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PHYSICAL REVIEW D VOLUME 7, NUMBER 10 15 MAY 1973

Nonuniqueness of Canonical Field Quantization in Riemannian Space-Time*

Stephen A. Fulling†
University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201
(Received 3 November 1972)

We point out and discuss an ambiguity which arises in the quantum theory of fields when the background metric is not explicitly Minkowskian–in other words, when an external gravitational field, real or apparent, is present. A general theory of a canonical neutral scalar field in a static universe, including the construction of a Fock space, is presented. It is applied to a portion of two-dimensional flat space-time equipped with a non-Cartesian space-time coordinate system with respect to which the metric is nonetheless static. The resulting particle interpretation of the field is shown to be different from the standard one in special-relativistic free-field theory. The ambiguity frustrates an attempt to define uniquely the energy-momentum tensor by the usual method of normal ordering. We discuss various suggestions for (1) distinguishing a unique correct quantization in a given physical situation, or (2) reinterpreting seemingly inequivalent theories as physically equivalent. In passing it is shown that the vacuum state and the energy density of a free field in a box with periodic boundary conditions differ from those associated with a region of the same size in infinite space; this result should be of interest outside the gravitational context.

I. INTRODUCTION

In recent years generalizations of elementary quantum field theory to Riemannian space-times[1] have begun to be applied seriously to various cosmological and astrophysical problems.[2-5] In this work it is necessary to proceed beyond the establishment of field equations and commutation relations[6] to the construction of some framework of *observables* and *quantum states*, and to give this apparatus a physical interpretation. This is traditionally done in terms of a Hilbert space of quantum-state vectors in which the fields are realized as operators.[7] The most common strategy in the Riemannian context is to choose (if possible) a coordinate system in which the field equation can be solved by separation of variables and to quantize the resulting "normal mode" structure in close analogy to the standard quantization of a free field in flat space. In the case of a static metric (e.g., Ref. 3) one is thus led to what appears to be a unique theory, which we outline in Sec. II A. In time-dependent problems (e.g., Refs. 2, 4, 5) the situation is less clear, since there is not an unambiguous division of the solutions of the field equation into positive- and negative-frequency parts; it has been suggested that the concept of *particle* loses some of its physical significance in such situations.[8-10]

Such constructions are not manifestly generally covariant. If a given space-time admits two or more of them, there is no guarantee that they will agree physically; if not, of course, at most one of them can be correct. In particular, any procedure which purports to apply to all Riemannian metrics of a certain form (e.g., static metrics) must yield physically sensible results when the metric considered is that of *ordinary flat space* equipped with a curvilinear coordinate system. One might hope to use this principle as a criterion for the correctness of the theories, or as a guide to the choice of the correct ansatz in the cases where ambiguities remain.

In this paper a neutral scalar field in a patch of two-dimensional Minkowski space is quantized according to the "unique" prescription for static metrics referred to above. It is shown that the resulting notions of *particles* and *vacuum state* are completely different from those of the standard Fock representation. This ambiguity affects the definition of the energy-momentum tensor, the principal observable of the system in a gravitational context. These observations have serious implications for the quantum theory of matter near the horizon of the Schwarzschild solution.

It is observed that similar problems arise even in the familiar situation of a free field in a box with periodic boundary conditions (compared to infinite space). Details of this case are given in the Appendix. Here only elementary free-field theory is involved, not the questions of gravitation and general covariance.

Two classes of possible resolutions of the paradox are discussed in general terms: (1) general criteria according to which the "heretical" quantization might be rejected, (2) restrictions on which elements of the theory are to be considered observable, with the aim of making both quantizations admissible. In particular, the nonstandard quantization in a subregion of Minkowski space is tentatively associated with physical conditions of reflection at the boundary of the region. Nevertheless, we emphasize the possibility that there is a certain arbitrariness in the choice of an operator representation of the formal field algebra because we believe that the difficulties of the *time-dependent* theory must and can be resolved in that spirit.

II. THE MODEL

A. Quantization of a Scalar Field in a Static Universe

We consider a Riemannian manifold of dimension $n=s+1$ with metric tensor $g_{\mu\nu}$ of signature $(+ - \cdots -)$ (s minus signs). Let $g=\det\{g_{\mu\nu}\}$. We assume that there is a coordinate system (not necessarily covering the entire space) with respect to which the metric is *static*: The components $g_{\mu\nu}$ are independent of the time coordinate x^0, and also $g_{0j}=0$ for $j\neq 0$.

It is assumed that the generalization of the Klein-Gordon equation to a Riemannian space-time (not necessarily static) is[11,12]

$$\Box_c\phi+m^2\phi=0\,, \tag{1}$$

where the covariant d'Alembertian (Laplace-Beltrami operator) is

$$\Box_c=|g|^{-1/2}\partial_\mu(|g|^{1/2}g^{\mu\nu}\partial_\nu) =g^{\mu\nu}\nabla_\mu\nabla_\nu\,, \tag{2}$$

∇ denoting the covariant derivative. A Lagrangian density which yields this equation upon variation of the action $\int d^nx\,\mathcal{L}$ is

$$\mathcal{L}=\tfrac{1}{2}|g|^{1/2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-m^2\phi^2)\,. \tag{3}$$

Following the canonical procedure[13] we define a conjugate momentum,

$$\pi=\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\quad(=|g|^{1/2}g^{00}\partial_0\phi \text{ if } g_{0j}=0)\,, \tag{4}$$

and a Hamiltonian,

$$H=\int d^sx(\pi\partial_0\phi-\mathcal{L})\,, \tag{5}$$

and postulate the canonical equal-time commutation relations

$$\begin{gathered}[\phi(t,x),\phi(t,y)]=[\pi(t,x),\pi(t,y)]=0,\\ [\phi(t,x),\pi(t,y)]=i\delta(x-y)\qquad [t=x^0,\;x=(x^1,\ldots,x^s)]\,.\end{gathered} \tag{6}$$

Hamilton's equations of motion for this system are then equivalent to Eqs. (1) and (4). The relations (6) can be shown to be formally covariant – that is, independent of the spacelike hypersurfaces chosen to represent "equal times" and of the coordinate systems chosen within them.[14] (It is understood that x^0 is a timelike coordinate in any coordinate system to which the canonical Hamiltonian formalism is to be applied.)

When the metric is static, Eq. (1) can be solved by separation of variables. Substituting

$$\phi(t,x)=\psi_j(x)e^{\pm iE_jt}\,, \tag{7}$$

one obtains the eigenvalue equation

$$|g|^{-1/2}g_{00}\partial_l(|g|^{1/2}g^{lk}\partial_k\psi_j)+g_{00}m^2\psi_j\equiv K\psi_j =E_j{}^2\psi_j \tag{8}$$

(where the Latin indices l and k range from 1 to s). The differential operator K is Hermitian with respect to the scalar product

$$(F_1,F_2)=\int d^sx|g|^{1/2}g^{00}F_1^*(x)F_2(x)\,. \tag{9}$$

It is also positive[15]:

$$\begin{aligned}(F,KF)&=\int d^sx\,F^*(x)\partial_j[|g|^{1/2}g^{jk}\partial_kF(x)]\\ &\quad+\int d^sx|g|^{1/2}m^2|F(x)|^2\\ &\geq-\int d^sx|g|^{1/2}g^{jk}\partial_jF^*(x)\partial_kF(x)\\ &\geq 0\,.\end{aligned}$$

Thus all the $E_j{}^2$ are non-negative; we take $E_j\geq 0$ by definition.

A positive Hermitian operator may not have a unique self-adjoint extension (hence a unique spectrum of generalized eigenvalues).[16] However, if the Cauchy problem is well-posed – that is, if a (classical) solution of Eq. (1) is uniquely determined throughout the region covered by the coordinate system by the values of $\phi(x)$ and $\pi(x)$ on any given hypersurface $t=\text{const}$ – then one expects that K, supplemented if necessary by geometrically obvious boundary conditions (such as periodicity in an angular coordinate), will be essentially self-adjoint. (Such a space will be called *Cauchy-complete.*) Failure of this condition is associated, roughly speaking, with the possibility of information (e.g., particles) leaking into or out of this region at spatial infinity at finite times.[17] Such situations will not concern us here, so we shall henceforth treat K as a self-adjoint operator.

In fact, for convenience it will be assumed (as usual in quantum mechanics) that the numbers in the spectrum can be classified as *point spectrum* σ_p or *continuous spectrum* σ_c (or both), and that a corresponding complete set of generalized eigenfunctions exists. That is, an arbitrary function in the Hilbert space determined by Eq. (9) can be expanded as

$$F(x)=\int d\mu(j)\tilde{f}(j)\psi_j(x)\,, \tag{10}$$

where the $\psi_j(x)$ are solutions of Eq. (8) and μ is a measure such that the scalar product (9) becomes

$$(F_1, F_2)=\int d\mu(j)\tilde{f}_1^*(j)\tilde{f}_2(j)\,; \tag{11}$$

if the eigenfunctions are given the most natural normalization, $d\mu(j)$ means

$$\sum_{j\in\sigma_p}+\int_{\sigma_c} dj\,.$$

In any case, consistency of Eqs. (9) and (11) leads to the orthonormality and completeness relations

$$\begin{aligned}&\int d^s x|g|^{1/2}g^{00}\psi_j^*(x)\psi_k(x)=\delta(j,k)\,,\\&\int d\mu(j)\psi_j^*(x)\psi_j(y)=[|g|^{1/2}g^{00}]^{-1}\delta(x-y)\,,\end{aligned} \tag{12}$$

where $\int d\mu(k)\delta(j,k)\tilde{f}(k)=\tilde{f}(j)$.

The general solution of Eq. (1) can now be written

$$\phi(t,x)=\int\frac{d\mu(j)}{(2E_j)^{1/2}}\,[a_j\psi_j(x)e^{-iE_jt}+a_j^\dagger\psi_j(x)e^{iE_jt}]\,, \tag{13}$$

where without loss of generality the ψ_j have been chosen to be real. (If σ_p includes the value $E_j=0$, it must be treated separately – see Ref. 10, Chap. VIII.) Equation (13) defines the *annihilation and creation operators* a_j and $a_j^\dagger$. Equations (4) and (6) are now equivalent to

$$[a_j, a_k]=0,\quad [a_j, a_k^\dagger]=\delta(j,k)\,. \tag{14}$$

The Hamiltonian (5) reduces formally to

$$H=\int d\mu(j)E_j[a_j^\dagger a_j+\tfrac{1}{2}\delta(j,j)]\,, \tag{15}$$

which diverges; analogy with the theory of the free field in flat space strongly suggests discarding the infinite c-number term $\frac{1}{2}\int d\mu(j)E_j\delta(j,j)$.

In the standard way[18] $\phi(t,x)$ can be represented as an operator-valued distribution in a Hilbert space (Fock space) spanned by a *vacuum*[19] vector $|0\rangle$, defined by the property

$$a_j|0\rangle=0 \text{ for all } j\,, \tag{16}$$

and *n-particle vectors* of the form

$$(n!)^{-1/2}\int d\mu(j_1)\cdots d\mu(j_n)\tilde{f}_n(j_1,\ldots,j_n)a_{j_1}^\dagger\cdots a_{j_n}^\dagger|0\rangle\,. \tag{17}$$

In this space the renormalized H,

$$H=\int d\mu(j)E_ja_j^\dagger a_j\,, \tag{15'}$$

is a positive self-adjoint operator which generates a unitary group which implements the time translation symmetry:

$$\phi(t,x)=e^{iHt}\phi(0,x)e^{-iHt}\,. \tag{18}$$

The operator $N_j\equiv a_j^\dagger a_j$ has a natural interpretation as the observable "number of particles in the state j." The particle numbers are constants of the motion; hence the theory can be interpreted as the description in second quantization of a system of an arbitrary number of noninteracting particles whose wave functions in first quantization obey Eq. (1).

A crucial element in this whole construction is the unambiguous division of the solutions of Eq. (1) into *positive-frequency* and *negative-frequency* functions, with temporal behavior of the type e^{-iE_jt} and e^{+iE_jt}, respectively. The single-particle wave functions mentioned above are restricted to have positive frequency.

B. Rindler Space

Let us apply the general theory of Sec. II A to a two-dimensional[20] space-time with time coordinate $x^0=v$ $(-\infty<v<\infty)$, space coordinate $x^1=z$ $(0<z<\infty)$, and metric

$$g_{00}=z^2,\quad g_{11}=-1,\quad g_{01}=0\,. \tag{19}$$

We choose the unit of length (or mass) so that $m=1$. The integration density in Eq. (8) is

$$|g|^{1/2}g^{00}=z^{-1}\,. \tag{20}$$

The wave equation (8) becomes a Bessel equation:

$$\left(z^2\frac{d^2}{dz^2}+z\frac{d}{dz}-m^2z^2+E_j^{\,2}\right)\psi_j(z)=0. \tag{21}$$

The solution of this self-adjoint eigenvalue problem has been given by Titchmarsh.[21] The spectrum of E_j^2 extends from 0 to $+\infty$ with unit multiplicity. One can therefore set $E_j = j$ $(0 \leq j < \infty)$ by definition. The eigenfunctions, normalized according to Eqs. (12) with $\delta(j, k) = \delta(j - k)$ (Dirac δ function), are

$$\psi_j(z) = \pi^{-1}[2j \sinh(\pi j)]^{1/2} K_{ij}(mz), \tag{22}$$

where K_{ij} is the Macdonald function (modified Bessel function) of imaginary *order*.

The expansion of the field in creation and annihilation operators [Eq. (13)] is

$$\phi(v, z) = \int_0^\infty \frac{dj}{(2j)^{1/2}} \psi_j(z)(e^{-ijv} a_j + e^{ijv} a_j^\dagger). \tag{23}$$

It can be inverted [with the aid of Eqs. (12) and (4)] to yield

$$a_j = 2^{-1/2}\left[j^{1/2} \int_0^\infty \frac{dz}{z} \psi_j(z)\phi(0, z) + i j^{-1/2} \int_0^\infty dz\, \psi_j(z) \pi(0, z) \right]. \tag{24}$$

If the theory is interpreted in terms of particles, the general (positive-frequency) wave function has the form

$$\psi(v, z) = \int_0^\infty dj\, f(j) \psi_j(z) e^{-ijv}. \tag{25}$$

A space with the metric (19) is flat. In fact, through the transformation

$$t = z \sinh v, \quad x = z \cosh v, \tag{26}$$

the space can be identified with the region $\{(t, x): |t| < x\}$ in a two-dimensional Minkowski space (Fig. 1). We shall call v and z *Rindler coordinates*, and the manifold covered by them *Rindler space*, because they have been discussed most thoroughly by Rindler[22]; he points out that the relation of this system to Cartesian coordinates is very similar to the relation between Schwarzschild and Kruskal coordinates[23] for the space surrounding an isolated point mass. Translation in the coordinate v (with z fixed) corresponds to a homogeneous Lorentz transformation in (t, x) space; this is the geometrical reason why the metric of flat space has an explicitly static form with respect to the curvilinear coordinates (v, z). It is easy to see geometrically [since the field equation (1) cannot propagate information faster than light] that the classical Cauchy problem should be well-posed for initial data on any hypersurface $v = \text{const}$. In keeping with our earlier remarks, the self-adjointness of the operator in Eq. (21) confirms this.

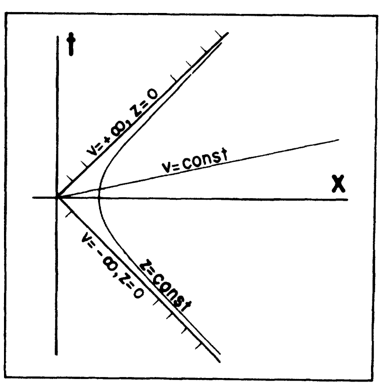

FIG. 1. Rindler coordinates in Minkowski space.

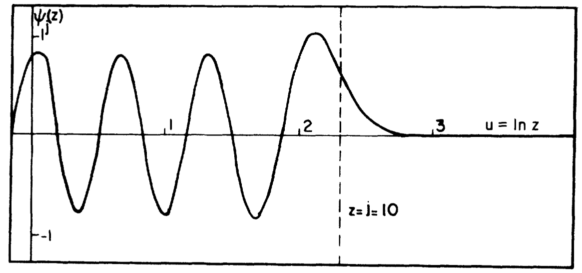

FIG. 2. The eigenfunction $\psi_{10}(z)$.

If we let $u = \ln z$, Eq. (21) becomes

$$-\frac{d^2\psi_j}{du^2} + e^{2u}\psi_j = j^2 \psi_j \quad [\psi_j = \psi_j(e^u)], \tag{27}$$

which has the form of a nonrelativistic Schrödinger equation with potential e^{2u}. A typical solution [cf. Eq. (22)] is graphed in Fig. 2. The qualitative behavior of wave packets formed out of such functions can be investigated as in ordinary quantum mechanics.[24] One finds that a packet incident from the left (large negative u) "scatters" and returns to the left, in a way consistent with the transcription of a straight particle path in Cartesian coordinates into the Rindler system (Fig. 3). Thus the theory based on Eq. (23) [or Eq. (25)] is a physically reasonable description of free particles in Rindler space in the quasiclassical limit of large j (i.e., short wavelengths or large distances from the coordinate singularity at $z = 0$).

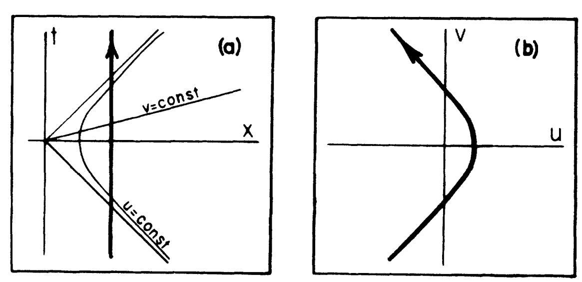

FIG. 3. Trajectory of a free particle in flat space. (a) Cartesian coordinates. (b) Rindler coordinates.

C. Inequivalence of the Particle Structures

Nevertheless, a wave function of the form (25) is *not* a positive-frequency solution of the Klein-Gordon equation in the usual sense when transcribed back into terms of t and x, but rather a superposition of positive- and negative-frequency solutions. Therefore, we are here considering a relativistic theory of free spinless particles which *differs* in its details from the usual one. This observation can also be made from the point of view of the field formalism. The standard expansion of the free field at $t=0$ and its conjugate momentum into annihilation and creation operators is

$$\phi(0, x)=\frac{1}{(2\pi)^{1/2}}\int\frac{dk}{(2\omega_k)^{1/2}}\left(e^{ikx}b_k+e^{-ikx}b_k^\dagger\right),$$
$$\pi(0, x)=\frac{-i}{(2\pi)^{1/2}}\int dk\left(\tfrac{1}{2}\omega_k\right)^{1/2}\left(e^{ikx}b_k-e^{-ikx}b_k^\dagger\right), \tag{28}$$

where $\omega_k=(k^2+m^2)^{1/2}$ and b_k is the annihilation operator for particles of momentum k. Now the field ϕ is a scalar object, but π, in general, is not. However, Eqs. (4) and (20) show that the conjugate momenta defined relative to the two coordinate systems are related by

$$\begin{aligned}\pi(0, z)&=\frac{1}{z}\frac{\partial\phi}{\partial v}(0, z)\\&=\frac{\partial\phi}{\partial t}(0, x)\\&=\pi(0, x),\end{aligned} \tag{29}$$

so that in the present case $\pi(0, z)$ and $\pi(0, x)$ stand for the same quantity. So one can substitute Eqs. (28) into Eq. (24), obtaining

$$\begin{aligned}a_j&=\frac{1}{2}\frac{1}{(2\pi)^{1/2}}\left\{\int_0^\infty dy\,\psi_j(y)\int_{-\infty}^{\infty}dk\,e^{iky}\left[\frac{1}{y}\left(\frac{j}{\omega_k}\right)^{1/2}+\left(\frac{\omega_k}{j}\right)^{1/2}\right]b_k\right.\\&\qquad\left.+\int_0^\infty dy\,\psi_j(y)\int_{-\infty}^{\infty}dk\,e^{-iky}\left[\frac{1}{y}\left(\frac{j}{\omega_k}\right)^{1/2}-\left(\frac{\omega_k}{j}\right)^{1/2}\right]b_k^\dagger\right\}\\&=\int_{-\infty}^{\infty}dk\,U(j, k)b_k+\int_{-\infty}^{\infty}dk\,V(j, k)b_k^\dagger,\end{aligned} \tag{30}$$

where[25]

$$\begin{aligned}U(j, k)&=\left[2\pi\omega_k(1-e^{-2\pi j})\right]^{-1/2}\left[(\omega_k+k)/m\right]^{ij},\\V(j, k)&=\left[2\pi\omega_k(e^{2\pi j}-1)\right]^{-1/2}\left[(\omega_k+k)/m\right]^{ij}.\end{aligned} \tag{31}$$

The crucial point is that the kernel $V(j, k)$ does not vanish, so that Eq. (30) for a_j includes creation operators. A vector which is annihilated by the b's is not annihilated by the a's, and vice versa. The vacuum of the Rindler-space theory is not the ordinary vacuum of the free field, one-particle states in one theory are not one-particle states in the other theory, and so on. The notion of a *particle* is completely different in the two theories. *The particles or quanta of the Rindler-Fock representation cannot be identified with the physical particles described by the usual quantum theory of the free field.*

The minimal conclusion which must be drawn from this observation is the following: *In the context of the general static universe treated in Sec. IIA, the particle concept does not have the full physical significance which it has in Minkowski space.* The theory of quantization in a static metric amounts to the following: Given a manifold with a timelike Killing vector, we have constructed a representation of the field algebra in which the symmetry generated by the Killing vector field is implemented by a group of unitary operators. Also, the generator of this unitary group has been required to be a positive operator, and we have used it like the Hamiltonian in special-relativistic theories. We found that the eigenstates of this operator can be labeled in a way which is quite similar to the particle structure of the states of the free field. This does not necessarily mean, however, that the associated "quanta" have anything to do with physical particles in an operational sense (e.g., entities which trigger detectors).

The phenomenon under study is reflected in the problem of generalizing to a Riemannian manifold the various elementary solutions and Green functions associated with the Klein-Gordon equation.[26] Some, but not all, of these are intrinsically determined by the manifold. The commutator

$$i\Delta(x_2, x_1)=[\phi(x_2), \phi(x_1)]\quad[x=(x^0, \ldots, x^3)] \tag{32}$$

is a c number calculable from Eqs. (1) and (6), and $\Delta_{\text{ret}}(x_2, x_1)$ and $\Delta_{\text{adv}}(x_2, x_1)$, the retarded and advanced solutions of

$$(\Box_c+m^2)G(x_2, x_1)=\delta(x_2, x_1) \tag{33}$$

(where $\Box_c$ operates on x_2 and δ is a scalar distribution, not a density), are uniquely defined by the purely geometrical restrictions on their supports. However, the positive-frequency part of $\Delta(x_2, x_1)$,

$$\Delta_+(x_2, x_1) = \langle 0|\phi(x_2)\phi(x_1)|0\rangle, \tag{34}$$

requires for its definition either a distinguished vacuum vector or a notion of positive frequency for either classical or operator solutions of Eq. (1). As we have seen, these elements are either absent (for time-dependent models) or ambiguous (as in the case of Rindler space). Similar remarks apply to the Feynman propagator or causal solution of Eq. (33),

$$-\Delta_F(x_2, x_1) = i\langle 0|T[\phi(x_2)\phi(x_1)]|0\rangle \tag{35}$$

(T denoting the time-ordered product). Hence our difficulty cannot be resolved merely by shifting attention from annihilation-creation operators to Green functions.

Note that equivalence of two Fock-like representations of a field algebra in the sense of this section is a stronger notion than unitary equivalence. Two representations could be unitarily equivalent (in other words, the Hilbert space and the field operators could be abstractly identical although the state vectors are labeled differently), and yet the particle structures could be different. In this situation the field theory is mathematically well-defined, but the particle interpretation is ambiguous. On the other hand, if the representations are unitarily inequivalent, then the field theory is not unique even mathematically. In the present case the unitary equivalence question is whether the Rindler representation of the subalgebra of the $\phi(x)$ and $\pi(x)$ at points on the positive x axis is unitarily equivalent to a subrepresentation of the representation of this subalgebra in the standard quantization. The answer appears to be negative, but the argument has not been made completely rigorous.[27]

III. IMPLICATIONS

A. Possible Restrictions on Permissible Quantizations

1. *General Discussion*

One's first reaction to the nonstandard theory of the free field presented above is to reject it outright. The viewpoint is the following: The free field in flat space is well understood. The physically relevant representation of the field algebra is the Fock representation of the operators b_k of Eqs. (28). If any other proposed theory disagrees with this one, so much the worse for that theory. More specifically, one might object that the Rindler coordinate system covers only a part of space-time, that it has a singularity at $z=0$ which has nothing to do with the intrinsic structure of the space, that it is not an inertial frame (in the sense that the curves of constant z are not geodesics). For these reasons, the critic would argue, it is not surprising that a naive imitation of the quantization of the free field leads to unphysical results in this context.

This conclusion, however, would be rather unwelcome from the point of view of the general problem of field quantization in static background metrics. The theory expounded in Sec. II A is a natural generalization from the theoretical ideas which have evolved in the study of free fields and of external potential problems. If it is wrong, it is not clear how to construct,[28] or to interpret physically, a quantum field theory in this situation, because in the general static case we do not have an underlying flat space to tell us what the "right answer" is.

As a particular example, consider the Schwarzschild metric, whose close resemblance to the Rindler model has already been mentioned. We are interested only in the region of the space-time where the Killing vector is timelike. This also is only a part of the maximally extended solution and terminates at a fictitious singularity, and the Schwarzschild coordinates are also not inertial. Rindler coordinates are just as appropriate for the description of the region of flat space which they cover as Schwarzschild coordinates are for the study of the space around a massive body outside the radius $r=2M$. If the theory fails in this test case, it probably must also be rejected for the Schwarzschild metric. It cannot be applied as a general method without additional conditions.

Thus one seeks either to "save" the theory based on Eq. (23) by showing that the elements in which it disagrees with the standard theory are without observational significance, or to outlaw it on the basis of the most tolerant criteria one can devise. We shall discuss in a preliminary way the first approach in Sec. III B and the second in this section.

To begin with, note that our dubious treatment of the free field in flat space is unusual in two ways: It deals with only a fragment of the space, and it employs a non-Cartesian coordinate system. Let us consider these features in turn.

2. *The Effect of Incompleteness of the Space*

It has been suggested[29] that field quantization should not be attempted on a manifold unless it is "the whole space" in some sense – for instance, that of geodesic completeness[30] or of maximal analytic extension. This criterion rules out the Schwarzschild solution in its static guise, as well as Rindler space, but admits some static space-times.

The quantum theory usually deals with phenomena that happen on a microscopic scale. It is hard to understand how the global structure of the universe can affect the physics inside a small Cauchy-complete region. Nevertheless, a decomposition of a field into modes appears unavoidably to involve global integral transformations, like Eq. (24) and the inverse of Eqs. (28).

Support for the idea that the global structure of a space-time affects the concept of particles comes from the observation (see Appendix) that the vacuum state (as a functional on the field algebra) of a free field in a box of length L with periodic boundary conditions is different from the vacuum of the field in infinite space, even for field operators with argument points confined to a Cauchy-complete region of dimension smaller than L. If one grants that the field in a box has a solid particle interpretation, one must conclude that the type of quantum state which is encountered experimentally in a region, along with the entire meaning of the particle language as applied to those states, depends upon the global setting of the region. Although the dynamics of the field is self-contained if the region is Cauchy-complete, the broader environment apparently determines what constitutes an "equilibrium state" or vacuum.[31]

One can imagine a perfectly reflecting wall placed at some small negative x in Fig. 1. In nonrelativistic quantum mechanics one knows that low-energy particles will avoid the wall (their wave functions must vanish there), and hence the particle density for small positive x will be lowered. A relativistic system presumably behaves similarly. Despite the argument from Cauchy completeness, therefore, scalar particles in the Rindler region with and without the wall constitute *different physical situations*. A quantization construction which is entirely internal to the region should not be expected to represent both, and, of course, may not represent either.

In fact, the a quantization in Rindler space [the Rindler-Fock representation, Eq. (23)] can be interpreted as that appropriate to the physical situation of an impenetrable wall located on the light cone $z=0$, or, more realistically, to the limiting case of a wall which accelerates along one of the curves $z=z_0$, where z_0 is a small positive constant. Let us consider the portion of Rindler space to the right of this curve and impose, for instance, the "perfect reflection" boundary condition[32] that $\psi_j(z_0)=0$. From Eq. (22) and the power series[33] of K_{ij} at small z, one sees that the values of j for which the boundary condition is satisfied are separated roughly by $\pi/|\ln z_0|$, which approaches zero as $z_0 \to 0$. The Rindler-Fock representation, in which all j are admissible, can thus be regarded as a limiting case of this situation. When a particle is reflected from a barrier moving at nearly the speed of light, the recoiling particle itself attains such a speed. In the limit $z_0 \to 0$ the reflected particles are confined to the light cone ($|v|=\infty$); this explains why reflected wave packets do not show up in our formalism, which describes only the interior of Rindler space.

This argument does not establish conclusively the physical interpretation of the Rindler-Fock representation, since it *assumes* the correctness of a similar quantization when $z_0 \neq 0$. It is, however, a suggestive demonstration of consistency. It would be interesting to test the unitary equivalence of these representations, based on separation of variables in Rindler coordinates, with various other quantization prescriptions for the same physical situations studied in their time-dependent Cartesian guise.

Note that, if the above interpretation is correct, field quantization in the exterior Schwarzschild solution regarded as a static space-time corresponds to the presence of a reflecting wall at the Schwarzschild radius. A physically correct quantization for the Schwarzschild-Kruskal black hole, therefore, must (at least in principle) deal with the entire space, which is time-dependent.

The author does not believe that the situation is well enough understood for one to assert with confidence that the quantization indicated in Eq. (13) corresponds to physical particles in all static metrics which *are* geodesically complete.

3. *Distinguished Coordinate Systems*

The argument leading to Eq. (30) is of a rather general type, and the kernel V vanishes only in very special cases (for example, a transformation between two Lorentz frames in flat space). Hence it is possible that different ways of "slicing" a manifold into *space* and *time* can produce different quantizations in the sense of Sec. II C, even if no global mutilation of the space is involved. (It is impossible to make precise and to test this conjecture in the absence of a definite prescription for constructing a unique physically relevant Fock space when the metric is not static.)

One can hope to reduce the ambiguity of quantization by demanding that the slicing be as nearly Cartesian as possible.[34] First, to define particle observables on a spacelike hypersurface, use the Gaussian coordinate system[35] based on that hypersurface and construct (analogously to Sec. II A of this paper) a Fock representation which diagonalizes the instantaneous Hamiltonian (5). Second, use as equal-time hypersurfaces only those which are spanned by radial geodesics through the observer's worldline. (The first requirement eliminates the Rindler coordinate system; the second is relevant in more general situations.)

This prescription was applied to two-dimensional closed de Sitter space in Ref. 10. Study of other examples is needed before its value can be judged.

4. *The Possibility of a Local, Manifestly Covariant Formulation*

The preceding considerations suggest that the troubles of field theory in Riemannian space-time are rooted in the formulation of the theory in terms of particular spacelike hypersurfaces treated as wholes. This approach indeed clashes with one's intuitive conception of particles as localized entities, and with the mathematical apparatus of local fields to describe them, but one seems to be drawn into it ineluctably by the canonical formalism. The situation may be summarized by two phrases familiar to readers of the works of Wheeler[36]: Physical intuition tells us that the universe is a vast haystack of particle paths, but the theoretical apparatus presently available to us forces us to treat the universe as a stack of automobile fenders.

One would like to have a satisfactory working theory in which all the observational elements are related to the fields by manifestly local and covariant constructions. This has not yet been done.

B. Observables and Algebraic Approaches

1. *The Field Algebra*

The Fock representation of the scalar field in a static space-time (Sec. II A) is a natural and convincing generalization of the quantization of the free field. For the reasons mentioned at the beginning of Sec. III A there is strong motivation for attempting to make physical sense out of this formalism in the case of Rindler space, if necessary modifying the preconceived ideas about field theory that would lead one to reject it. One must accept the fact, however, that the quanta of this Fock representation do not correspond to physical particles. One must find something else to play the role of the basic observables of the theory (in this special case and in the general case of a quantized field in a Riemannian space-time).

An excellent-looking candidate is the field itself. The time evolution of the field (or one of its expectation values) from given initial values is given by a classical formula for the solution of the Cauchy problem [cf. Eqs. (23) and (24)], which is independent of any representation of the field by operators. A similar statement holds for a product of n fields (which obeys the field equation with respect to each of the n arguments). If the fundamental dynamical problem is the prediction of the outcome of field measurements at later times on the basis of known results of measurements at earlier times, then a quantum state becomes just an intermediary apparatus which summarizes (idealized) earlier measurements. Then it is not necessary to insist on a unique "physical" representation or an absolute definition of number-of-particles observables. Any representation (which may involve a labeling of the quantum states by a particle-like structure) is potentially physically relevant, although some will be more useful than others. For instance, in a static universe the theory of Sec. II A, which exploits the time translation symmetry, might be favored. (Rindler space is a very special case in which there is another way of looking at the space which makes additional symmetries manifest and, consequently, leads to a more useful notion of particle.)

An approach to the problem in terms of local algebras[37] is thus indicated. For rigorous work it is necessary to replace the algebra of fields by a C^* algebra of bounded observables; for a neutral scalar field this has been done in a variety of ways.[38,39] A *state* of a physical system is then taken as any functional on the algebra of observables which can be interpreted as an expectation value; it does not have to be related to a vector in a particular Hilbert space. [The primary requirement is *positivity:* $\omega(A^\dagger A) \geq 0$ for all A in the algebra.] Each state is related as a vector state,

$$\omega(A) = \langle\psi|A|\psi\rangle \quad (|\psi\rangle \in \mathcal{H}),$$

to some representation of the algebra as operators in a Hilbert space $\mathcal{H}$, but there are many unitarily inequivalent representations. It can be argued, however, that all the faithful representations are physically equivalent, because every representation contains a state which is consistent with any given set of practical observations.[40] That is, given a list of results of a finite set of measurements, these results can be reproduced to arbitrary accuracy by a density matrix – a weighted average of the expectation values with respect to certain vectors – in any one of these Hilbert spaces. [This does not mean that the vector $|0\rangle$ of Eq. (16) is an approximation to the usual vacuum of the free field, but that there are other vectors in the Hilbert space spanned by the vectors (17) which approximate the vacuum with respect to any given finite set of observables.]

2. *The Energy-Momentum Tensor*

A shortcoming of the algebraic approach just proposed is that the physical interpretation is still unclear. We do not, after all, have meters to measure the strength of the pion field, for instance. In fact, charged fields and spinor fields cannot be observables at all, becuase they are not invariant under gauge transformations and 2π rotations, respectively.[41]

Attempting to introduce particle observables evidently leads back to the original problem: In fact, as previously mentioned, it is not clear that particle concepts have any operational meaning in situations of strong space-time curvature anyway. In the theory of interacting fields in Minkowski space, one defines particles only in the asymptotic limit of large particle separation.[42] The asymptotic approach can also be applied to external potential problems when the potential is bounded (or rapidly decreasing) in space or time. In gravitational problems, however, the gravitational field is usually of macroscopic extent compared to the physical processes studied, so that asymptotic concepts are not applicable.

A more promising idea is to study as the primary observables the various *currents* – the quadratic (or higher order) combinations of the fields which appear in interaction Lagrangians and Hamiltonians. In particular, any matter field couples to the gravitational field through its energy-momentum tensor. Since an observation must take place through some physical interaction, one would expect these currents, if anything, to be directly observable.

The obstacle here is that a product of field operators at the same space-time point is not generally a well-defined operator in any representation. To obtain a meaningful object one ordinarily subtracts an infinite c number from an infinite formal expression for a current, either by calculating and subtracting a vacuum expectation value or by normal-ordering the original expression (moving creation operators to the left of annihilation operators and discarding c-number terms). Thus we are again confronted with the ambiguity of Sec. II C.

As an example, consider the neutral scalar field's energy density, or time-time component of the energy-momentum tensor, at $t=0$ in Minkowski space, which may be expressed covariantly in terms of η^μ, the unit vector normal to the hypersurface $t=0$:

$$\eta_\mu T^{\mu\nu}\eta_\nu = \tfrac{1}{2}[\pi^2 + (\nabla\phi)^2 + m^2\phi^2]. \qquad (36)$$

(We emphasize that no transformation of contravariant tensor components from Cartesian to Rindler coordinates is involved in what follows. The point is to compare the *same* physical quantity with respect to two definitions of normal ordering.) Suppose that the fields in Eq. (36) are expressed in terms of the operators b_k and $b_k^\dagger$ of Eqs. (28) and that the resulting expression is normal-ordered; this is the standard energy density for the free field. If the b's are now reexpressed in terms of the a's [cf. Eq. (30)], the expression will not be in normal order with respect to the a's, but will contain a complicated c-number term.[43] This term (the expectation value in the a vacuum $|0\rangle$ of the b-normal-ordered version of $\eta_\mu T^{\mu\nu}\eta_\nu$) is not a determinate expression, since it involves multiple divergent integrals of varying phase. Unlike the case of the momentum density (considered below), there is no visible justification for setting it equal to zero. An analogous expression for the free field in a box is discussed in the Appendix.

The situation is different for the momentum density at $t=0$,

$$\eta_\nu T^{\mu\nu}\zeta_\nu = \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\pi + \pi\frac{\partial\phi}{\partial x}\right), \qquad (37)$$

where ζ_ν is a unit vector in the x direction. This expression is automatically normal-ordered with respect to both the a's and the b's. A formal calculation like that described above will consequently yield a vacuum expectation value which can quite reasonably be said to vanish (although cancellation of divergent integrals will be involved). This phenomenon seems to be characteristic of currents corresponding to physical quantities (like momentum and charge, but not energy) which can be carried with opposite signs by particles. For instance, it holds for the electromagnetic current of the charged scalar field if it is written in the symmetrized form

$$j_\mu = \tfrac{1}{2}i(\phi^\dagger\partial_\mu\phi - \partial_\mu\phi^\dagger\phi + \partial_\mu\phi\phi^\dagger - \phi\partial_\mu\phi^\dagger). \qquad (38)$$

Thus there is available an intrinsic algebraic definition of this current as a finite operator-valued distribution, independent of normal ordering with respect to any Fock-like representation of the field.

The crucial point is that no such intrinsic definition seems to exist for the on-diagonal components of the energy-momentum tensor. (Another current with this property is the scalar $\bar\psi\psi = \psi^\dagger\gamma^0\psi$ associated with a spin-$\frac{1}{2}$ Fermion field.) This fact is a serious obstacle to a purely algebraic interpretation of the quantum field formalism in situations where the gravitational effect of the field itself is important.

One would like, in the spirit of the remark at the end of Sec. III A, to have a purely local and manifestly covariant definition of the energy density at a point, avoiding the introduction of normal modes and the manipulation of divergent integrals. An argument in the Appendix indicates that this cannot be a very simple matter; the energy density at a point depends not only on the geometry of space-time near the point, but on the global structure of the space.

3. *Some Final Speculations*

The considerations of the Appendix and of Sec. IIIA bring to mind the calculation by Casimir[44] of the force between uncharged conducting plates in terms of differences between divergent expressions for the vacuum energy of the electromagnetic field for various positions of the plates. In the course of such a calculation, a cutoff is temporarily introduced at high frequency. That the final result is finite – in other words, independent of the cutoff as the cutoff is removed – seems to be a special feature of the *electromagnetic* field.[45] Furthermore, the type of cutoff used is motivated by the physical fact that no material is a perfect conductor at arbitrarily high frequencies. (The cutoff is based on frequency or wavelength rather than quantum number, and hence different numbers of modes may effectively contribute to the cutoff vacuum energy expressions corresponding to different plate separations.) Nevertheless, one wonders if the gravitational effect of quantized matter could be calculated in a similar way (the gravitational field playing the role of the plates). The Planck length, 10^{-33} cm, might provide a natural cutoff. Such an approach could probably be related to the proposals for renormalization of the gravitational interaction by Utiyama and DeWitt (Ref. 1) and Sakharov.[46]

The success of calculations of the Casimir type suggests that divergent terms in $T^{\mu\nu}$ have physical significance and should be taken seriously. Perhaps a new mathematical concept is needed for a correct treatment of this problem, just as similar needs of theoretical physics have been filled by the inventions of the "infinitesimal" calculus, densely defined unbounded operators in Hilbert space, and distributions.

Note added in proof. In the conventional theory of the Klein-Gordon equation a Fourier transform different from Eq. (10) is used; the form of the scalar product in x space for positive-energy solutions of the equation is consequently different from Eq. (9). The function $F(x)$ defined by Eq. (10) expresses such a solution in the spectral representation of the Newton-Wigner position operator. For further details see Ref. 10, Secs. VIII.3–VIII.4. The author thanks A. Ashtekar for bringing attention to this possible source of confusion.

ACKNOWLEDGMENTS

The author has benefited from discussions with many members and visitors of the mathematical physics and relativity research groups at Princeton University. In particular (in addition to Ref. 29) he wishes to thank A. S. Wightman for directing the thesis research, L. Parker for advice in the preparation of the paper, and J. E. Roberts for an enlightening discussion of the algebraic approach to quantum field theory. B. S. DeWitt stressed the relevance of Casimir's work and suggested that the Rindler-Fock representation could be interpreted in terms of an accelerating impenetrable wall.

APPENDIX: VACUUM STATE AND ENERGY DENSITY OF THE FREE FIELD – BOX vs INFINITE SPACE

We consider a torus universe (i.e., a box with periodic boundary conditions on the field) of length L. For algebraic simplicity we consider only one space dimension. The spatial coordinate x ranges from $-\frac{1}{2}L$ to $\frac{1}{2}L$. Let I be the interval $a<x<b$ ($-\frac{1}{2}L<a$, $b<\frac{1}{2}L$) of the x axis, and let D be the domain of dependence[47] of I (see Fig. 4). Then the Cauchy problem for the Klein-Gordon equation is well-posed in D for initial values on I. (In fact, one could prescribe initial values on any hypersurface of the form indicated by the dashed line in Fig. 4, so D is Cauchy-complete by the definition given in Sec. IIA.) Thus a scalar field in D presents a self-contained dynamical problem, and one would not expect the quantum theory of that field to depend on whether D is regarded as a subregion of the box universe or as a subregion of Minkowski space. Nevertheless, it will be shown[48]

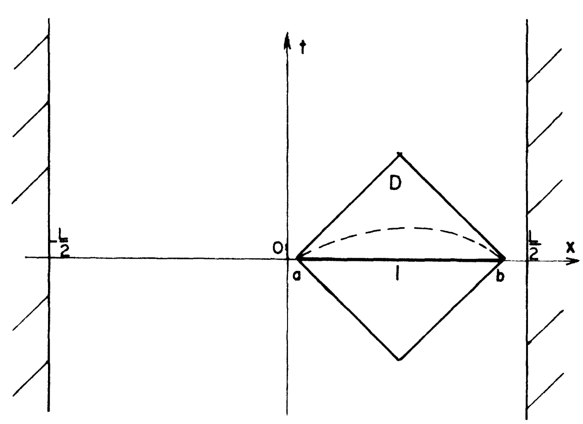

FIG. 4. A Cauchy-complete region D based on an interval $I=[a,b]$ between $x=-\frac{1}{2}L$ and $x=\frac{1}{2}L$ can be regarded as part of either a total Minkowski space or a torus universe of length L.

that the concepts of vacuum state and energy density for points in D depend on the space in which the field is canonically quantized.

First we show that the vacuum state of the free neutral scalar field in the box is not the same as that of the field in all of space. It clearly suffices to show that the vacuum states are different for boxes of different length, say 2π and 4π. The meaning of the assertion is that the functional on the field algebra (restricted to fields with arguments in the smaller box) defined by taking the expectation value with respect to the vacuum vector (of the standard Fock representation for a field with periodic boundary conditions) is different for $L=2\pi$ and $L=4\pi$.

We shall calculate explicitly

$$E_1(g,g)=\langle 0_L|\phi(g)\phi(g)|0_L\rangle,$$

where $\phi(g)=\int dx\, \phi(0,x)g(x)$ and (for example) $g(x)=1$ for $-\frac{1}{2}\pi<x<\frac{1}{2}\pi$ and $g(x)=0$ elsewhere. From the Fock representation of the field of mass $m\neq 0$ [cf. Eqs. (28)],

$$\phi(0,x)=L^{-1/2}\sum_k \omega_k^{-1/2}(e^{ikx}a_k^{(L)}+e^{-ikx}a_k^{(L)\dagger}) \qquad (k=0,\pm 2\pi/L,\ldots), \tag{39}$$

where the $a_k^{(L)}$ annihilate $|0_L\rangle$, one finds that

$$E_L(g,g)=\frac{2\pi}{L}\sum_k (2\omega_k)^{-1}\hat{g}(k)^2,$$

where the Fourier transform $\hat{g}(k)$ is $(2/\pi)^{1/2}k^{-1}\times\sin(\frac{1}{2}k\pi)$. It follows that

$$E_{2\pi}(g,g)=\frac{\pi}{4m}+\sum_k(\pi\omega_k k^2)^{-1}$$

and

$$E_{4\pi}(g,g)=\frac{\pi}{8m}+\sum_k(2\pi\omega_k k^2)^{-1}+\sum_k[4\pi\omega_k(\tfrac{1}{2}k)^2]^{-1},$$

where, in all the sums, $k=\pm1,\pm3,\ldots$. As $m\to 0$ the first terms in these expressions dominate. Since these differ from each other by a factor of 2, $E_{2\pi}(g,g)$ is not equal to $E_{4\pi}(g,g)$ for sufficiently small m. [Since $E_L(g,g)$ is analytic in m, they also cannot coincide for large m except possibly at some discrete points.]

The same argument applies to any function g whose $k=0$ Fourier coefficient, $\hat{g}(0)$, does not vanish, and to boxes of any lengths.

Some speculations on the physical interpretation of this result are offered in Ref. 10, Sec. IX.7.

If the box of length 2π is embedded in the infinite universe as the interval $[-\pi,\pi]$, then the field algebra of the box is a subalgebra of the total field algebra, and in analogy to Sec. II C one can solve for the box annihilation operators which appear in Eq. (39) in terms of the b operators of Eq. (28). One finds [dropping the index (L)] that

$$a_k=\int_{-\infty}^{\infty}dp\,\bar{u}(k-p)\left\{\left[\frac{1}{2}\left(\frac{\omega_k}{\omega_p}\right)^{1/2}+\left(\frac{\omega_p}{\omega_k}\right)^{1/2}\right]b_p+\frac{1}{2}\left[\left(\frac{\omega_k}{\omega_p}\right)^{1/2}-\left(\frac{\omega_p}{\omega_k}\right)^{1/2}\right]b_{-p}^\dagger\right\}, \tag{40}$$

where $\bar{u}(p)=(\pi p)^{-1}\sin(\pi p)$.

Next we investigate how the difference between the Fock representations affects the normal-ordered energy density $T^{00}(x)$ [Eq. (36)]. To postpone the appearance of ill-defined multiple integrals, we calculate first

$$T^{00}(g,g)=\tfrac{1}{2}\int dy_1\int dy_2\, g(y_1)g(y_2)\left[\pi(y_1)\pi(y_2)+\frac{\partial\phi}{\partial x}(y_1)\frac{\partial\phi}{\partial x}(y_2)+m^2\phi(y_1)\phi(y_2)\right], \tag{41}$$

where $g(y)$ is a smooth real function peaked at $y=x$. One would hope to define $T^{00}(x)$ by performing some operation such as normal ordering on the quantity (41) and then taking a limit as $g(y)$ approaches $\delta(y-x)$. We are interested in

$$\langle T^{00}(g,g)\rangle\equiv\langle T^{00}(g,g)\rangle_b-\langle T^{00}(g,g)\rangle_a,$$

where the terms on the right are the expectation values of $T^{00}(g,g)$ with respect to the vacuum states of the corresponding Fock representations. Alternatively, $\langle T^{00}(g,g)\rangle$ is the expectation value with respect to the b vacuum of the a-normal ordering of the quantity in Eq. (41). In Ref. 10 a calculation of $\langle T^{00}(g,g)\rangle$ defined in the latter way, based on Eq. (40), produced an indisputably infinite expression, given one choice of the (rather ambiguous) ordering of the multiple integrations involved. However, the following approach, which leads to a different conclusion, appears sounder.

The quantities on the right of Eq. (41) are finite, since they are sums of squares of norms of vectors such as $\int dy\, g(y)\pi(y)|0_a\rangle$. (Each Fock vacuum is in the domain of the smeared field operators of its own representation.) Explicit calculation yields

$$\langle T^{00}(g,g)\rangle_a = \tfrac{1}{2}\sum_{k=-\infty}^{\infty}\omega_k|\hat{g}(k)|^2,$$

$$\langle T^{00}(g,g)\rangle_b = \tfrac{1}{2}\int_{-\infty}^{\infty}dk\,\omega_k|\hat{g}(k)|^2,$$

where

$$\hat{g}(k) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{-iky}g(y)\,.$$

Hence

$$\langle T^{00}(g,g)\rangle = \tfrac{1}{2}\int_{-\infty}^{\infty}dk\left[\omega_k|\hat{g}(k)|^2 - \omega_{\{k\}}\,|\hat{g}(\{k\})|^2\right]$$

$$\equiv \sum_{k=-\infty}^{\infty}A(k)\,, \qquad (42)$$

where $\{k\}$ is the integer nearest to k and $A(k)$ is the integral over the interval of unit length centered on k.

As $g(y)$ peaks, $|\hat{g}(k)|^2$ approaches $(2\pi)^{-1}$. If we simply make this substitution in Eq. (42), we find, since ω_k is a convex function of k for $m\neq 0$, that each $A(k)$ is positive. The sum (42) converges in this case to a positive number, since the leading term in $A(k)$ is proportional to $d^2\omega_k/dk^2 \sim k^{-3}$. The analogous sum in three-dimensional k space would diverge to $+\infty$.

This argument is not rigorous, since it would be better to study the behavior of expression (42) for an arbitrary sequence of cutoff functions $\hat{g}(k)$ within the space of Fourier transforms of functions of compact support. Nevertheless, we have established that $\langle T^{00}(x)\rangle_b - \langle T^{00}(x)\rangle_a$ is an ill-defined object, and the burden of proof is surely upon him who would assert that it is zero. The author hopes to make a more thorough study of the physical significance of $T^{\mu\nu}$ in models of this type in another paper.

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*Based in part on portions of a dissertation presented to the Physics Department of Princeton University in partial fulfillment of the requirements for the degree of Doctor of Philosophy. Research supported in part by AFOSR Contract No. F44620-71-C-0108 with Princeton University. Preparation of paper supported by University of Wisconsin-Milwaukee Graduate School.

†NSF Graduate Fellow, Princeton University, 1967–71.

[1]We are not considering here the problem of quantization of the gravitational field. The metric of space-time is regarded as classical and, as far as the quantum field theory proper is concerned, externally prescribed. The classical dynamics of the metric may be imposed later in a self-consistency approximation by inserting an expectation value of the quantum energy-momentum tensor into the Einstein equations [M. R. Potier, Compt. Rend. 243, 939 (1956); C. Møller, in *Les Théories Relativistes de la Gravitation*, proceedings of a conference, Royaumont, 1959 (Centre National de la Recherche Scientifique, Paris, 1962), pp. 21–29; R. Utiyama and B. S. DeWitt, J. Math. Phys. 3, 608 (1962); S. Bonazzola *et al.*, Refs. 3 below; L. Parker and S. Fulling, Phys. Rev. D 7, 2357 (1973).

[2]L. E. Parker, Ph.D. thesis, Harvard University, 1966 (unpublished); Phys. Rev. Letters 21, 562 (1968); Phys. Rev. 183, 1057 (1969); Phys. Rev. D 3, 346 (1971); Phys. Rev. Letters 28, 705 (1972).

[3]S. Bonazzola and F. Pacini, Phys. Rev. 148, 1269 (1966); R. Ruffini and S. Bonazzola, Phys. Rev. 187, 1767 (1969).

[4]A. A. Grib and S. G. Mamaev, Yad. Fiz. 10, 1276 (1969) [Sov. J. Nucl. Phys. 10, 722 (1970)]; Yad. Fiz. 14, 800 (1971) [Sov. J. Nucl. Phys. 14, 450 (1972)].

[5]Ya. B. Zel'dovich, ZhETF Pis. Red. 12, 443 (1970) [JETP Lett. 12, 307 (1970)]; Ya. B. Zel'dovich and A. A. Starobinsky, Zh. Eksp. Teor. Fiz. 61, 2161 (1971) [Sov. Phys. JETP 34, 1159 (1972)].

[6]See, e.g., A. Lichnerowicz, *Propagateurs et commutateurs en relativité générale* (Institut de Hautes Etudes Scientifiques, Publications Mathématiques, 1961), No. 10.

[7]See, however, Sec. IIIB below.

[8]These points have been emphasized by Parker in the first and third of Refs. 2.

[9]G. T. Moore, J. Math. Phys. 11, 2679 (1970) (a study of the problem of the electromagnetic field in a cavity bounded by moving mirrors).

[10]S. A. Fulling, Ph.D. dissertation, Princeton University, 1972 (unpublished), especially Chap. X.

[11]Units such that $c=1$ and $\hbar=1$ are used throughout.

[12]It has been argued, primarily on the grounds of conformal invariance, that the equation should include a term $[(n-2)/[4(n-1)]R\phi$, where $R\equiv R^{\mu}_{\mu}$ is the scalar curvature [R. Penrose, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964), pp. 565–566; Proc. Roy. Soc. (London) A284, 159 (1965), especially Sec. 2; N. A. Chernikov and E. A. Tagirov, Ann. Inst. Henri Poincaré A 9, 109 (1968); I. I. Tugov, Ann. Inst. Henri Poincaré A 11, 207 (1969); C. G. Callan, Jr., S. Coleman, and R. Jackiw, Ann. Phys. (N.Y.) 59, 42 (1970); see also Ref. 5 and L. Parker, Phys. Rev. 7, 976 (1973)]. This controversy does not affect the argument of this paper, since R vanishes in flat space and the numerical factor vanishes in two-dimensional space. See also footnote 15 below.

[13]See, e.g., E. L. Hill, Rev. Mod. Phys. 23, 253 (1951), and any standard textbook on quantum field theory. Note that in this context one must use ordinary derivatives, delta functions, etc., rather than covariant ones. The canonical formalism is more general than Riemannian geometry.

[14]H. K. Urbantke, Nuovo Cimento 63B, 203 (1969); Ref. 10, Sec. VII.2.

[15]This conclusion need not hold if the "conformal" term proportional to $R\phi$ is added to the field equation (see Ref. 12). Indeed, a counterexample is provided by a static Robertson-Walker universe of the hyperbolic type,

with sufficiently large (negative) R and sufficiently small m. In such a case one encounters "jelly modes" like those studied in other external potential problems by L. I. Schiff, H. Snyder, and J. Weinberg, Phys. Rev. 57, 315 (1940), and by B. Schroer and J. A. Swieca, Phys. Rev. D 2, 2938 (1970).

[16]See M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, Vol. 1 (*Functional Analysis*) (Academic, New York, 1972), Chaps. 7 and 8. An operator is called *essentially self-adjoint* if it has a unique self-adjoint extension.

[17]An example is open de Sitter space regarded as a static universe — see R. Geroch, J. Math. Phys. 11, 437 (1970), and Ref. 10, Secs. III.5–III.6 and V.8. The latter reference shows that the question of Cauchy incompleteness when the mass is not zero is slightly more complicated than one might think.

[18]Details are presented in Ref. 10, Chap. VIII. A more abstract treatment of quantum field systems with time translation invariance is M. Weinless, J. Func. Anal. 4, 350 (1969), and the work of I. E. Segal cited there.

[19]This term does not here connote invariance under the inhomogeneous Lorentz group.

[20]The restriction to dimension 2 is inessential. It makes possible a simple exact solution.

[21]E. C. Titchmarsh, *Eigenfunction Expansions Associated with Second-order Differential Equations*, 2nd edition, Part I (Clarendon, Oxford, 1962), Sec. 4.15.

[22]W. Rindler, Am. J. Phys. 34, 1174 (1966); *Special Relativity*, 2nd edition (Oliver and Boyd, Edinburgh, 1966), pp. 41–43; *Essential Relativity* (Van Nostrand Reinhold, New York, 1969), pp. 61–64, 184–195.

[23]M. D. Kruskal, Phys. Rev. 119, 1743 (1960).

[24]Reference 10, Sec. IX.2.

[25]The integrals are evaluated in Ref. 10, Sec. IX.3. Equation (30) is a generalized Bogoliubov transformation, as is Eq. (40) below.

[26]See, e.g., J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965), Appendix C.

[27]Reference 10, Sec. IX.3 and Appendix F.

[28]In the case of unitary inequivalence (see last paragraph of Sec. II C).

[29]The author is grateful to K. Kuchař, L. H. Ford, L. Parker, A. S. Wightman, and E. Lubkin for discussions on this point. (The following remarks do not necessarily represent their views.)

[30]See R. Geroch, Ann. Phys. (N.Y.) 48, 526 (1968).

[31]A more extensive discussion of this point is contained in Ref. 10, Sec. IX.7.

[32]Other boundary conditions are possible, corresponding to reflection of wave packets with a change of phase. Cf. A. S. Wightman, in *Cargèse Lectures in Theoretical Physics: High Energy Electromagnetic Interactions and Field Theory*, 1964, edited by M. Lévy (Gordon and Breach, New York, 1967), Sec. 8. All these (physically distinct) models lead to the same Rindler-space theory when $z_0 \to 0$, since in that limit the reflected wave packet remains at "infinity," as explained below.

[33]E.g., N. J. Vilenkin, *Special Functions and the Theory of Group Representations* (American Mathematical Society, Providence, R.I., 1968), p. 270; or Ref. 10, p. 276.

[34]Reference 10, Appendix D and Secs. III. 3 and X.8.

[35]R. Adler, M. Bazin, and M. Schiffer, *General Relativity* (McGraw-Hill, New York, 1965), Sec. 2.4. A more abstract characterization of the method is that the time derivative which enters the definitions of π and H should correspond to the future-directed unit vector field normal to the hypersurface.

[36]R. F. Marzke and J. A. Wheeler, in *Gravitation and Relativity*, edited by H.-Y. Chiu and W. F. Hoffmann (Benjamin, New York, 1964), p. 42; J. A. Wheeler, in *Relativity, Groups and Topology*, edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964), p. 346.

[37]I. E. Segal, Ann. Math. 48, 930 (1947); R. Haag and D. Kastler, J. Math. Phys. 5, 848 (1964); G. G. Emch, *Algebraic Methods in Statistical Mechanics and Quantum Field Theory* (Wiley-Interscience, New York, 1972).

[38]See, e.g., I. E. Segal, *Mathematical Problems of Relativistic Physics* (American Mathematical Society, Providence, R. I., 1963); D. Kastler, Commun. Math. Phys. 1, 14 (1965); J. Manuceau, Ann. Inst. Henri Poincaré A 8, 139 (1968); G. F. Dell'Antonio, Commun. Math. Phys. 9, 81 (1968); J. Slawny, Commun. Math. Phys. 24, 151 (1972).

[39]It should be emphasized that, except for these technicalities, the canonical postulates [Eqs. (6) plus the equations of motion] define a unique and covariant algebraic structure. The argument of this paper does not show that there is anything wrong with the canonical *formalism;* the trouble begins (in the present exposition) after Eq. (14).

[40]The mathematical basis for the claim is a theorem of J. M. G. Fell, Trans. Am. Math. Soc. 94, 365 (1960). See, e.g., Emch (Ref. 37), pp. 97–110. Compare an observation in the context of the canonical formalism by A. Komar, Phys. Rev. 133, B542 (1964).

[41]In these cases a subalgebra of observables can be isolated abstractly from the C^* algebra associated with the field [see, e.g., I. F. Wilde, thesis, University of London, 1971 (unpublished), Chap. 2], but this does not really solve the physical problem.

[42]The connection between fields and asymptotic particle observables has been studied by H. Araki and R. Haag, Commun. Math. Phys. 4, 77 (1967), and by O. Steinmann, Commun. Math. Phys. 7, 112 (1968).

[43]Reference 10, Sec. IX.5 [Eq. (5.3)].

[44]H. B. G. Casimir, Proc. Kon. Ned. Akad. Wetenschap. 51, 793 (1948). An excellent recent review is T. H. Boyer, Ann. Phys. (N.Y.) 56, 474 (1970). The same method has been used to calculate electromagnetic forces in other physical situations (see Boyer's paper).

[45]T. H. Boyer, Phys. Rev. 185, 2039 (1969), and Ref. 46, pp. 498–499.

[46]A. D. Sakharov, Dokl. Akad. Nauk SSSR 177, 70 (1967) [Sov. Phys. Dokl. 12, 1040 (1968)].

[47]See R. Geroch, Ref. 15.

[48]Details of the calculations appear in Ref. 10, Appendix G.