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Scattering of Electromagnetic Radiation from a Black Hole*†

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The problem of propagation of electromagnetic waves in a gravitational field can be reduced to the problem of wave propagation in a material medium in flat space-time. This method is used here to calculate the scattering cross section of a Schwarzschild black hole and the polarization of the scattered wave. It is shown that the polarization of the wave (as measured at infinity) is not affected by the presence of the spherically symmetric gravitational field of the black hole. The light scattered from a rotating or "live" black hole is, however, expected to be polarized for an incident unpolarized plane wave. This may offer a new way to detect a live black hole.

I. INTRODUCTION

Electromagnetic and gravitational radiations offer powerful means to detect a black hole. Such a collapsed object may exist as the unseen component of an x-ray-emitting binary or the nucleus of some radio-galaxy. An account of the scattering of gravitational radiation from a Schwarzschild black hole has been given by Vishveshwara.[1] In this paper we calculate the cross section for the scattering of electromagnetic waves from a Schwarzschild black hole and study the polarization properties of the scattered radiation. We also discuss qualitatively the behavior of electromagnetic waves in a Kerr background. The results may offer a new way to detect a black hole under suitable circumstances.

The equation for the propagation of electromagnetic waves in a gravitational field can be obtained from the following formulation of the field equations.[2] Let $F^{\mu\nu}$ be the antisymmetric tensor of the electromagnetic field.[3] The equations that govern electromagnetic fields in a Riemannian space are

$$F^{\mu\nu}{}_{;\nu}=4\pi j^{\mu}$$

and

$$F_{\mu\nu;\sigma}+F_{\nu\sigma;\mu}+F_{\sigma\mu;\nu}=0\ ,$$

where J^{μ} is the electric current vector. Consider a definite coordinate frame where $-ds^2=g_{\mu\nu}dx^{\mu}dx^{\nu}$ and define the dual tensor $H^{\mu\nu}\equiv(-g)^{1/2}F^{\mu\nu}$, and $I^{\mu}\equiv(-g)^{1/2}J^{\mu}$. In this notation Maxwell's equations take the form $H^{\mu\nu}{}_{,\nu}=4\pi I^{\mu}$ and $F_{\mu\nu,\sigma}+F_{\nu\sigma,\mu}+F_{\sigma\mu,\nu}=0$. A space and time decomposition in a Cartesian coordinate system such that $F_{\mu\nu}\to(\vec{E},\vec{B})$, $H^{\mu\nu}\to(-\vec{D},\vec{H})$, and $I^{\mu}\to(\rho,\vec{J})$ yields

$$\vec{\nabla}\cdot\vec{B}=0\ ,\quad \vec{\nabla}\times\vec{E}=-\partial\vec{B}/\partial t\ ,$$

$$\vec{\nabla}\cdot\vec{D}=0\ ,\quad \vec{\nabla}\times\vec{H}=\partial\vec{D}/\partial t+4\pi\vec{J}\ .$$

Here $t=x^0$, $(\vec{x})_i=x^i$, and $\partial\rho/\partial t+\vec{\nabla}\cdot\vec{J}=0$. These equations look like Maxwell's equations in a flat space-time.

The "constitutive equations" $H^{\mu\nu}=(-g)^{1/2}g^{\mu\rho}g^{\nu\sigma}\times F_{\rho\sigma}$ can be written as

$$D_i=\epsilon_{ik}E_k-(\vec{G}\times\vec{H})_i \tag{1}$$

and

$$B_i=\mu_{ik}H_k+(\vec{G}\times\vec{E})_i\ , \tag{2}$$

where

$$\epsilon_{ik}=\mu_{ik}=-(-g)^{1/2}\frac{g^{ik}}{g_{00}} \tag{3}$$

and

$$G_i = -\frac{g_{0i}}{g_{00}} . \tag{4}$$

Therefore a complete description of the electromagnetic fields in a gravitational field can be obtained through the solution of Maxwell's equations in flat space-time but in a medium which is specified through the above constitutive equations. These latter equations are similar to those in a moving bi-anisotropic medium.

In this formulation the conformal invariance of Maxwell's equations is reflected in the fact that (ϵ_{ik}) and $\vec{G}$ are independent of a conformal factor. Conversely, experiments using scattering and absorption of electromagnetic waves to determine the background gravitational field can fix the metric only up to a conformal factor. This can be shown simply by noting that such experiments determine (ϵ_{ik}) and $\vec{G}$ and hence also

$$g_{0i}/g_{00} = -G_i \tag{4}$$

and

$$g_{ik}/g_{00} = -\lambda(\epsilon^{-1})_{ik} + G_i G_k , \tag{5}$$

where $\lambda = (-g)^{1/2}/(g_{00})^2$ can be obtained from the solution of the equation

$$-\det(g_{\mu\nu}/g_{00}) = \lambda^2 . \tag{6}$$

We note that this formulation of Maxwell's equations in a gravitational field offers a way to discuss the dispersion of electromagnetic waves in a gravitational "medium." Let $J^\mu = 0$ and consider the Fourier transforms of $\vec{E}$, $\vec{B}$, $\vec{D}$, and $\vec{H}$, e.g., $\vec{E}(\vec{x}, t) = \int \vec{E}(\vec{k}, \omega)\exp(i\vec{k}\cdot\vec{x} - i\omega t)d\vec{k}d\omega$. Let

$$\vec{F}^{\pm} = \vec{E} \pm i\vec{H} \tag{7}$$

and

$$\mathcal{K}_{ij} = \epsilon_{ij} - ie_{ijl}G_l , \tag{8}$$

where e_{ijl} is the totally antisymmetric symbol with $e_{123} = 1$. Then we see that in matrix notation

$$\mathcal{L}F^{\pm} = \pm\omega \int \mathcal{K}(\vec{k} - \vec{k}', \omega - \omega')F^{\pm}(\vec{k}', \omega')d\vec{k}'d\omega' , \tag{9}$$

where $\mathcal{L}_{ij} = -e_{ijl}k_l$ and $(\mathcal{K}_{ij})$ are Hermitian matrices. For a conformally flat space-time equation (9) implies that $\vec{F}^{\pm} = 0$ unless $\omega = |\vec{k}|$. For a general space-time, this linear homogeneous integral equation will have a nonzero solution only for a definite dispersion relation $\omega = \omega(\vec{k})$. Thus only a wave that satisfies this dispersion relation can propagate in the gravitational field. We note that the relation $\omega = \omega(\vec{k})$ is reasonable for a constant gravitational field where (ϵ_{ij}) and $\vec{G}$ are time-independent.

II. SCATTERING FROM A SCHWARZSCHILD BLACK HOLE

The solution of Einstein's equations corresponding to the exterior field of a massive spherical body is given by the Schwarzschild metric

$$-ds^2 = -\left(1 - \frac{2\mu}{r}\right)dt^2 + \left(1 - \frac{2\mu}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) , \tag{10}$$

where μ is the mass of the body. Maxwell's equations can be cast into a simple form if this metric is written in isotropic coordinates. Consider the coordinate ρ defined by $r \equiv \mu + \rho + \mu^2/4\rho$. We note that r is left invariant when ρ is replaced by $\mu^2/4\rho$. Therefore we shall choose as the basic coordinate $\mu/2 < \rho < \infty$ which corresponds to $2\mu < r < \infty$ ("upper sheet"; lesser positive values of ρ give the "lower sheet" of the 3-geometry). Let $g(\rho) = r(\rho)/\rho$ and $f(\rho) = 1 - 2\mu/r(\rho)$, then

$$-ds^2 = -f(\rho)dt^2 + g^2(\rho)(d\rho^2 + \rho^2 d\theta^2 + \rho^2\sin^2\theta d\phi^2) . \tag{11}$$

We now introduce the Cartesian coordinates $t = x^0$, $x^1 = \rho\sin\theta\cos\phi$, $x^2 = \rho\sin\theta\sin\phi$, and $x^3 = \rho\cos\theta$. Hence

$$-ds^2 = -f(\rho)dt^2 + g^2(\rho)(\delta_{ij}dx^i dx^j) .$$

As far as electromagnetic phenomena are concerned we can think of space-time as Minkowskian, occupied by a "medium" which is characterized by dielectric and magnetic permeability tensor $\epsilon_{ik} = \mu_{ik} = n(\rho)\delta_{ik}$ where

$$n(\rho) = g(\rho)/[f(\rho)]^{1/2} > 1 . \tag{12}$$

For large ρ, $n(\rho)$ has the interpretation of the index of refraction of the gravitational field and $n(\rho) \to 1$ as $\rho \to \infty$. The Maxwell's equations can then be written as

$$\frac{1}{i}\vec{\nabla}\times\vec{F}^{\pm} = \pm n\frac{\partial\vec{F}^{\pm}}{\partial t} . \tag{13}$$

Let us now consider ingoing 2^J-pole radiation of frequency ω

$$\vec{F}^{\pm}_{JM}(\vec{\rho}, t) = \sum_{\lambda=e,m,0} F^{\pm(\lambda)}_{JM}(\rho, \omega)\vec{Y}^{(\lambda)}_{JM}(\hat{\rho})e^{-i\omega t} , \tag{14}$$

where $\vec{Y}^{(e)}_{JM}$ and $\vec{Y}^{(m)}_{JM}$ are the transverse and $\vec{Y}^{(0)}_{JM}$ the longitudinal vector spherical harmonics.[4]

When one puts Eq. (14) into (13) one obtains

$$-[J(J+1)]^{1/2}F^{\pm(m)}_{JM} = \pm\omega\rho n(\rho)F^{\pm(0)}_{JM} , \tag{15}$$

$$-\frac{d}{d\rho}[\rho F^{\pm(m)}_{JM}] = \pm\omega\rho n(\rho)F^{\pm(e)}_{JM} , \tag{16}$$

and

$$\frac{d}{d\rho}\left[\rho F^{\pm(e)}_{JM}\right]-[J(J+1)]^{1/2}F^{\pm(0)}_{JM}=\pm\,\omega\rho n(\rho)F^{\pm(m)}_{JM}\ . \tag{17}$$

We now introduce a new coordinate x; $dx/d\rho\equiv n(\rho)$ and a new function $h(x)\equiv\rho(x)n(\rho(x))$. Then $x(\rho)$ can be written in terms of $r(\rho)$ as

$$x=r+2\mu\ln(r/2\mu-1) \tag{18}$$

so that as $r\to\infty$ (2μ), $x\to+\infty$ $(-\infty)$. Define a new radial function

$$X^{\pm(\lambda)}_{JM}(x,\omega)=\rho(x)F^{\pm(\lambda)}_{JM}(\rho(x),\omega)\ .$$

In terms of this function the radial part of the wave equation reads

$$d^2X^{\pm(m)}_{JM}/dx^2+[\omega^2-2U_J(x)]X^{\pm(m)}_{JM}=0\ , \tag{19}$$

$$X^{\pm(e)}_{JM}=\mp(1/\omega)dX^{\pm(m)}_{JM}/dx\ , \tag{20}$$

and

$$X^{\pm(0)}_{JM}=\mp(1/\omega)[J(J+1)]^{1/2}h^{-1}(x)X^{\pm(m)}_{JM}\ . \tag{21}$$

Equation (19) is similar to the Schrödinger equation in one dimension for a particle of unit mass and energy $\frac{1}{2}\omega^2$ in a potential

$$U_J(x)=\frac{1}{2}\,\frac{J(J+1)}{r^2(x)}\left(1-\frac{2\mu}{r(x)}\right)\ . \tag{22}$$

For $x\to\infty$,

$$U_J(x)\to\sim\tfrac{1}{2}J(J+1)/x^2$$

and for $x\to-\infty$,

$$U_J(x)\to\frac{J(J+1)}{8\mu^2}\,e^{x/2\mu}\ .$$

The maximum of $U_J(x)$ occurs at $r(x_{\max})=3\mu$ (Fig. 1).

It is now simple to calculate reflection coefficients for ingoing spherical waves. We have

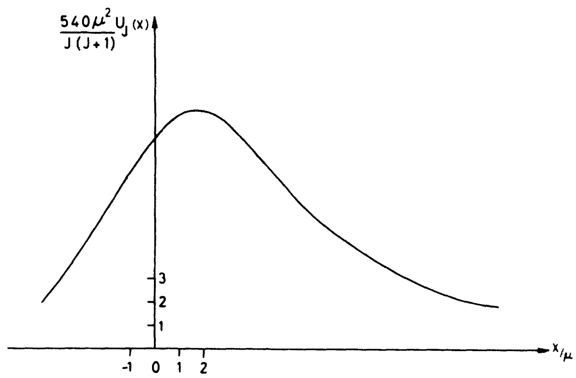

FIG. 1. The effective potential for electromagnetic waves in the Schwarzschild background.

$$X^{\pm(m)}_{JM}\underset{x\to\infty}{\sim}A^{\pm(m)}_{JM}\exp(-ikx)+B^{\pm(m)}_{JM}\exp(ikx)$$
$$(k=\omega)\ , \tag{23}$$

$$X^{\pm(m)}_{JM}\underset{x\to-\infty}{\sim}C^{\pm(m)}_{JM}\exp(-ikx) \tag{24}$$

and Eq. (20) implies

$$X^{\pm(e)}_{JM}\underset{x\to\infty}{\sim}\pm iA^{\pm(m)}_{JM}\exp(-ikx)\mp iB^{\pm(m)}_{JM}\exp(ikx) \tag{25}$$

and

$$X^{\pm(e)}_{JM}\underset{x\to-\infty}{\sim}\pm iC^{\pm(m)}_{JM}\exp(-ikx)\ . \tag{26}$$

Equation (21) implies that $X^{\pm(0)}_{JM}(x\to\pm\infty)\to0$ as expected. Since as $x\to\infty$, $x\to\rho$, we see that ingoing magnetic and electrical spherical waves are reflected with an amplitude $B^{(m)}_{JM}/A^{(m)}_{JM}$ $(B^{+(m)}_{JM}/A^{+(m)}_{JM}=B^{-(m)}_{JM}/A^{-(m)}_{JM})$, and a reflection coefficient $R_J=|B^{(m)}_{JM}/A^{(m)}_{JM}|^2$. The amount of radiation that goes down the black hole is then given by the transmission coefficient

$$T_J=1-R_J=|C^{(m)}_{JM}/A^{(m)}_{JM}|^2\ .$$

Therefore the scattering amplitude for the magnetic 2^J-pole radiation is

$$a^{(m)}_{JM}=-(-1)^JR_J^{\,1/2}e^{2i\delta_J} \tag{27}$$

and for the electric 2^J-pole radiation is $a^{(e)}_{JM}=-a^{(m)}_{JM}$. We note that the potential barrier vanishes for $J=0$ and U_J goes up as J^2 for large J. Therefore the reflection coefficient R_J increases with J for a given value of the frequency. The complex form of the potential does not allow an exact analytic solution for $a^{(m)}_{JM}$. Here we present only limiting formulas for R_J. For large values of J such that $2J>\omega a_0>1$, where a_0 is the width of the potential and is of the order of μ, the quasiclassical approximation yields

$$R_J(\omega)=1-\exp\left(-2\left|\int_{\rho_1}^{\rho_2}[J(J+1)/\rho^2-\omega^2n^2]^{1/2}d\rho\right|\right)\ , \tag{28}$$

where ρ_1 and ρ_2 are the turning points, that is solutions of

$$\omega\rho n(\rho)=[J(J+1)]^{1/2}\ .$$

For x sufficiently close to $x_{\max}$ we can write

$$U_J(x)\simeq U_J(x_{\max})-\tfrac{1}{2}k_0(x-x_{\max})^2\ ,\quad k_0>0\ .$$

Let

$$\epsilon=[\omega^2/2-U_J(x_{\max})]/(k_0)^{1/2}$$

then the reflection coefficient "above" or "below" the barrier (but near the top of the barrier) is given by[5]

$$R \simeq 1/[1+\exp(2\pi\epsilon)] . \tag{29}$$

When $\omega^2 < 2U_J(x_{\max})$ such that $\exp(2\pi\epsilon) \ll 1$, then $R \simeq 1-\exp(-2\pi|\epsilon|)$ and we recover the quasiclassical barrier-penetration formula given above.

III. SCATTERING OF PLANE WAVES

A plane wave can be expressed as a superposition of spherical waves; therefore we can use the results of the previous section to study the scattering of a plane wave of arbitrary polarization from a Schwarzschild black hole. Let us consider an ingoing plane wave of frequency ω. It will be enough to consider right circularly polarized ($\vec{F}^- = 0$) and left circularly polarized ($\vec{F}^+ = 0$) photons. Therefore the solution of the wave equation $\vec{\nabla}\times\vec{F}^\pm = \pm n\omega\vec{F}^\pm$ can be written as

$$\vec{F}^\pm \underset{\rho\to\infty}{\sim} \vec{\chi}_{\pm1}\exp(ikx\cos\theta) + \vec{\mathfrak{F}}^\pm(\vec{k})\exp(ikx)/\rho , \tag{30}$$

where the initial plane wave is incident along the x^3 axis, $k=\omega$, and $\vec{k}=(k, \theta, \phi)$. We have used the orthonormal vectors $\vec{\chi}_0 = i\hat{x}^3$ and $\vec{\chi}_{\pm1} = (\hat{x}^2 \mp i\hat{x}^1)/(2)^{1/2}$. In the asymptotic formula for $\vec{F}^\pm$, we have written x in the exponents instead of ρ to take due account of the long-range nature of the gravitational interaction.[6] The vectors $\vec{\mathfrak{F}}^\pm$ can in general be written as

$$\vec{\mathfrak{F}}^\pm = \sum_{JM\,\sigma=e,m} f_{JM}^{\pm(\sigma)}(k)\vec{Y}_{JM}^{(\sigma)}(\hat{k}) . \tag{31}$$

To obtain expressions for $f_{JM}^{(\sigma)}$ we expand $\exp(ikx\cos\theta)$ as follows:

$$\underline{1}\exp(ikx\cos\theta) = \frac{4\pi}{kx}\sum_{Jlm}(i)^l\,\vec{Y}_{Jl}^M(\hat{k}) \times u_l(kx)\vec{Y}_{Jl}^{M*}(\hat{x}^3) ,$$

where $\underline{1}$ is the unit tensor and $u_l(z) = zJ_l(z)$. From

$$u_l(z) \underset{z\to\infty}{\sim} \sin(z - l\pi/2)$$

we get

$$\vec{\chi}_{\pm1}\exp(ikx\cos\theta) \underset{\rho\to\infty}{\sim} \frac{4\pi}{k\rho}\sum_{JM\,\sigma=e,m}\vec{Y}_{JM}^{(\sigma)}(\hat{k})u_J^{(\sigma)}(kx)v_{JM\pm}^{(\sigma)} , \tag{32}$$

where

$$u_J^{(e)} = [\exp(ikx) + (-1)^J\exp(-ikx)]/(2i) , \tag{33}$$

$$u_J^{(m)} = [\exp(ikx) - (-1)^J\exp(-ikx)]/(2i) , \tag{34}$$

and

$$v_{JM\pm}^{(\sigma)} = \vec{Y}_{JM}^{*(\sigma)}(\hat{x}^3)\cdot\vec{\chi}_{\pm1} . \tag{35}$$

Therefore

$$v_{JM\pm}^{(e)} = i(-i)^J\delta_{\pm1,M}[(2J+1)/8\pi]^{1/2} ,$$

$$v_{JM\pm}^{(m)} = \mp(-i)^J\delta_{\pm1,M}[(2J+1)/8\pi]^{1/2}$$

and putting Eqs. (30)–(35) together with (23)–(27) we obtain the following formulas for the scattering amplitudes:

$$f_{JM}^{\pm(m)} = \pm(i)^{J-1}(2\pi/k)[(2J+1)/8\pi]^{1/2} \times[(-1)^J + a_{JM}^{(m)}]\delta_{\pm1,m} , \tag{36}$$

$$f_{JM}^{\pm(e)} = \mp i f_{JM}^{\pm(m)} . \tag{37}$$

The spherical symmetry of the problem allows us to put $\phi=0$ in the following without any loss in generality. Then we can choose the outgoing polarization basis such that

$$\vec{\chi}_{\pm1}^{(k)} = \tfrac{1}{2}(1\pm\cos\theta)\vec{\chi}_1 + \tfrac{1}{2}(1\mp\cos\theta)\vec{\chi}_{-1} \pm(1/\sqrt{2})\sin\theta\vec{\chi}_0 , \tag{38}$$

$$\vec{\chi}_0^{(k)} = -(1/\sqrt{2})\sin\theta\vec{\chi}_1 + (1/\sqrt{2})\sin\theta\vec{\chi}_{-1} + \cos\theta\vec{\chi}_0 . \tag{39}$$

It can be shown (Appendix) that

$$\vec{\mathfrak{F}}^\pm = -(i/4k)(1+\cos\theta)Q(\mu k, \cos\theta)\vec{\chi}_{\pm1}^{(k)} , \tag{40}$$

where

$$Q(\mu k, \cos\theta) = \sum_{J\geq1}(2J+1)[R_J^{1/2}\exp(2i\delta_J) - 1] \times Q_J(\cos\theta) . \tag{41}$$

Let $z=\cos\theta$; then $Q_J(z)$ is defined as follows:

$$Q_J(z) \equiv 2[J(J+1)]^{-1} \times\frac{d^2}{dz^2}[(J+1)(2J+1)^{-1}P_{J-1}(z) + 2(2J+1)^{-1}P_{J+1}(z) - P_J(z)] , \tag{42}$$

where $P_J(z)$ is the Legendre polynomial of order J. The $Q_J(z)$, $J=1,2,3,\ldots$ are polynomials of order $(J-1)$ with $Q_J(1)=1$, i.e., $Q_1(z)=1$, $Q_2(z)=2z-1$, $Q_3(z)=(15z^2-10z-1)/4$, etc. These polynomials form a complete set and are orthogonal with $(1+z)$ as the weight function, that is

$$\int_{-1}^{1}(1+z)^2 Q_J(z)Q_{J'}(z)dz = 8(2J+1)^{-1}\delta_{JJ'} . \tag{43}$$

Let ρ_i be the density matrix of the initial plane wave and F the scattering matrix. F is a 2×2 matrix with elements F_{RR}, F_{RL}, F_{LR}, and F_{LL}. F_{RL}, say, is the amplitude for scattering of an incident left-circularly polarized plane wave to an outgoing right-circularly polarized spherical wave. From Eq. (40) we can see that F is a multiple of the unit matrix and hence the scattering cross section is

(for any initial polarization)

$$d\sigma_{\rm scatt}/d\Omega = (1/16k^2)(1+\cos\theta)^2|\,Q(\mu k, \cos\theta)|^2 \tag{44}$$

and the final density matrix ρ_f is the same as ρ_i, $\rho_f = \rho_i$, so that the polarization properties of the initial wave are unchanged.

The absorption cross section is given by

$$\sigma_{\rm abs} = \pi k^{-2} \sum_{J\geq 1} (2J+1)(1-R_J) \ . \tag{45}$$

From (45) we recover the usual expression

$$\sigma_{\rm abs} = 27\pi\mu^2$$

in the limit of geometrical optics. Thus for a given large k value ($k\mu \gg 1$) partial waves go down the black hole ($R_J = 0$) or do not ($R_J = 1$) according as the classical impact parameter $b = J/k$ is less than or greater than the critical impact parameter $b = (27)^{1/2}\mu$.

From (44) we recover the Rutherford law of scattering

$$d\sigma_{\rm scatt}/d\Omega \approx 16\mu^2/\theta^4 \tag{46}$$

for small-angle scattering of photons (large impact parameter) in the same semiclassical limit. For $k\mu \gg 1$ and $\theta \ll 1$, the main contribution to the sum in (41) comes from $J \gg k\mu$ (in fact from J close to J_0, $J_0 + \frac{1}{2} = 4k\mu/\theta$). The phase shift and the reflection coefficient for such high J values are then $\delta_J \sim 2k\mu \ln(J+\frac{1}{2})$ and $R_J \sim 1$. Hence

$$(1+\cos\theta)Q(\mu k, \cos\theta) \approx (16\mu k/\theta^2)e^{is} \ , \tag{47}$$

where $s(\theta) = 4\mu k[\ln(4\mu k/\theta) - 1]$. This analysis neglects photons which make a 2π (or 4π or 6π, etc.) loop around the black hole, an approximation which is the more justified the smaller the scattering angle (predominance of Rutherford scattering over "orbiting").

When one turns to 180° scattering, one might expect no back scattering from a first look at (44) because of the factor $(1+\cos\theta)^2$ and the fact that $Q_J(-1) = (-1)^{J+1}J(J+1)/2$. However, this result stands in complete contrast with the most elementary concepts of black-hole physics.[7] A photon can take a $\pi = 180°$ (or 3π or 5π, etc.) loop around a black hole and return straight back to the source. Moreover, this scattering gives rise to a glory.[8] Thus geometrical optics predicts infinity for the scattering cross section at $\cos\theta = -1$. The apparently contradictory predictions of zero and infinity can be reconciled when more knowledge of R_J and δ_J is available.

IV. ELECTROMAGNETIC WAVES IN A KERR GRAVITATIONAL FIELD

The Kerr solution to Einstein's equations describes the geometry of space-time outside a rotating collapsed object. In Schwarzschild-like coordinates the metric is

$$\begin{aligned} -ds^2 &= \Sigma\Delta^{-1}dr^2 + \Sigma d\theta^2 \\ &\quad + \Sigma^{-1}\sin^2\theta\,[a\,dt - (r^2+a^2)d\phi]^2 \\ &\quad - \Sigma^{-1}\Delta[dt - a\sin^2\theta\, d\phi]^2 \ , \end{aligned} \tag{48}$$

where $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2\mu r + a^2$, μ is the mass, and μa the angular momentum of the black hole about the x^3 axis. The stationary limit surfaces occur at $r_\pm = \mu \pm (\mu^2 - a^2\cos^2\theta)^{1/2}$. The region outside $r = r_+$ is the subject of all the remarks that follow. We again introduce the Cartesian coordinate system ($t = x^0, x^1, x^2, x^3$) and write the metric as $-ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The electromagnetic properties of the Kerr field can be described by the dielectric "tensor" $\epsilon_{ik}(x) = -(-g)^{1/2}g^{ik}/g_{00}$ and the "vector" $G_i(x) = -g_{0i}/g_{00}$. We can therefore write

$$e_{ijk}\frac{\partial F_k^\pm}{\partial x^j} = (\pm i\epsilon_{ik} - e_{ijk}G_j)\frac{\partial F_k^\pm}{\partial t} \ . \tag{49}$$

Assuming solutions that vary with time as $e^{-i\omega t}$ we get

$$\left(\frac{1}{i}\vec{\nabla} - \omega\vec{G}\right)\times\vec{F}^\pm = \mp i\omega\epsilon\cdot\vec{F}^\pm \ , \tag{50}$$

where $(\epsilon\cdot\vec{F}^\pm)_i = \epsilon_{ik}F_k^\pm$. The solution of this equation in the general case is complicated and hence we shall limit the discussion to some qualitative remarks. We know that in a (conformally) flat space-time the equations of electromagnetic wave propagation are

$$\frac{1}{i}\vec{\nabla}\times\vec{F}^\pm = \pm\frac{\partial\vec{F}^\pm}{\partial t}$$

which can easily be written in Dirac form. For waves of frequency ω, we get $\vec{\nabla}\times\vec{F}^\pm = \pm\omega\vec{F}^\pm$. In the Schwarzschild geometry the corresponding equation was shown to be

$$\vec{\nabla}\times\vec{F}^\pm + \omega V_\pm\vec{F}^\pm = \pm\omega\vec{F}^\pm \ , \tag{51}$$

where $V_\pm = \pm(1-n)$ has the interpretation of gravitational potential. The corresponding equation for the Kerr metric is the relation (50) where besides the tensor gravitational potential we have an extra vector potential $\vec{G}$. Hence the motion of electromagnetic waves in the Kerr geometry is similar to that in the Schwarzschild case except for the inclusion of a "gravitational magnetic" field.

We have ignored the effect of electromagnetic radiation on the black hole itself in all of the above discussions, that is the change in the mass or angular momentum of the collapsed body in the process of scattering has been neglected. It is clear that an electromagnetic perturbation of the Kerr (or Schwarzschild) field produces only a second-order correction to the metric. For a charged black hole however the correction to the metric will in general be of the first order and cannot be ignored.

We have seen that a Schwarzschild black hole usually absorbs electromagnetic radiation. A rotating black hole is also capable of absorbing electromagnetic waves. However there are arguments which suggest the possibility that such an object can also amplify electromagnetic radiation of the right frequency ω and component of angular momentum in the direction of the rotation of the black hole. The black hole itself instead of gaining energy loses it in the process.[9,10] Recently Teukolsky[11] has produced separable equations for the radiative part of the electromagnetic perturbation of the Kerr field, and has thereby rendered tractable the scattering problem for rotating black holes.

V. ASTROPHYSICAL IMPLICATIONS

In the scattering of electromagnetic radiation by a material medium, the effect of matter usually dominates that of the gravitational field of the medium. A black hole, however, scatters electromagnetic waves by its gravitational field only, provided that it is not accreting fresh matter. We have shown that a spherically symmetric black hole does not affect the polarization of light as measured at infinity with respect to the local polarization basis. In contrast the discussion of the electromagnetic scattering in the Kerr background tells one that incident right circularly polarized photons must scatter differently from left-circularly polarized ones due to the interaction of the photon spin with the gravitational "magnetic field." Let a right circularly polarized plane wave ($\vec{F}^- = 0$) be incident on a Kerr black hole. We have seen that the electromagnetic equations break up into an equation for $\vec{F}^+$ and another for $\vec{F}^-$, so that the scattered wave will also have $\vec{F}^- = 0$ and far away from the black hole it is again right circularly polarized. Similarly an incident left circularly polarized plane wave will always have $\vec{F}^+ = 0$, and the scattered wave will be left circularly polarized at infinity. The scattering amplitude, however, will be different in the two cases. Thus one must expect partial polarization in the scattered light when an unpolarized plane wave is incident on a Kerr black hole.

It has been suggested that a collapsed star may exist as the unseen component of a binary star system.[12] Thorne and Trimble[13] investigated known binaries. However, they reported that no definitive identification was possible. If the light scattered from a black hole is polarized, then a binary system with a black-hole component might show a periodically variable polarization due to electromagnetic scattering from the gravitational field of the black hole. Observation of polarized radiation from some eclipsing binaries has revealed variable polarization.[14] Due to its small size, however, a black hole cannot perhaps provide effective eclipsing. This variable polarization has been interpreted to be mainly due to Thomson scattering in a nonspherical gaseous envelope. Such envelopes can be formed by accretion of matter, which is perhaps the dominant dynamical interaction in a close binary system. Some of the accreted matter will probably form a thin, hot layer in the equatorial plane of the dense component of the binary.[15]

Consider a single-line binary where the invisible component is a massive rotating collapsed star and negligible accretion is occurring (Fig. 2). The visible component gives off radiation, some of which is scattered from the black hole and received by the observer. The scattering is dominated by the gravitational field if the average electron density around the black hole is much smaller than a critical value N_c for which a crude estimate can be given as follows. Most of the radiation is emitted in wavelengths that are very short compared with the size of the black hole ($\mu > 2M_\odot$). Thus the scattering cross section is proportional to μ^2. N_c is obtained from $N_c\mu^3\sigma_T \sim \mu^2$, where $\sigma_T = (8\pi/3)(e^2/m_e c^2)^2$ is the Thomson cross section. Thus for $\mu \sim 10M_\odot$, we get $N_c \sim 10^{18}$ electrons/cm^3. In other words, gravitational scattering will dominate over Thomson scattering if the number density of electrons is less than 10^{18}/cm^3, a condition that is not difficult to meet.[16] Thus the polarization properties of the light scattered from a

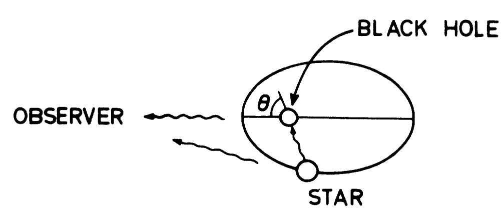

FIG. 2. The radiation received by the observer is mainly due to the normal star, but contains a contribution from the radiation scattered from the Kerr black hole. Hence if the observer is in the plane of the binary the observed polarization is the product of the fraction of the intensity scattered by the black hole and the intrinsic polarization of the scattered light.

black hole may provide a further clue in the continuing search for black holes.

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APPENDIX

Let us define the quantities f_J, $\mathfrak{M}_J$, and $\mathfrak{N}_J$ as follows:

$$f_J=(-i)^J[\pi(2J+1)/2]^{1/2}k^{-1}[R_J^{1/2}\exp(2i\delta_J)-1]\,, \tag{A1}$$

$$\mathfrak{M}_J=[\vec{Y}^{(e)}_{J1}+i\vec{Y}^{(m)}_{J1}]\cdot\vec{\chi}^{(k)*}_1\,, \tag{A2}$$

and

$$\mathfrak{N}_J=[\vec{Y}^{(e)}_{J1}+i\vec{Y}^{(m)}_{J1}]\cdot\vec{\chi}^{(k)*}_{-1}\,. \tag{A3}$$

One can then show that

$$\vec{\mathfrak{F}}^+=\sum_J f_J(\mathfrak{M}_J\vec{\chi}^{(k)}_1+\mathfrak{N}_J\vec{\chi}^{(k)}_{-1}) \tag{A4}$$

and

$$\vec{\mathfrak{F}}^-=\sum_J (-1)^{J+1}f_J(\mathfrak{M}_J\vec{\chi}^{(k)}_1+\mathfrak{N}_J\vec{\chi}^{(k)}_{-1})^*\,. \tag{A5}$$

We want to prove that $\mathfrak{N}_J=0$. To do this we need to define polynomials $S_J(z)$, $J=1,2,3,\ldots$ with $S_J(1)=1$ as follows:

$$S_J(z)=\tfrac{1}{2}[(J+1)(2J+1)^{-1}P_{J-1}(z)+J(2J+1)^{-1}P_{J+1}(z)+P_J(z)]\,; \tag{A6}$$

then one can show that

$$\mathfrak{M}_J=(i)^{J-1}[(2J+1)/8\pi]^{1/2}(1+z^{-1})\times[S_J(z)-\tfrac{1}{4}(1-z)^2Q_J] \tag{A7}$$

and

$$\mathfrak{N}_J=(i)^{J-1}[(2J+1)/8\pi]^{1/2}(1-z^{-1})\times[S_J(z)-\tfrac{1}{4}(1+z)^2Q_J]\,. \tag{A8}$$

It now remains to show that $S_J(z)=\frac{1}{4}(1+z)^2Q_J(z)$. Let $P_J'=dP_J/dz$ and recall the following relations between Legendre polynomials:

$$(1-z^2)P_J''=2zP_J'-J(J+1)P_J\,, \tag{A9}$$

$$(J+1)P_J=P_{J+1}'-zP_J'\,, \tag{A10}$$

$$JP_J=zP_J'-P_{J-1}'\,, \tag{A11}$$

$$(1-z^2)P_J'=[J(J+1)/(2J+1)](P_{J-1}-P_{J+1})\,, \tag{A12}$$

and

$$(2J+1)zP_J=JP_{J-1}+(J+1)P_{J+1}\,. \tag{A13}$$

Relations (A10) and (A11) imply

$$(J+1)P_{J-1}'+JP_{J+1}'=(2J+1)zP_J'. \tag{A14}$$

We differentiate (A14) and insert it into $Q_J(z)$ to get

$$Q_J(z)=[2/J(J+1)][P_J'-(1-z)P_J'']\,, \tag{A15}$$

then

$$(1+z)Q_J(z)=[2/J(J+1)][(1-z)P_J'+J(J+1)P_J]$$

using the relation (A9). Now

$$\tfrac{1}{4}(1+z)^2Q_J(z)=[2(2J+1)]^{-1}\times[(2J+1)(1+z)P_J+P_{J-1}-P_{J+1}] \tag{A16}$$

if we use the relation (A12). Finally by using (A13) we can put $S_J(z)$ into the form $S_J(z)=\frac{1}{4}(1+z)^2\times Q_J(z)$. It then follows that

$$\mathfrak{M}_J=(i)^{J-1}[(2J+1)/8\pi]^{1/2}(1+z)Q_J(z)\,. \tag{A17}$$

When we put this equation back into (A4) and (A5) we obtain Eq. (40).

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Maximally Slicing a Black Hole*

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Analytic and computer-derived solutions are presented of the problem of slicing the Schwarzschild geometry into asymptotically-flat, asymptotically-static, maximal spacelike hypersurfaces. The sequence of hypersurfaces advances forward in time in both halves ($u \geq 0$, $u \leq 0$) of the Kruskal diagram, tending asymptotically to the hypersurface $r = \frac{3}{2}M$ and avoiding the singularity at $r = 0$. Maximality is therefore a potentially useful condition to impose in obtaining computer solutions of Einstein's equations.

We report (1) the results of a computer study of the problem of maximally slicing the Schwarzschild geometry and (2) an analytic solution of the same problem. The computer study was part of a much larger program currently under way at Texas to study the behavior of colliding black holes. Data obtained in such studies will be of interest to two groups of physicists: those who wish to learn something about the gravitational energy radiated to infinity and those who are curious about details of the growth and coalescence of event horizons and the dynamical processes taking place near the singularities hidden inside. In order to obtain data of use to both groups one must employ coordinate systems that guarantee the following:
(A) The metric must pass smoothly to a familiar metric (e.g., spherical Minkowskian) at infinity.
(B) The hypersurfaces x^0 = constant must be spacelike and must penetrate the event horizon(s).
(C) The metric must remain nonsingular for the duration of the computation.

It has long been speculated that a good way of securing these conditions is to require the hypersurfaces x^0 = constant to be maximal.[1] The maximality condition is expressed by the statement

$$\gamma^{ij}K_{ij}=0\,, \tag{1}$$

where, for each x^0, γ^{ij} and K_{ij} are, respectively, the (contravariant) 3-metric and second fundamental form on the corresponding hypersurface. When (1) holds, Einstein's vacuum equations reduce to

$$K_{ij}K^{ij} = {}^{(3)}R \equiv \gamma^{ij}\,{}^{(3)}R_{ij}\,, \tag{2}$$

$$K^{ij}{}_{.j}=0\,, \tag{3}$$

$$K_{ij,0}=\alpha\,{}^{(3)}R_{ij}-\alpha_{.ij}-2\alpha K_{ik}K^{k}{}_{j}+\mathcal{L}_{\beta}K_{ij}\,, \tag{4}$$

where, for each x^0, ${}^{(3)}R_{ij}$ is the curvature tensor of the corresponding hypersurface, α and β_i are, respectively, the lapse function and shift vectors relating this hypersurface to its neighbors, dots denote covariant differentiation in the hypersurface, and indices are raised and lowered by means of the 3-metric.[2]

Equation (1) may be treated on a par with the so-called initial-data constraints (2) and (3). If these three equations are imposed on an initial hypersurface then they will automatically be maintained on each succeeding hypersurface by the dynamical equation (4) together with the following condition on the lapse function[3]:

$$\alpha_{.i}{}^{i}={}^{(3)}R\alpha\,. \tag{5}$$

In an effort to determine whether the maximality condition in fact secures conditions (A), (B), (C) above, and whether solution of Eqs. (1)–(5) is practical on the computer, four of us (S.C., B.D., L.S., E.T.) undertook to run a test on the simplest nontrivial example: the Schwarzschild black hole.

Constraints (1), (2), and (3) were satisfied by choosing the initial hypersurface to be the u axis in the familiar Kruskal plane.[4] Equations (4) and