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Metric of Two Spinning Charged Sources in Equilibrium

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Using the approach of Israel, Wilson, and Perjes we give the explicit form of the metric corresponding to two identical Kerr-Newman sources in equilibrium under their mutual electromagnetic and gravitational forces, with their spins oppositely oriented along a given axis. Symmetries, the complete analytic extension, the limit of infinite separation of the sources, and the two types of solution with vanishing separation are discussed.

A class of static solutions of the Einstein-Maxwell equations corresponding to an equilibrium distribution of charged sources was found in 1947 by Majumdar[1] and Papapetrou.[2] These solutions have recently been generalized to the stationary case (sources with spin) by Israel and Wilson[3] and Perjes[4] (IWP). The first-order set of partial differential equations characterizing the IWP solutions must be solved to obtain the metric explicitly.

One member of the class of IWP solutions is the Kerr-Newman solution with charge equal to mass (in units with $G=c=1$) as has been proved in that case by direct integration of the IWP differential equations.[3,4] It is of interest to find the solution for several Kerr-Newman sources. The result would provide a new explicit exact solution of the Einstein-Maxwell equations. It could, for example, serve as the basis for an analysis of solutions with slightly different parameters by means of perturbation or related techniques.

The problem of forces and torques between spinning objects has been extensively treated.[5,6,7] All these analyses, however, are based on the assumption that one of the bodies is a test particle whose own effect can be neglected. The solution we present in this paper allows for this interaction in the realistic case of a two-body problem, and provides a standard against which results obtained for test particles may be judged. It is interesting that the solution here presented demonstrates the balance (correct to all multipole orders) of the forces between two spinning sources.

In the present paper, we give the explicit stationary metric corresponding to two identical Kerr-Newman sources in equilibrium under their mutual gravitational and electromagnetic forces, with their spins oppositely oriented along a given axis. We confirm that the result obtained has the expected properties. To our knowledge, this is the first explicit nonstatic two-body solution of the Einstein-Maxwell equation. It is of interest in connection with certain aspects of coalescence, such as changes in singularity structure and possible horizon formation, and the uniqueness of the final configuration, as well as in analyzing the force balance and the electromagnetic field, which can only be given explicitly in this highly relativistic regime now that the analytic expression for the metric is known.

The IWP class of metrics is written in the standard form

$$ds^2 = -f^{-1}\gamma_{mn}dx^m dx^n + f(\omega_m dx^m + dt)^2 , \tag{1}$$

where f, ω_m, and γ_{mn} are independent of t. The line element $\gamma_{mn}\, dx^m dx^n$ corresponds to a Euclidean 3-space (in arbitrary coordinates). Any function U of the spatial coordinates which satisfies Laplace's equation in the flat 3-space generates an IWP metric by means of the equations

$$f = |U|^{-2} \tag{2}$$

and

$$\epsilon^{npq}\partial_p \omega_q = i\gamma^{1/2} f^{-1}\gamma^{nm}\partial_m \ln(U/U^*) , \tag{3}$$

where U^* is the complex conjugate of U. The electromagnetic field follows from γ_{mn}, U, and ω_p by means of the expression given in Refs. 3 and 4.

The solution corresponding to two identical Kerr-Newman sources, with charge equal to mass, lying in equilibrium on the z axis at $\pm b$, each with angular momentum per unit mass a pointing toward the origin, is generated by[3,4]

$$U = 1 + \frac{m}{R_1} + \frac{m}{R_2} , \tag{4}$$

with

$$R_1^{\,2} = x^2 + y^2 + (z - b - ia)^2 \tag{5}$$

and

$$R_2^{\,2} = x^2 + y^2 + (z + b + ia)^2 . \tag{6}$$

In order to obtain the explicit form of the metric, one must solve Eq. (3) for ω_j. We use spherical coordinates

$$x + iy = r \sin\theta\, e^{i\phi} , \qquad z = r\cos\theta , \tag{7}$$

in terms of which

$$R_1^{\,2} = r^2 - 2lr\cos\theta + l^2 \tag{8}$$

and

$$R_2^{\,2} = r^2 + 2lr\cos\theta + l^2 , \tag{9}$$

where

$$l = b + ia . \tag{10}$$

The axial symmetry implies that the right-hand side of Eq. (3) vanishes when the index $n = \phi$. Therefore, the coordinate t can be chosen such that ω_r and ω_θ vanish, since the addition of a scalar function of x^m to t changes ω_j by a gradient. Then Eq. (3) reduces to

$$\frac{\partial\omega_\phi}{\partial\theta} = -\mathrm{Im}\left\{2mr^2\sin\theta\left(1 + \frac{m}{R_1^*} + \frac{m}{R_2^*}\right) \times\left[r\left(\frac{1}{R_1^{\,3}} + \frac{1}{R_2^{\,3}}\right) - l\cos\theta\left(\frac{1}{R_1^{\,3}} - \frac{1}{R_2^{\,3}}\right)\right]\right\}, \tag{11}$$

and

$$\frac{\partial\omega_\phi}{\partial r} = \mathrm{Im}\left[2mlr\sin^2\theta\left(1 + \frac{m}{R_1^*} + \frac{m}{R_2^*}\right)\left(\frac{1}{R_1^{\,3}} - \frac{1}{R_2^{\,3}}\right)\right] , \tag{12}$$

where Im denotes the imaginary part. One can check directly that (11) and (12) satisfy the integrability condition

$$\frac{\partial}{\partial r}\left(\frac{\partial}{\partial\theta}\,\omega_\phi\right) = \frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial r}\,\omega_\phi\right) .$$

After a very lengthy calculation, one obtains the integral of (11) in the form

$$\omega_\phi = -\mathrm{Im}\left\{\frac{2m}{R_1}[l - r\cos\theta] + \frac{im^2(r^2 - l^2)}{2a(r^2 - |l|^2)}\frac{R_1^*}{R_1} + \frac{m^2}{2b}\frac{(r^2 - l^2)}{(r^2 + |l|^2)}\frac{R_1^*}{R_2}\right\} + (\text{all previous terms with } a \to -a,\ b \to -b, \text{ where } l = b + ia) . \tag{13}$$

No additive function of r appears, for reasons discussed further on. Note that the branches of the square-root functions R_1 and R_2 can be chosen independently. Equations (1), (2), (4), and (13) give the explicit expression for the two-body metric in this case.

For the following discussion, it is convenient to introduce two redundant sets of oblate spheroidal coordinates, (r_1, θ_1, ϕ_1) and (r_2, θ_2, ϕ_2), centered at $z = +b$ and $z = -b$, respectively. Each point (x, y, z) is related to the new coordinates as follows:

$$\begin{aligned} x + iy &= [(r_1 - m)^2 + a^2]^{1/2}\sin\theta_1\, e^{i\phi} , \\ z &= b + (r_1 - m)\cos\theta_1 , \\ x + iy &= [(r_2 - m)^2 + a^2]^{1/2}\sin\theta_2\, e^{i\phi} , \\ z &= -b + (r_2 - m)\cos\theta_2 . \end{aligned} \tag{14}$$

Now R_1 and R_2 assume the simple forms

$$R_1 = r_1 - m - ia\cos\theta_1 \tag{15}$$

and

$$R_2 = r_2 - m + ia\cos\theta_2 . \tag{16}$$

Replacing $(r_j - m, \theta_j)$ by $(m - r_j, \pi - \theta_j)$ leaves (x, y, z) unchanged, but does change the sign of R_j ($j = 1$ or 2), thus changing the branch of the square-root function.

In the new coordinates, (13) becomes

$$\omega_\phi = \frac{(m^2 - 2mr_1)a\sin^2\theta_1}{(r_1 - m)^2 + a^2\cos^2\theta_1} - \frac{(m^2 - 2mr_2)a\sin^2\theta_2}{(r_2 - m)^2 + a^2\cos^2\theta_2} + m^2aF[(2b)^{-1}(r^2 + |l|^2)H - (G + bH)], \tag{17}$$

where

$$F = \{[(r_1 - m)^2 + a^2\cos^2\theta_1]^{-1} - [(r_2 - m)^2 + a^2\cos^2\theta_2]^{-1}\}(r^2 + |l|^2)^{-1}, \tag{18a}$$

$$G = (r_1 - m)(r_2 - m) + a^2\cos\theta_1\cos\theta_2, \tag{18b}$$

$$H = (r_2 - m)\cos\theta_1 - (r_1 - m)\cos\theta_2, \tag{18c}$$

and

$$\begin{aligned} r^2 + |l|^2 &= (r_1 - m)^2 + a^2(1 + \sin^2\theta_1) + 2b^2 + 2b(r_1 - m)\cos\theta_1 \\ &= (r_2 - m)^2 + a^2(1 + \sin^2\theta_2) + 2b^2 - 2b(r_2 - m)\cos\theta_2 . \end{aligned} \tag{18d}$$

Each of the first two terms in (17) is the ω_ϕ of a single Kerr-Newman source with $q = m$.[3,4]

Equation (12) requires that on the symmetry axis, where $\sin\theta$ vanishes, ω_ϕ is a constant, which must be set equal to zero so as to ensure an asymptotically Minkowskian metric. The right-hand side of Eq. (17) vanishes on the symmetry axis, so that it is correct as it stands, without an added function of r. To obtain that result, one notes that the first two terms of Eq. (17) clearly vanish on the symmetry axis. Moreover, if one solves for r_1 in terms of r_2 by means of Eq. (14), and notes that $\cos^2\theta_2 = 1$ on the symmetry axis, then one finds that the factor in square brackets in Eq. (17) vanishes. Furthermore, one confirms by inspection that (17) vanishes as expected when $a = 0$.

There are four regions: I, in which $(r_1 - m)$ and $(r_2 - m)$ (equal, in coordinate-free form, to $\mathrm{Re}R_1$ and $\mathrm{Re}R_2$, respectively) are positive; II_1 and II_2, in which they have opposite signs; and III, in which they are both negative. One would expect by symmetry (since the spins are oppositely directed) that, in regions I and III, ω_ϕ would vanish in the equatorial plane ($z = 0$), as well as when the two particles "coalesce" ($b = 0$). In the former case, Eq. (14) implies that $r_1 = r_2$ and $\cos\theta_1 = -\cos\theta_2$, from which it follows that ω_ϕ vanishes. In the latter case, it is necessary to consider the limit as b approaches zero, since b^{-1} appears in the third term of Eq. (17). For small b, one finds from Eq. (14) that

$$r_1 = r_2 + O(b),$$

$$\cos\theta_1 = \cos\theta_2 + O(b),$$

$$F = O(b),$$

and

$$H = O(b),$$

whence

$$\omega_\phi = O(b).$$

As a final check on the metric, fix (r_1, θ_1, ϕ_1) and let b become large, effectively carrying the second source off to infinity. One finds from Eq. (14) that in the above limit

$$|(r_2 - m)| = O(b),$$

$$\theta_2 = O(b),$$

and

$$r^2 + |l|^2 = 2b^2 + O(b),$$

so that all terms but the first in Eq. (17) vanish as b^{-1}, leaving

$$\omega_\phi = \frac{(m^2 - 2mr_1)a\sin^2\theta_1}{(r_1 - m)^2 + a^2\cos^2\theta_1}, \tag{19}$$

which is the ω_ϕ for a single Kerr-Newman source with $q = m$, and angular momentum per unit mass a in the $-z$ direction. We now turn to a discussion of further properties of our metric.

Rings of infinite redshift occur where $g_{tt} = 0$, or $U = \infty$. At the first ring $r_1 - m = 0$ and $\theta_1 = \frac{1}{2}\pi$, while at the second ring $r_2 - m = 0$ and $\theta_2 = \frac{1}{2}\pi$. In order to get, say, from region I to II_1, one must change the sign of $r_1 - m$, which entails passing through the interior of the first ring.

Naked-ring singularities occur at $U = 0$, where one component of the stress-energy tensor,

$$8\pi T_{tt} = \left(\gamma^{mn}\frac{\partial U}{\partial x^m}\frac{\partial U^*}{\partial x^n}\right)|U|^{-6}, \tag{20}$$

diverges.[8] Apart from exceptional cases such as $b = 0$ (see below) or $a = 0$, which is treated by Hartle and Hawking,[9] the real and imaginary parts of $U = 0$ will give two independent equations for the two unknowns, e.g., r_1, θ_1, thus determining a set of rings. We shall also see in the following that T_{tt} may sometimes diverge at a ring of infinite redshift, where $U = \infty$.

The circle r_1, θ_1, t = constant is a closed timelike line if

$$-g_{\phi\phi} = f^{-1}\gamma_{\phi\phi} - f(\omega_\phi)^2 < 0 . \tag{21}$$

This inequality is indeed usually satisfied near a ring singularity where $f=|U|^{-2}$ approaches $+\infty$, ω_ϕ does not vanish, and

$$\gamma_{\phi\phi}=[(r_1-m)^2+a^2]\sin^2\theta_1$$

is finite. In this paper we investigate the so-called electrovac solutions, in which source-free electromagnetic fields provide the stress-energy. It is customarily assumed that if corresponding exterior solutions were found in nature, closed timelike lines and singularities not surrounded by event horizons would not occur, thanks to an interior solution; that is to say, the electrovac condition would not hold everywhere.

The problem of dynamical coalescence is a very difficult one, which may be conveniently split into two parts, namely (a) change in singularity structure and horizon formation, and (b) emission of gravitational radiation. Consideration of the quasistationary approach of our two sources through a sequence of equilibrium configurations, corresponding to successively smaller values of b, may throw light on part (a) of the more general problem of dynamical coalescence. The condition that the charge equal the mass is necessary to allow part (a) to be focused upon by use of the electromagnetic field to produce equilibrium.

The Majumdar-Papapetrou geometry corresponding to many Reissner-Nordström black holes, each with charge equal to mass, has the metric[1,2,9]

$$ds^2=U^{-2}dt^2-U^2\gamma_{mn}\,dx^m\,dx^n\,, \tag{22}$$

where

$$U=1+\sum_i(m_i/r_i)\,.$$

If the masses are equal for a system of N such black holes, the area S of a spherical surface enclosing them satisfies $S\geq 4\pi(Nm)^2$. The equal sign is only possible in the limit that the horizons $(r_i=0)$ touch one another. If coalescence were to occur, the horizon of the resulting black hole would have area $4\pi(Nm)^2$. Examples given by Brill and Lindquist[10] in the analysis of time-symmetric solutions of the Einstein-Maxwell equations suggest that the formation of a horizon may normally have to be accompanied by radiation of energy with consequent binding of the set of black holes. It is interesting that quasistatic coalescence of our two sources does not yield a Reissner-Nordström black hole. Rather, one obtains the following results.

When $b=0$, the two rings of infinite redshift coincide. It is then impossible to change the signs of r_1-m and r_2-m independently. There are two types of solutions, according to whether the interior of the redshift ring connects regions I and III or II_1 and II_2. In the former case,

$$\begin{aligned}&r_1-m=r_2-m\,,\\&\cos\theta_1=\cos\theta_2\,,\\&U=1+m[(R_1)^{-1}+(R_1^*)^{-1}]\,.\end{aligned}\tag{23}$$

Since U is real-valued, ω_ϕ vanishes. The solution is static. Expansion of the axially symmetric U in powers of the spherical radius defined by Eq. (7) shows that it differs from the Reissner-Nordström form $U=1+2m/r$ (for mass $=\pm$charge $=2m$) by higher multipoles which depend on even integral powers of a.

In this static solution, solving $U=0$ for the radius of the singularity yields in general two roots:

$$r_1=\begin{cases}-(m^2-a^2\cos^2\theta_1)^{1/2} & \text{and}\\ +(m^2-a^2\cos^2\theta_1)^{1/2}, & \text{if } a\cos\theta_1\neq 0\,.\end{cases}\tag{24}$$

When $a\cos\theta_1=0$, the second root is spurious. Assuming $a\neq 0$, a gap in the singularity's "surface" thus occurs at $r_1-m=0$, $\theta_1=\frac{1}{2}\pi$ – the location of the ring of infinite redshift. Although the singularity has vanishing surface area according to Eqs. (1) and (2), its structure is much more complicated than that of the point singularity in the Reissner-Nordström solution.

For the other type of solution with $b=0$,

$$\begin{aligned}&r_1-m=m-r_2\,,\\&\cos\theta_1=-\cos\theta_2\,,\\&U=1+2iam\cos\theta_1[(r_1-m)^2+a^2\cos^2\theta_1]^{-1}\,.\end{aligned}\tag{25}$$

Expanding U in powers of r_1 and comparing with that for a single Kerr-Newman particle allows the mass and angular momentum to be read off directly.[4] The coefficient of $(r_1)^{-1}$ is the mass and the imaginary part of the coefficient of $-\cos\theta_1$ $(r_1)^{-2}$ is the angular momentum (taken positive along the $+z$ direction). In this case, the mass vanishes and the angular momentum is $-2am$.

U does not vanish anywhere. But Eq. (20) enables one to show that the redshift ring is a singularity: T_{tt} diverges as one approaches this ring in the equatorial plane from outside ($\theta_1=\frac{1}{2}\pi$, $r_1-m\neq 0$). In Eq. (17), the "interaction" term vanishes, leaving

$$\omega_\phi=\frac{4m(m-r_1)a\sin^2\theta_1}{(r_1-m)^2+a^2\cos^2\theta_1}\,.\tag{26}$$

The values of the electric and magnetic field at a point in space-time become well-defined only when a local observer is specified. It is natural to choose an observer with velocity $u^\nu=dx^\nu/ds$ along the symmetry directions $\partial/\partial\phi$, $\partial/\partial t$ of the field such that $g_{0\phi}$ vanishes with respect to his local comoving frame.[11] This choice is clearly suitable for any axially symmetric stationary geometry; in the ergosphere of a Kerr-Newman black hole it yields, as desired, a timelike world line. The nonzero components of u^ν are

$$\begin{aligned} u^t &= \frac{dt}{ds} \\ &= (g_{tt} - g_{t\phi}{}^2/g_{\phi\phi})^{-1/2}, \\ u^\phi &= \frac{d\phi}{ds} \\ &= -(g_{t\phi}/g_{\phi\phi})(g_{tt} - g_{t\phi}{}^2/g_{\phi\phi})^{-1/2}. \end{aligned} \tag{27}$$

It follows that all components of u_ν vanish except for u_t. The electric and magnetic fields defined, respectively, by the Lorentz force on a unit electric and magnetic monopole moving with velocity u^ν are given by

$$E^\nu = F^{\nu t} u_t, \quad B^\nu = {}^*F^{\nu t} u_t, \tag{28}$$

where

$${}^*F^{\mu\nu} = \tfrac{1}{2}(-g)^{-1/2}[\mu\nu\rho\sigma] F_{\rho\sigma}.$$

Here the completely antisymmetric symbol $[\mu\nu\rho\sigma]$ satisfies $[0123] = +1$, it being understood that t corresponds to 0.

Those components of $F_{\mu\nu}$ not given in the following[3] can be readily derived from these:

$$F_{tn} = \partial_n A_t, \tag{29a}$$

$$F^{mn} = f\gamma^{-1/2}\epsilon^{mnp}\partial_p\Phi, \tag{29b}$$

$$F^{nt} = \omega_m F^{mn} + F_{tm}\gamma^{mn}. \tag{29c}$$

The two potentials A_t and Φ are determined by

$$A_t + i\Phi = e^{i\alpha}(1 - U^{-1}), \tag{30}$$

where the real parameter α effects a duality transformation.

Let a caret over a quantity denote the value when $\alpha = 0$. From Eq. (30),

$$\begin{aligned} A_t &= \hat{A}_t \cos\alpha - \hat{\Phi}\sin\alpha, \\ \Phi &= \hat{\Phi}\cos\alpha + \hat{A}_t \sin\alpha. \end{aligned} \tag{31}$$

For the two-source solution described here, it is straightforward to prove that

$$\begin{aligned} E^\mu &= \hat{E}^\mu \cos\alpha + \hat{B}^\mu \sin\alpha, \\ B^\mu &= \hat{B}^\mu \cos\alpha - \hat{E}^\mu \sin\alpha. \end{aligned} \tag{32}$$

To show this, start by using oblate spheroidal coordinates (t, r_1, θ_1, ϕ) for which $\omega_m = \delta_m^\phi \omega_\phi$, and γ_{mn} is diagonal. Then, from Eqs. (28) and (29), $E^\phi = B^\phi = E^t = B^t = 0$. Relations of the form (32) can be derived for the r_1, θ_1 components. The same linear relation holds trivially for the vanishing ϕ and t components, and being tensorial, (32) must then be true in arbitrary coordinates.

The static $b = 0$ solution goes asymptotically to the Reissner-Nordström solution for large radii. It has no magnetic field unless a duality transformation is performed, in which case it will acquire a magnetic monopole moment.

The stationary $b = 0$ solution is more unusual. Although the mass and charge vanish, the solution possesses angular momentum and, for $\alpha = 0$, a magnetic dipole moment. Expanding in powers of r_1, one obtains through Eqs. (25) and (30) with $\alpha = 0$

$$\begin{aligned} \hat{A}_t &= (2a^2m^2\cos^2\theta_1) r_1{}^{-4} + O(r_1{}^{-5}), \\ \hat{\Phi} &= (am\cos\theta_1) r_1{}^{-2} + O(r_1{}^{-3}). \end{aligned} \tag{33}$$

The resulting electromagnetic fields are

$$\begin{aligned} \hat{E}^{r_1} &= -4m^2a^2(1 + \cos^2\theta_1) r_1{}^{-5} + O(r_1{}^{-6}), \\ \hat{E}^{\theta_1} &= (4m^2a^2 \sin\theta_1\cos\theta_1) r_1{}^{-6} + O(r_1{}^{-7}), \\ \hat{B}^{r_1} &= (2am\cos\theta_1) r_1{}^{-3} + O(r_1{}^{-4}), \\ \hat{B}^{\theta_1} &= (am\sin\theta_1) r_1{}^{-4} + O(r_1{}^{-5}). \end{aligned} \tag{34}$$

To compare these with the results of classical electromagnetism, one must project at each point onto an orthonormal pair of vectors transported with the preferred observer and directed along the r_1, θ_1 coordinate axes. At large radii, this effectively multiplies the θ_1 components by r_1. To lowest order, the magnetic field is a dipole field with moment am directed along the $+z$ direction. If, however, the fields are expressed in terms of

TABLE I. Two types of solutions with $b = 0$. Upper sign in table refers to r_1, θ_1 coordinates, lower to r_2, θ_2, and $\hat{z}$ denotes unit vector in $+z$ direction.

Type	Coordinate relations	Mass	Angular momentum	Charge ($\alpha = 0$)	Magnetic dipole moment ($\alpha = 0$)
1. Static	$r_1 - m = r_2 - m$ $\theta_1 = \theta_2$	$2m$	0	$2m$	0
2. Stationary	$r_1 - m = m - r_2$ $\theta_1 = \pi - \theta_2$	0	$\mp 2am\hat{z}$	0	$\pm am\hat{z}$

r_2, θ_2 coordinates, a must be changed to $-a$, and the dipole moment changes sign. The electric field does not have the requisite angular dependence to be an octopole field, which may be derived by taking the gradient of $U \sim (5\cos^3\theta - 3\cos\theta)r^{-4}$. Mathematically, this arises both because $\hat{A}_t$ is not octopolar[12] and because, thanks to $\omega_\phi \neq 0$, $F^{\phi n}$ contributes to F^{nt} in Eq. (29c).

Table I summarizes the two types of solutions with $b = 0$.

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