xi^{-1}(\theta_\alpha\lambda)_i(\theta_\alpha\lambda)_j]\phi_j'-\bar\psi[\gamma^\mu\partial_\mu+m]\psi+\omega_\alpha^*[\delta_{\alpha\beta}\Box^2-\xi^{-1}\mu^2{}_{\alpha\beta}]\omega_\beta\,, \tag{A9}$$

where M^2, m, and μ^2 are the matrices defined by Eq. (2.17)–(2.19). The propagators of the A, ϕ', ψ, and ω fields are thus defined by the differential equations

$$[\delta_{\alpha\beta}\delta_\nu^\mu\Box^2-(1-\xi)\delta_{\alpha\beta}\partial_\nu\partial^\mu-\mu^2{}_{\alpha\beta}\delta_\nu^\mu]G^A_{\mu\beta,\lambda\gamma}(x,y)=-\delta^4(x-y)\eta_{\nu\lambda}\delta_{\alpha\gamma}\,, \tag{A10}$$

$$[\delta_{ij}\Box^2-M^2{}_{ij}+\xi^{-1}(\theta_\alpha\lambda)_i(\theta_\alpha\lambda)_j]G^\phi_{jk}(x,y)=-\delta^4(x-y)\delta_{ik}\,, \tag{A11}$$

$$[\gamma^\mu\partial_\mu+m]_{nm}G^\psi_{ml}(x,y)=\delta^4(x-y)\delta_{nl}\,, \tag{A12}$$

$$[\delta_{\alpha\beta}\Box^2-\xi^{-1}\mu^2{}_{\alpha\beta}]G^\omega_{\beta\gamma}(x,y)=-\delta^4(x-y)\delta_{\alpha\gamma}\,, \tag{A13}$$

with the usual causal boundary conditions. We write the various G's in terms of momentum-space propagators Δ:

$$G(x,y)\equiv(2\pi)^{-4}\int d^4k\,\Delta(k)e^{ik\cdot x}\ .$$

Equations (2.24)–(2.27) then follow directly from (A10)–(A13), except that to solve (A11) we must make use of (2.17) and (2.20) as well.

The effective interaction here is just the part of $\mathcal{L}_{\rm eff}$ not quadratic in ϕ', A, ψ, or ω:

$$\mathcal{L}_{\rm eff}'=\mathcal{L}_{\rm eff}-\mathcal{L}_{\rm eff}^{\rm quad}. \tag{A14}$$

This immediately yields Eq. (2.21).

APPENDIX B: CONSTRAINTS ON SCALAR INTERACTIONS

The interactions (2.21) involve unknown scalar-field coupling constants f_{ijk} and f_{ijkl}. Very fortunately, it turns out that the only ones of these coupling constants that we actually need to know for most purposes are those determined by certain constraints, which arise from the invariance properties of the polynomial $P(\phi)$.

The constraints we need may be derived by a method originally developed by Glashow and myself[22] while preparing a paper on SU(3)×SU(3)-symmetry breaking. Let θ_A denote the imaginary antisymmetric matrix representing any of the generators of the group of symmetries of the polynomial $P(\phi)$. This group always contains the gauge group G, so that θ_A may be one of the θ_α matrices, but in some cases[4] $P(\phi)$ is invariant under a group $\bar G$ larger than G, and the set of θ_A may include matrices linearly independent of the θ_α. The invariance of $P(\phi)$ requires that

$$0=\frac{\partial P(\phi)}{\partial\phi_i}(\theta_A\phi)_i\ . \tag{B1}$$

Differentiating successively with respect to ϕ_j, ϕ_k, and ϕ_l gives

$$0=\frac{\partial^2P(\phi)}{\partial\phi_i\partial\phi_j}(\theta_A\phi)_i+\frac{\partial P(\phi)}{\partial\phi_i}(\theta_A)_{ij}\,, \tag{B2}$$

$$0=\frac{\partial^3P(\phi)}{\partial\phi_i\partial\phi_j\partial\phi_k}(\theta_A\phi)_i+\frac{\partial^2P(\phi)}{\partial\phi_i\partial\phi_j}(\theta_A)_{ik}+\frac{\partial^2P(\phi)}{\partial\phi_i\partial\phi_k}(\theta_A)_{ij}\,, \tag{B3}$$

and

$$0=\frac{\partial^4 P(\phi)}{\partial\phi_i\partial\phi_j\partial\phi_k\partial\phi_l}(\theta_A\phi)_i+\frac{\partial^3 P(\phi)}{\partial\phi_i\partial\phi_j\partial\phi_k}(\theta_A)_{il}+\frac{\partial^3 P(\phi)}{\partial\phi_i\partial\phi_j\partial\phi_l}(\theta_A)_{ik}+\frac{\partial^3 P(\phi)}{\partial\phi_i\partial\phi_k\partial\phi_l}(\theta_A)_{ij}\ . \tag{B4}$$

Setting $\phi=\lambda$ in (B2) gives

$$M^2\theta_A\lambda=0\,, \tag{B5}$$

of which (2.20) is a special case. Setting $\phi=\lambda$ in (B3) and (B4) gives our desired constraints:

$$0=f_{ijk}(\theta_A\lambda)_i+[M^2,\theta_A]_{jk}\ , \tag{B6}$$

$$0=f_{ijkl}(\theta_A\lambda)_i+f_{ijk}(\theta_A)_{il}+f_{ijl}(\theta_A)_{ik}+f_{ikl}(\theta_A)_{ij}\ . \tag{B7}$$

From (B5)–(B7) we may derive further useful relations. Multiplying (B6) by $(\theta_B\lambda)_k$ [where θ_B is any other matrix generating a symmetry of $P(\phi)$] and using (B5), we find

$$f_{ijk}(\theta_A\lambda)_i(\theta_B\lambda)_k=-(M^2\theta_A\theta_B\lambda)_j\ . \tag{B8}$$

Also, multiplying (B7) by $(\theta_B\lambda)_j$ and using (B6), we find

$$f_{ijkl}(\theta_A\lambda)_i(\theta_B\lambda)_j=[\theta_A,[\theta_B,M^2]]_{kl}-f_{ikl}(\theta_A\theta_B\lambda)_i\ . \tag{B9}$$

APPENDIX C

We have based our quantization procedure in the text on a shifted field ϕ_i' defined as $\phi_i-\lambda_i$, where λ_i is the *zeroth-order* vacuum expectation value of ϕ_i given as an extremum of the polynomial $P(\phi)$. If instead we defined a shifted field

$$\tilde{\phi}_i=\phi_i-\langle\phi_i\rangle_0\,, \tag{C1}$$

with $\langle\phi_i\rangle_0$ the *true* vacuum expectation value of ϕ_i, then all tadpole graphs would have to cancel, and therefore

$$-i(2\pi)^4\left(\frac{\partial P(\phi)}{\partial\phi_i}\right)_{\phi=\langle\phi\rangle_0}+\tilde{T}_i=0\,, \tag{C2}$$

where $\tilde{T}_i$ is the sum of tadpole graphs with one or more loops, calculated using the shifted field (C1) to define our quantization procedure.

In the one-loop approximation, $\langle\phi_i\rangle_0$ will be close to an extremum λ_i of $P(\phi)$,

$$\langle\phi_i\rangle_0=\lambda_i+\delta\lambda_i\ , \tag{C3}$$

and $\tilde{T}_i$ will be small, given to an adequate approximation by the tadpole T_i calculated in Sec. III. In this approximation (C2) becomes simply

$$-i(2\pi)^4 M^2{}_{ij}\delta\lambda_j+T_i=0\ . \tag{C4}$$

Now, (C4) puts an immediate constraint on the tadpole T_i: If M^2 has an eigenvector u with eigenvalue zero,

$$M^2u=0\,, \tag{C5}$$

then, multiplying (C4) by u_i, we must have

$$u_iT_i=0. \tag{C6}$$

[In particular, we must have

$$(\theta_A\lambda)_iT_i=0\,,$$

where θ_A is the generator of any symmetry of the polynomial.] As long as (C6) is satisfied, (C4) may be solved to give

$$\delta\lambda_i=-i(2\pi)^{-4}(M^{-2})_{ij}T_j\ , \tag{C7}$$

with an appropriate interpretation of M^{-2}. But (C6) allows us to replace M^{-2} with the propagator Δ^ϕ, so

$$\delta\lambda_i=-i(2\pi)^{-4}\Delta^\phi_{ij}(0)T_j\ . \tag{C8}$$

Thus the effect of one-loop corrections to the vacuum expectation value of ϕ_i is precisely the same as the effect of the tadpole graphs encountered when we use a shifted field $\phi_i'\equiv\phi_i-\lambda_i$, as in the text.

APPENDIX D

For the sake of completeness, and as an exercise in the use of the formal apparatus developed here, in this appendix we will prove the following rather obvious theorem:

Let $F(z)$ be any function of a single variable z, analytic in some region of the z plane. Let $F(M^2)$, $F(m)$, or $F(\mu^2)$ be the matrix obtained by substituting M^2, m, or μ^2 for z in the power series expansion of $F(z)$. Then this matrix has a G-invariant trace, in the sense that

$$(\theta_\alpha\lambda)_i\frac{\partial}{\partial\lambda_i}\mathrm{Tr}F=0\ .$$

Using Eqs. (3.2), (3.3), and (3.4) gives immediately

$$\frac{\partial}{\partial\lambda_i}\mathrm{Tr}F(M^2)=f_{jki}F'(M^2)_{kj}\ , \tag{D1}$$

$$\frac{\partial}{\partial\lambda_i}\mathrm{Tr}F(m)=\tfrac{1}{4}(\Gamma_i)_{mn}F'(m)_{nm}\,, \tag{D2}$$

$$\frac{\partial}{\partial\lambda_i}\mathrm{Tr}F(\mu^2)=(\{\theta_\gamma,\theta_\beta\}\lambda)_iF'(\mu^2)_{\beta\gamma}\,, \tag{D3}$$

a prime denoting differentiation with respect to the argument. [The factor $\frac{1}{4}$ appears in (D2) because the trace on the left involves a sum over particle labels only, while the trace on the right includes a sum over Dirac indices as well.] Multiplying (D1) by $(\theta_\alpha\lambda)_i$ and using Eq. (B6) gives

$$(\theta_\alpha\lambda)_i \frac{\partial}{\partial\lambda_i}\,\mathrm{Tr}F(M^2) = \mathrm{Tr}\{[\theta_\alpha, M^2]F'(M^2)\}$$
$$= \mathrm{Tr}\{\theta_\alpha[M^2, F'(M^2)]\}$$
$$= 0\,. \qquad \text{(D4)}$$

Multiplying (D2) by $(\theta_\alpha\lambda)_i$ and using (2.14), (2.11), and (2.19) gives

$$(\theta_\alpha\lambda)_i \frac{\partial}{\partial\lambda_i}\,\mathrm{Tr}F(m) = \tfrac{1}{4}\,\mathrm{Tr}\{\gamma_4[t_\alpha, \gamma_4\Gamma_i\lambda_i]\,F'(m)\}$$
$$= \tfrac{1}{4}\,\mathrm{Tr}\{\gamma_4[t_\alpha, \gamma_4 m]\,F'(m)\}$$
$$= \tfrac{1}{4}\,\mathrm{Tr}\{[t_\alpha, m]\,F'(m)\}$$
$$= \tfrac{1}{4}\,\mathrm{Tr}\{t_\alpha[m, F'(m)]\}$$
$$= 0\,. \qquad \text{(D5)}$$

Finally, multiplying (D3) by $(\theta_\alpha\lambda)_i$ gives

$$(\theta_\alpha\lambda)_i \frac{\partial}{\partial\lambda_i}\,\mathrm{Tr}F(\mu^2) = (\{\theta_\gamma, \theta_\beta\}\lambda)_i(\theta_\alpha\lambda)_i\,F'(\mu^2)_{\beta\gamma}$$
$$= -(\theta_\alpha\{\theta_\gamma, \theta_\beta\}\lambda)_i F'(\mu^2)_{\beta\gamma}\,. \qquad \text{(D6)}$$

But using (2.5) and (2.17), we have

$$\lambda_i(\theta_\alpha\theta_\gamma\theta_\beta\lambda)_i = iC_{\alpha\gamma\delta}\,\mu^2{}_{\delta\beta} + iC_{\alpha\beta\delta}\,\mu^2{}_{\gamma\delta} + \lambda_i(\theta_\gamma\theta_\beta\theta_\alpha\lambda)_i\,,$$

and therefore

$$\lambda_i(\theta_\alpha\{\theta_\gamma, \theta_\beta\}\lambda)_i = iC_{\alpha\gamma\delta}\,\mu^2{}_{\delta\beta} + iC_{\alpha\beta\delta}\,\mu^2{}_{\gamma\delta}\,. \qquad \text{(D7)}$$

Using (D7) in (D6) gives then

$$(\theta_\alpha\lambda)_i \frac{\partial}{\partial\lambda_i}\,\mathrm{Tr}F(\mu^2) = \mathrm{Tr}\{[\mathcal{T}_\alpha, \mu^2]\,F'(\mu^2)\}$$
$$= \mathrm{Tr}\{\mathcal{T}_\alpha[\mu^2, F'(\mu^2)]\}$$
$$= 0\,, \qquad \text{(D8)}$$

where $\mathcal{T}_\alpha$ is the matrix

$$(\mathcal{T}_\alpha)_{\beta\gamma} = -iC_{\alpha\beta\gamma}\,.$$

Thus $F(M^2)$, $F(m)$, and $F(\mu^2)$ have G-invariant traces, as was to be proved.

Precisely the same method can be used to show that $\mathrm{Tr}F(M^2)$ is invariant under any group of transformations which leave the polynomial $P(\phi)$ invariant, and that $\mathrm{Tr}F(m)$ is invariant under any group of transformations which leave the Yukawa term $\gamma_4\Gamma_i\lambda_i$ invariant. However, $\mathrm{Tr}F(\mu^2)$ is in general not expected to be invariant under any group of transformations larger than the fundamental gauge group G.

APPENDIX E: CALCULATION OF THE FERMION SELF-ENERGY

This appendix will describe in detail how the various terms in $\Sigma(p)$ are calculated. For this purpose, it is very convenient to diagonalize the μ^2 matrix,[11] with

$$\mu^2{}_{\alpha\beta}C_{\beta N} = \mu_N{}^2 C_{\alpha N}\,,$$
$$\sum_N C_{\alpha N}C_{\beta N} = \delta_{\alpha\beta}\,,$$
$$C_{\alpha N}C_{\alpha M} = \delta_{NM}\,,$$

and write

$$\bar{t}_N = C_{\alpha N}t_\alpha\,.$$

Let us now consider the individual terms in $\Sigma(p)$.

A1. The *A*1 term is given by Eq. (4.4) as

$$\Sigma^{(A1)}(p) = \frac{i}{(2\pi)^4}\sum_N \int d^4k\,\gamma^\mu \bar{t}_N\left[\frac{-i\gamma_\lambda(p-k)^\lambda + m}{(p-k)^2 + m^2}\right]\gamma_\mu \bar{t}_N (k^2 + \mu_N{}^2)^{-1}\,.$$

Even though m is a matrix and μ_N is a number, we can combine denominators here in the usual way, and write

$$\Sigma^{(A1)}(p) = \frac{i}{(2\pi)^4}\sum_N \int_0^1 dx \int d^4k\,\gamma^\mu \bar{t}_N[-i\gamma_\lambda(p-k)^\lambda + m][(k-px)^2 + p^2x(1-x) + m^2x + \mu_N{}^2(1-x)]^{-2}\gamma_\mu \bar{t}_N\,.$$

We are only interested here in the one-loop correction to the fermion mass matrix, and to this order p^2 in the denominator may be replaced with $-m^2$. Shifting the variable of integration $k \to k + px$ and picking up a surface term (which in the end will prove irrelevant) gives

$$\Sigma^{(A1)}(p) = \frac{i}{(2\pi)^4}\sum_N \Big\{\tfrac{1}{4}\pi^2\gamma^\mu \bar{t}_N\gamma_\lambda p^\lambda \gamma_\mu \bar{t}_N$$
$$+ \int_0^1 dx \int d^4k\,\gamma^\mu \bar{t}_N[-i\gamma_\lambda(p^\lambda(1-x) - k^\lambda) + m][k^2 + m^2x^2 + \mu_N{}^2(1-x)]^{-2}\gamma_\mu \bar{t}_N\Big\}.$$

The term proportional to $\gamma_\lambda k^\lambda$ drops out upon symmetric integration, while the term proportional to $\gamma_\lambda p^\lambda$ and m may be rewritten, using

$$\gamma^\mu \bar{t}_N \gamma_\lambda p^\lambda \gamma_\mu = \gamma_4 \bar{t}_N \gamma_4 \gamma^\mu \gamma_\lambda p^\lambda \gamma_\mu$$
$$= -2\gamma_4 \bar{t}_N \gamma_4 \gamma_\lambda p^\lambda$$
$$= -2\gamma_\lambda p^\lambda \bar{t}_N$$
$$\to -2im\bar{t}_N ,$$
$$\gamma^\mu \bar{t}_N m \gamma_\mu = \gamma_4 \bar{t}_N \gamma_4 \gamma^\mu m \gamma_\mu$$
$$= 4\gamma_4 \bar{t}_N \gamma_4 m ,$$

and therefore

$$\Sigma^{(A1)}(p) = \frac{i}{(2\pi)^4} \sum_N \left\{ -\tfrac{1}{2} i\pi^2 m \bar{t}_N \bar{t}_N + \int_0^1 dx \int d^4k \left[-2m\bar{t}_N(1-x) + 4\gamma_4 \bar{t}_N \gamma_4 m\right]\left[k^2 + m^2x^2 + \mu_N{}^2(1-x)\right]^{-2} \bar{t}_N \right\} .$$

Cutting off the k integral at Λ and discarding terms of order $1/\Lambda^2$, we find

$$\Sigma^{(A1)}(p) = -\frac{\pi^2}{(2\pi)^4} \sum_N \left\{ -\tfrac{1}{2} m \bar{t}_N \bar{t}_N + \int_0^1 dx\left[-2m\bar{t}_N(1-x) + 4\gamma_4 \bar{t}_N \gamma_4 m\right] \ln\left(\frac{\Lambda^2}{m^2x^2 + \mu_N{}^2(1-x)}\right) \bar{t}_N \right\} . \tag{E1}$$

$A\phi$. The $A\phi$ term is given by Eq. (4.11) as

$$\Sigma^{(A\phi)}(p) = \frac{i}{(2\pi)^4} \sum_N \int d^4k\, \gamma_4[\gamma_4 m, \bar{t}_N]\left[-i\gamma_\lambda(p-k)^\lambda + m\right]$$
$$\times\left[(p-k)^2 + m^2\right]^{-1} \gamma_4[\gamma_4 m, \bar{t}_N](k^2)^{-1}(k^2 + \mu_N{}^2)^{-1} .$$

Combining denominators, this may be written

$$\Sigma^{(A\phi)}(p) = \frac{2i}{(2\pi)^4} \int_0^1 dy \int_0^y dx \sum_N \int d^4k\, \gamma_4[\gamma_4 m, \bar{t}_N]\left[-i\gamma_\lambda(p-k)^\lambda + m\right]$$
$$\times\left[(k-px)^2 + p^2x(1-x) + m^2x + \mu_N{}^2(1-y)\right]^{-3} \gamma_4[\gamma_4 m, \bar{t}_N] .$$

Replacing p^2 in the denominator with $-m^2$, shifting $k \to k + px$, and performing a convergent symmetric integration gives

$$\Sigma^{(A\phi)}(p) = -\frac{\pi^2}{(2\pi)^4} \int_0^1 dy \int_0^x dx \sum_N \int d^4k\, \gamma_4[\gamma_4 m, \bar{t}_N]\left[-i\gamma_\lambda p^\lambda(1-x) + m\right]\left[m^2x^2 + \mu_N{}^2(1-y)\right]^{-1} \gamma_4[\gamma_4 m, \bar{t}_N] .$$

To eliminate the $\gamma_\lambda p^\lambda$ term, we note that

$$\gamma_4[\gamma_4 m, \bar{t}_N]\gamma_\lambda p^\lambda = m\bar{t}_N \gamma_\lambda p^\lambda - \gamma_4 \bar{t}_N \gamma_4 m \gamma_\lambda p^\lambda$$
$$= m\gamma_\lambda p^\lambda \gamma_4 \bar{t}_N \gamma_4 - \gamma_\lambda p^\lambda \bar{t}_N m$$
$$\to im^2 \gamma_4 \bar{t}_N \gamma_4 - im\bar{t}_N m$$
$$= im[\gamma_4 m, \bar{t}_N]\gamma_4 .$$

Interchanging the order of integration and evaluating the y integral, we have then

$$\Sigma^{(A\phi)}(p) = -\frac{\pi^2}{(2\pi)^4} \sum_N \frac{1}{\mu_N{}^2} \int_0^1 dx \left\{ m[\gamma_4 m, \bar{t}_N]\gamma_4(1-x) + \gamma_4[\gamma_4 m, \bar{t}_N] m \right\}$$
$$\times\left\{ \ln(m^2x^2) - \ln\left[m^2x^2 + \mu_N{}^2(1-x)\right] \right\} \gamma_4[\gamma_4 m, \bar{t}_N] . \tag{E2}$$

AT. The AT term is given by Eq. (4.12) as

$$\Sigma^{(AT)} = \frac{i}{2(2\pi)^4} \sum_N \gamma_4[\bar{t}_N, [\bar{t}_N, \gamma_4 m]] \int d^4k (k^2)^{-1}(k^2 + \mu_N{}^2)^{-1} .$$

Cutting off the integral at Λ and discarding terms of order Λ^{-2}, we easily find

$$\Sigma^{(AT)} = -\frac{\pi^2}{2(2\pi)^4}\sum_N \gamma_4[\bar{t}_N,[\bar{t}_N,\gamma_4 m]]\ln\left(\frac{\Lambda^2}{\mu_N{}^2}\right) . \tag{E3}$$

ϕ1. To calculate the $\phi 1$ term it is convenient to introduce a complete orthonormal set of eigenvectors u_K, with

$$M^2 u_K = M_K{}^2 u_K ,$$
$$u_{Ki}u_{Li} = \delta_{KL} ,$$
$$\sum_K u_{Ki}u_{Kj} = \delta_{ij} ,$$
$$\bar{\Gamma}_K \equiv u_{Ki}\Gamma_i .$$

Equation (4.7) then gives

$$\Sigma^{(\phi 1)}(p) = -\frac{i}{(2\pi)^4}\sum_K\int d^4k(k^2+M_K{}^2)^{-1}\bar{\Gamma}_K\left[\frac{-i\gamma_\lambda(p-k)^\lambda+m}{(p-k)^2+m^2}\right]\bar{\Gamma}_K .$$

Combining denominators in the usual way, we have

$$\Sigma^{(\phi 1)}(p) = -\frac{i}{(2\pi)^4}\int_0^1 dx\int d^4k\bar{\Gamma}_K[-i\gamma_\lambda(p-k)^\lambda+m][(k-px)^2+p^2x(1-x)+m^2x+M_K{}^2(1-x)]^{-2}\bar{\Gamma}_K .$$

Again we replace p^2 in the denominator with $-m^2$ and shift the variable of integration, and find

$$\Sigma^{(\phi 1)}(p) = -\frac{i}{(2\pi)^4}\sum_K\left\{\tfrac{1}{4}\pi^2\bar{\Gamma}_K\gamma_\lambda p^\lambda\bar{\Gamma}_K+\int_0^1 dx\int d^4k\,\bar{\Gamma}_K[i\gamma_\lambda(p^\lambda(1-x)-k^\lambda)+m][k^2+m^2x^2+M_K{}^2(1-x)]^{-2}\bar{\Gamma}_K\right\}.$$

The term proportional to $\gamma_\lambda k^\lambda$ drops out upon symmetric integration, while the other terms may be rewritten, making the substitution

$$\bar{\Gamma}_K\gamma_\lambda p^\lambda = \gamma_\lambda p^\lambda\gamma_4\bar{\Gamma}_K\gamma_4 - im\gamma_4\bar{\Gamma}_K\gamma_4 ,$$

and therefore

$$\Sigma^{(\phi 1)}(p) = -\frac{i}{(2\pi)^4}\sum_K\left\{\tfrac{1}{4}\pi^2 im\gamma_4\bar{\Gamma}_K\gamma_4\bar{\Gamma}_K+\int_0^1 dx\int d^4k[-(1-x)m\gamma_4\bar{\Gamma}_K\gamma_4+\bar{\Gamma}_K m][k^2+m^2x^2+M_K{}^2(1-x)]^{-2}\bar{\Gamma}_K\right\}.$$

Cutting off the k integral at Λ and discarding terms of order $1/\Lambda^2$, we find

$$\Sigma^{(\phi 1)} = \frac{\pi^2}{(2\pi)^4}\sum_K\left\{\tfrac{1}{4}m\gamma_4\bar{\Gamma}_K\gamma_4\bar{\Gamma}_K+\int_0^1 dx[-(1-x)m\gamma_4\bar{\Gamma}_K\gamma_4+\bar{\Gamma}_K m]\ln\left(\frac{\Lambda^2}{m^2x^2+M_K{}^2(1-x)}\right)\bar{\Gamma}_K\right\}. \tag{E4}$$

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