[11] I_{VV} for $\nu_{lab} \leq 0.45$ GeV due to $P_{33}(1236)$ has the following values resulting from different analyses: 260 μb with Walker's fit, 210 μb with the Moorhouse-Oberlack fit, and 237 μb with the Pfeil-Schwela fit.

[12] The Pfeil-Schwela analysis is limited to the first resonance region and to 0^+, 1^-, and 1^+ partial waves.

[13] H. D. I. Abarbanel and M. L. Goldberger, Phys. Rev. 165, 1594 (1968).

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Multiperipheral Cluster Model*

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A multiperipheral cluster model of high-energy inelastic collisions is discussed. A physical justification for the model is given, in that it accounts for "Ericson fluctuations" in the inelastic cross sections. These incoherent resonance contributions are ignored in most multi-Regge models. The major assumption is made that the ratio of the average fluctuation cross sections between different channels is governed by phase space. This allows a description of the cluster decay processes via the statistical bootstrap theory of Hagedorn and Frautschi. A simplified mathematical formulation of the model is studied, and it is shown that, subject to certain resistrictions on the vertex functions for cluster production, Feynman scaling occurs. The relationship with other models of inelastic processes is discussed. It is shown that the model includes both the ordinary multi-Regge model and the diffractive excitation models of Hwa and of Jacob and Slansky, as limiting cases, and that the approach is equivalent to, but is a refinement on, Hagedorn's thermodynamic model. A search for a power-behaved "tail" in the multiplicity distributions at high energies would provide an important test of the model.

I. INTRODUCTION

Two of the most popular models of high-energy inelastic hadron collisions are the multiperipheral or multi-Regge model, in various guises, and the thermodynamic model of Hagedorn (see, for example, the recent review by Frazer *et al.*[1]). These two models are almost orthogonal to each other in approach, and their main results concern quite different areas of interest. It is our object to show how a "multiperipheral cluster model"[2] can combine the essential features of them both, and to give a physical justification for the new model. Similar ideas have been expressed by Ranft and Ranft.[3]

In the multiperipheral cluster model, the total inelastic cross section is taken to be made up of a sum of terms,

$$\sigma_{\text{inel}} = \sum_n \sigma_n , \tag{1}$$

as represented diagrammatically in Fig. 1. Here σ_n represents the cross section for production of n "clusters," via a multiperipheral mechanism (i.e., repeated exchanges of Reggeons). Thus the coherent, or dynamical, part of the cross section in any given channel is assumed to be given by a multi-Regge model. Each "cluster," on the other hand, will be identified with the *incoherent*, or non-Regge, contributions of intermediate resonance states decaying according to the statistical bootstrap model of Hagedorn[4] and Frautschi.[5] The cluster may therefore have a variable mass. At the lower limit of its mass range it will consist of a single stable hadron (e.g., a pion), while at high mass it consists of an unstable resonance which eventually gives rise to a cloud of stable decay products, all moving with limited momenta in the center-of-mass system of the cluster. The multiplicity of decay products is thus proportional to the mass of the cluster (which is also called a "fireball" by Hagedorn[4]). This decay process has recently been studied in detail by Frautschi and the present author.[6,7]

A justification for this model is provided in Sec. II, on the grounds that it accounts for the incoherent resonance contributions to hadron cross sections, commonly called "Ericson fluctuations" in nuclear physics, which were recently discussed by Frautschi.[8] These fluctuation terms are due to random (incoherent) variations in the resonance coupling strengths and spacings: They average to zero in the reaction *amplitudes*, over suitably large energy intervals, but provide important

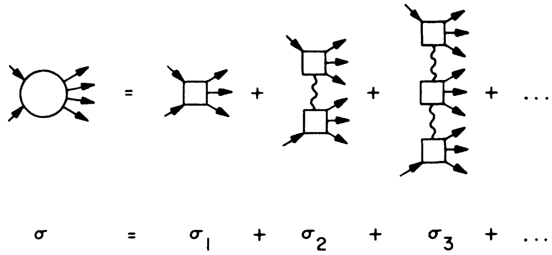

FIG. 1. Multiperipheral cluster model of a high-energy inelastic collision. Each box represents a "cluster," i.e., the incoherent production and decay of hadron resonances.

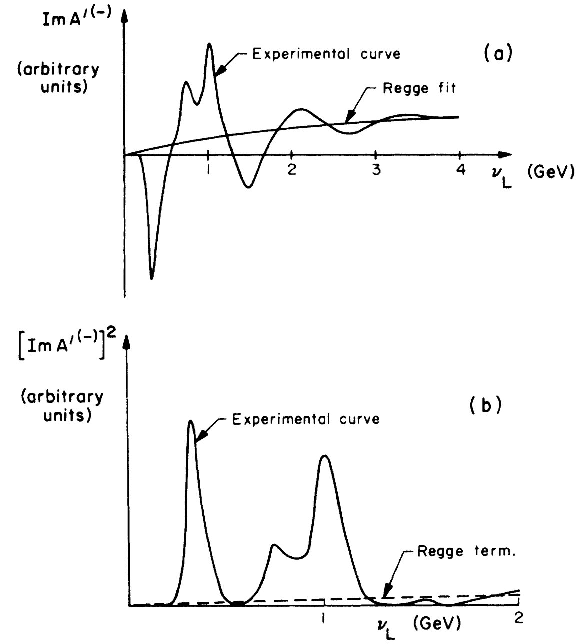

FIG. 2. (a) The amplitude $\mathrm{Im}A'^{(-)}$ at $t=0$ in πN charge-exchange scattering, and Regge fit. (b) $[\mathrm{Im}A'^{(-)}(t=0)]^2$ for πN charge-exchange scattering and Regge term.

positive contributions to the *cross sections* at low subenergies. These terms are left out of consideration in the usual multi-Regge models, and there is no "double counting" involved when we add them in. If one assumes that at a given energy the fluctuation cross sections in various channels are governed by statistical laws, then they are equivalent to the cross sections resulting from cluster production as in the model above.

In Sec. III a comparison is made with the thermodynamic model, and various advantages of the cluster model are summarized. A simplified mathematical model is constructed in Sec. IV, in which a simple power behavior is assumed for the dependence of the vertex functions on the cluster mass. Provided that this power lies within certain bounds, one finds that the cluster model exhibits all the characteristic features of both the thermodynamic and multiperipheral models, but with various more or less important quantitative modifications. Some consequences are explored in Sec. V, and our conclusions are summarized in Sec. VI. It is emphasized that accurate measurements of the tail of the multiplicity distribution in inelastic collisions will provide an important test of the model. We do not attempt to make any detailed comparisons with data.

II. COMPARISON WITH THE MULTI-REGGE MODEL

In any multiperipheral scheme, one has to answer the two basic questions[9]: What is exchanged, and what is produced at the vertices? We have no new answers to the first question, and it is likely to prove just as vexing as ever.[1] But let us consider the second one carefully.

In the usual forms of the multiperipheral model, one considers only the production of one type of final-state particle at each vertex, such as pions or occasionally ρ mesons (two π's). It is recognized that the true multi-Regge kinematic region in which this provides an accurate description of the scattering process contains only a small fraction of all possible events[10]: Most events occur where the subenergies of various groups of the produced particles are low, and the cross sections exhibit the usual resonance bumps. But the hope is expressed[11] that the treatment will also provide a reasonable *average* description even in the resonance region. This assumption rests on the hypothesis of "duality"[12]: the idea that Regge exchanges and direct-channel resonance terms provide equally good alternative descriptions, on the average, for at least the imaginary part of a nondiffractive amplitude.

This argument has been criticized before on several grounds.[13] But perhaps the main reason for its failure is the fact that we are dealing with cross sections and not amplitudes. Consider, for example, the famous amplitude $\mathrm{Im}A'^{(-)}$ occurring in πN charge-exchange scattering. As shown by Igi and Matsuda and by Dolen, Horn, and Schmid,[14] the Regge term does provide a good average description of the amplitude at $t=0$, even in the low-energy region where isolated resonance bumps appear:

$$\mathrm{Im}A'^{(-)} = \mathrm{Im}A_{\mathrm{Regge}} + \mathrm{Im}\delta A\,, \qquad (2)$$

where δA is a term fluctuating about zero [Fig. 2(a)]. But when one squares the amplitude to form a differential cross section, one finds

$$(\mathrm{Im}A'^{(-)})^2 = (\mathrm{Im}A_{\mathrm{Regge}})^2 + (\mathrm{Im}\delta A)^2 + (\text{interference terms averaging to zero}), \tag{3}$$

and at energies up to 2 GeV or so it is the *fluctuation* term $(\mathrm{Im}\delta A)^2$ which is of paramount importance [Fig. 2(b)]. It is such fluctuation terms which we hope to describe by introducing the cluster concept.

Fluctuations of this sort are well known in nuclear physics, and methods for treating them have been developed by Ericson.[15] The approach has been extended to hadron physics in a recent paper by Frautschi,[8] in which the statistical bootstrap model is employed to predict the resonance level density. Consider, for example, a two-particle scattering reaction in which we shall suppose that only resonance terms contribute (the discussion can easily be generalized). The scattering amplitude can be expressed as the sum of a coherent term (determined by the dynamics of the reaction process) and an *incoherent* (or fluctuation) term

$$A(\theta) = A^C(\theta) + A^F(\theta). \tag{4}$$

Here $A^C(\theta)$ arises from the constructive interference of the resonances, and is determined by their *average* coupling strength and level density, while $A^F(\theta)$ arises from additional random fluctuations about these average values.

Upon squaring the amplitude, and averaging over a suitable range of energies (several fluctuation lengths), one finds the average "differential cross section" is

$$\langle\sigma\rangle \equiv \langle|A(\theta)|\rangle^2 = |A^C(\theta)|^2 + \langle|A^F(\theta)|\rangle^2 \equiv \sigma^C + \sigma^F, \tag{5}$$

with the interference terms averaging to zero. The quantities σ^C and σ^F obviously correspond to the first two terms on the right-hand side of Eq. (3), for the example illustrated in Fig. 2.

As the energy increases, the ratio σ^F/σ^C will tend to fall like $1/N$, where N is the number of resonances coupling to this channel at a given energy, i.e.,

$$N \simeq \Gamma\rho, \tag{6}$$

where Γ is the average resonance width, and ρ is the resonance level density. The reason for the fall is statistical; the random resonance contributions to A^F cancel each other with increasing efficiency as their number rises. This fact was made use of by Frautschi[8] to explain the rapid decrease of the K^-p backward elastic cross section as the energy rises.

We will now make the major assumption that at a given energy the ratio of the fluctuation cross sections between *different* channels is governed by statistical laws (i.e., phase space). Then the sum over all available channels of the fluctuation cross sections is equal to the term σ_1 in Eq. (1) and Fig. 1. The average fluctuation cross section for any given final state α is given by

$$\sigma_\alpha^F(E) = \sigma_1(E)\,\frac{\phi_\alpha(E)}{\phi_{\mathrm{tot}}(E)}, \tag{7}$$

where $\phi_\alpha(E)$ is the phase space available to the final-state constituents (which may consist of any hadrons, stable *or* unstable) within the standard resonance volume V,[5] and $\phi_{\mathrm{tot}}(E)$ is the total phase space available at energy E:

$$\phi_{\mathrm{tot}}(E) = \sum_\alpha \phi_\alpha(E). \tag{8}$$

In the statistical bootstrap model, the asymptotic behavior of $\phi_{\mathrm{tot}}(E)$ is[16] (neglecting angular momentum conservation)

$$\phi_{\mathrm{tot}}(E) \sim cE^{-3}e^{E/kT_0}, \tag{9}$$

where T_0 is Hagedorn's "maximum temperature."[4]

This assumption is familiar in the case where *no* coherent terms σ^C are present, and the resonance couplings can be presumed to be completely random in all channels; it is equivalent to Bohr's "compound nucleus" model[17] in nuclear physics. The statistical bootstrap analog of Bohr's model in hadron physics was recently used by the present author[6] to predict branching ratios in $N\bar{N}$ annihilation reactions at rest, with a reasonable degree of success. The model succeeded in explaining the high multiplicity of final-state pions, and the narrow multiplicity distribution, as the result of a "cascading" decay process through resonance intermediate states; the low branching ratios into individual two-body final states were explained by the strong statistical competition from the rapidly increasing density of final states given by Eq. (9). Such a treatment is equivalent to dropping all terms except σ_1 in Eq. (1).

Given this statistical hypothesis, the rest of the series of terms σ_n in Eq. (1) follow from the usual multiperipheral-model assumption, that Reggeon-particle and Reggeon-Reggeon scattering processes have the same characteristics as particle-particle scattering.

The resulting model treats the fluctuation cross sections at a given energy just as one would describe the incoherent production and decay of resonances of that energy if no coherent terms were present. They are therefore represented by the production of a "cluster," or a resonance of the appropriate mass, which then decays statistically. The model therefore shares the virtue of the original statistical bootstrap model, in that it treats the production and decay of all hadrons on the same statistical footing, whether they be stable final-state particles or unstable resonances. It should be noted, however, that a statistical treatment of this kind, where one averages over many resonances, is strictly applicable only in the intermediate and high-energy regions where the resonances are strongly overlapping. At low energies, where the resonances are separable, they should properly be dealt with individually.

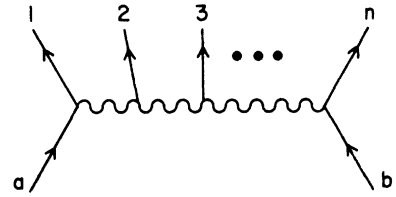

FIG. 3. Multiperipheral diagram for n-body production.

III. COMPARISON WITH THE THERMODYNAMIC MODEL

The thermodynamic model was pioneered by Hagedorn[4] several years ago, and he and his collaborators have elaborated on the subject in a series of papers,[18] and performed many detailed fits to high-energy data. The incoming particles are pictured as two blobs of hadronic matter, which collide in such a way that afterwards one finds a *distribution* of "hot" material moving with various velocities in the longitudinal direction. It is assumed that no "turbulence" occurs, i.e., that no transverse momentum is exchanged during the collision. The distribution of longitudinal velocities is taken as an input in this model.

The hadronic matter "heated" in the collision by the conversion of kinetic energy then decays by emission of particles, in the way predicted by the statistical bootstrap model.[4,5] Hagedorn uses the methods of thermodynamics (i.e., the canonical ensemble) rather than phase space (the microcanonical ensemble) in order to discuss this process, but the basic results are the same. The important physical consequences are the following:

(i) A prediction is made for the transverse-momentum spectrum which should have a thermodynamic Boltzmann form controlled by the temperature T_0. Fits of this type to high-energy data work rather well, with $kT_0 \simeq 140-160$ MeV.

(ii) Production rates for heavy particles are predicted. The statistical competition implies that the probability of producing a given heavy particle (or particle pair) decreases exponentially with its mass. Production rates have been calculated[18,19] for $\bar{K}$, $\bar{p}$, $\bar{d}$ and even $\overline{\mathrm{He}}^3$ particles on this basis, and they agree well with experiment.

Now this picture is essentially equivalent to the multiperipheral cluster model. The advantages of the cluster model are as follows:

(i) It provides a popular and plausible explanation for the longitudinal-momentum correlations, via the multiperipheral-exchange mechanism, which is a natural extension of the processes observed to occur in two-body reactions. In the thermodynamical model, on the other hand, the correlations had to be accounted for by seemingly *ad hoc* assumptions. The implied equivalence between the two mechanisms has already been investigated by Ranft and Ranft.[3]

(ii) It allows one to incorporate the effects of energy and momentum conservation, which are neglected in the thermodynamic approach.

(iii) It allows one to describe a reaction via phase space, and to avoid thermodynamic calculational techniques which may lead to incorrect results for this rather unique model.[20]

(iv) It provides a more detailed description of the reaction process, and allows one to make statistical predictions for the cross sections in individual exclusive channels, whereas the thermodynamic model can only predict *inclusive* properties.[21]

The major *disadvantage* of the cluster model with respect to the thermodynamic model is likely to be the increased difficulty of making realistic calculations. This means that for many practical purposes the thermodynamic model will remain more useful.

IV. CONSTRUCTION OF A SIMPLIFIED MULTIPERIPHERAL CLUSTER MODEL

In order to exhibit some of the basic qualitative features of such a model, we consider a simplified version patterned after a discussion of the Chew-Pignotti model[22] given by DeTar.[23] Ignore all spins, internal quantum numbers, and transverse momenta, and consider a diagram in which two incoming particles a and b give rise to n clusters in the final state as in Fig. 3. The cross section for this process will be taken as

$$\sigma_n \propto \frac{(2g^2)^n}{s} \prod_{i=1}^{n} \int dp_i dE_i dm_i^2\, \delta^{(+)}(E_i^2 - p_i^2 - m_i^2)\,\delta\left(\sum_1^n p_i - p_a - p_b\right) \times \delta\left(\sum_1^n E_i - E_a - E_b\right) m_i^{2\omega} \prod_{j=1}^{n-1} \left(\frac{s_{j,j+1}}{m_j^2 m_{j+1}^2}\right)^{2\alpha_R}. \tag{10}$$

In this equation, $1/s$ is a flux factor, g^2 represents a vertex coupling constant, and the variables p_i, E_i, and m_i^2 are the longitudinal momentum, energy, and mass squared of the ith cluster, respectively. The bracketed factors give the Reggeized dependence of the cross section on the subenergy and masses of the jth and $(j+1)$th clusters, where we have

$$s_{j,j+1}=(E_j+E_{j+1})^2-(p_j+p_{j+1})^2 . \tag{11}$$

The constant α_R is an "average" Regge trajectory value. Finally, the factors $m_i^{2\omega}$ represent an assumed power dependence of the vertex couplings on the mass of the produced clusters. Such a power behavior seems natural, but we have no theoretical justification for it whatever. The coefficient ω has nothing to do with any Regge trajectory α. The vertex behavior is determined by some unknown mechanism of "cluster production."

Let us now transform to the rapidity variables

$$y_i=\sinh^{-1}(p_i/m_i) , \tag{12}$$

and specialize to the laboratory frame in which the target a is at rest, and b moves with rapidity Y so that

$$s\simeq m_a m_b e^Y . \tag{13}$$

Then apply the "strong-ordering approximation"[24]

$$E_{i+1}\gg E_i\gg m_i . \tag{14}$$

(As noted by DeTar,[23] the strong-ordering assumption is unrealistic, but allows one to derive some useful conclusions by simple means.) This allows the replacements

$$s_{12}s_{23}\cdots s_{n-1,n}/(m_2^2m_3^2\cdots m_{n-1}^2)\simeq s \tag{15}$$

and

$$\delta\Big(\sum_i p_i-p_a-p_b\Big)\,\delta\Big(\sum_i E_i-E_a-E_b\Big)\to\frac{2}{s}\,\delta(y_1-\ln(m_1/m_a))\,\delta(Y-y_n-\ln(m_n/m_b)) . \tag{16}$$

We shall also let the lower limit to the mass spectrum be m_0, and suppose for simplicity that

$$m_a=m_b=m_0\simeq m_1\simeq m_n . \tag{17}$$

Then the cross section can be written

$$\begin{aligned}\sigma_n &\propto g^{2n}s^{2\alpha_R-2}\prod_{i=1}^{n}\int_{m_0^2}dm_i^2\int dy_i\,(m_i^2)^{\omega-2\alpha_R}\delta(y_1)\delta(Y-y_n)\\ &\simeq g^{2n}s^{2\alpha_R-2}\int^{Y}dy_n\int^{y_n}dy_{n-1}\cdots\int^{y_3}dy_2\,\delta(Y-y_n)\prod_{i=1}^{n}\int_{m_0^2}dm_i^2\,(m_i^2)^{\omega-2\alpha_R}\\ &=s^{2\alpha_R-2}\,\frac{(g^2\ln s)^{n-2}}{(n-2)!}\prod_{i=1}^{n}\int_{m_0^2}dm_i^2\,(m_i^2)^{\omega-2\alpha_R}\qquad (n\geq 2) .\end{aligned} \tag{18}$$

Now suppose that

$$\Delta\equiv\omega-2\alpha_R+1<0 ; \tag{19}$$

then the mass integrations can be done immediately, giving

$$\sigma_n\propto\frac{\left(\dfrac{m_0^{2\Delta}}{-\Delta}\right)^2 s^{2\alpha_R-2}\left[g^2\,\dfrac{m_0^{2\Delta}}{(-\Delta)}\,\ln s\right]^{n-2}}{(n-2)!}\qquad (n\geq 2) . \tag{20}$$

This result is identical to that obtained in the Chew-Pignotti model[22] by DeTar,[23] except that the coupling constant g^2 has been replaced by a new value

$$g'^2\equiv g^2m_0^{2\Delta}/(-\Delta) . \tag{21}$$

So provided that the constant Δ defined by Eq. (19) is less than zero, only low cluster masses are important in the partial cross sections, which then behave in the same way as in the Chew-Pignotti model.

From the Poisson distribution[25] of σ_n in Eq. (20), one then deduces that the average *cluster* multiplicity is

$$\langle n\rangle=g'^2\ln s , \tag{22}$$

and the total cross section is

$$\sigma_t=\sum_n\sigma_n\propto s^{2\alpha_R-2+g'^2} . \tag{23}$$

But $g'^2 \to \infty$ as Δ tends to zero from below. So the boundedness condition (19) is in fact necessary in order that σ_t should not violate the Froissart bound.

The average cluster mass can also be deduced:

$$\langle m \rangle = \frac{\sum_n \sum_{i=1}^{n} \int dm_i \frac{d\sigma_n}{dm_i} m_i}{\sum_n \sum_{i=1}^{n} \int dm_i \frac{d\sigma_n}{dm_i}}$$

$$= m_0 \frac{\Delta}{\Delta + \frac{1}{2}} \qquad (24)$$

provided that

$$\Delta + \tfrac{1}{2} < 0 . \qquad (25)$$

But the average number of pions produced per cluster is proportional to its mass, in the statistical bootstrap model. So the over-all average number of pions produced is

$$\langle n_\pi \rangle \simeq c \langle m \rangle \langle n \rangle = c m_0 \left(\frac{\Delta}{\Delta + \frac{1}{2}} \right) g'^2 \ln s . \qquad (26)$$

If $\Delta + \frac{1}{2} > 0$, however, the average cluster mass grows like $s^{\Delta + (1/2)}$, and so does the multiplicity $\langle n_\pi \rangle$. That would violate the Feynman scaling hypothesis,[26] which appears to be a feature of the real world, and so we will exclude this case also by assuming that the bound (25) is observed.

Provided that this bound is respected, one can go on to calculate the single-particle inclusive rapidity distribution, as done by DeTar.[23] One finds that scaling occurs, and that the produced pions are, on the average,[27] uniformly spaced in rapidity with a density proportional to $c m_0 g'^2 \times [\Delta/(\Delta + \frac{1}{2})]$.

It has thus been demonstrated that under condition (25) the multiperipheral cluster model gives results which are qualitatively the same as in the ordinary Chew-Pignotti model.[22] The main effect of including resonance production (i.e., the fluctuation terms) via the cluster model is to increase the multiplicity of produced pions. Let us give an example, for purely illustrative purposes (we cannot pretend that our simplified model is at all realistic in quantitative terms). Suppose that the total cross section σ_t is constant, and that $\alpha_R = \frac{1}{2}$ for the leading Regge exchanges; then from Eq. (23) it follows that $g'^2 = 1$. Next take $c m_0 = 1$, corresponding to the fact that at the threshold mass m_0 each cluster consists of a single pion; then the slope of the multiplicity versus $\ln s$ is

$$\frac{d\langle n_\pi \rangle}{d(\ln s)} = \frac{\Delta}{\Delta + \frac{1}{2}} , \qquad (27)$$

which increases monotonically as Δ increases from $-\infty$ to $-\frac{1}{2}$. Recent CERN Intersecting Storage Rings (ISR) data, for instance, give[28]

$$\frac{d\langle n_\pi \rangle}{d(\ln s)} = 2.26 , \qquad (28)$$

which would correspond to $\Delta \simeq -1$ by Eq. (27).

V. CONSEQUENCES

The value of the parameter Δ is of interest for several reasons.

A. Distribution of Cluster Masses

The value of Δ determines the relative probability $P(m)$ of producing a cluster of given mass. From Eq. (18) one finds

$$P(m) = \frac{\sum_n \sum_{i=1}^{n} \frac{d\sigma_n}{dm_i}(m)}{\sum_n \sum_{i=1}^{n} \sigma_n}$$

$$= \left(-\frac{2\Delta}{m_0} \right) \left(\frac{m}{m_0} \right)^{2\Delta - 1} . \qquad (29)$$

The upper bound (25) therefore implies that

$$P(m) = o(m^{-2}) \qquad (30)$$

(assuming g'^2 is finite and nonzero). This is to be contrasted with the diffractive excitation models of Hwa,[29] and of Jacob and Slansky,[30] in which clusters are assumed to be produced with probability $P(m) \propto m^{-2}$. Such behavior is not allowed if clusters can be produced by a multiperipheral mechanism with constant total cross section, and scaling is to hold.

The function $P(m)$ is of practical interest in making refinements to the predictions of Hagedorn's thermodynamic model.[4,18,19] It will affect the production rates of heavy particles in a high-energy collision, for instance. Hagedorn[19] assumes that the production rate $R(m')$ for a particle of mass m' is given by

$$R(m') \propto \int d^3p \exp[-(m'^2 + \vec{p}^{\,2})^{1/2}/kT_0] , \qquad (31)$$

which is the thermodynamic result appropriate to the decay of a cluster of infinite mass. A more detailed model of cluster production such as the present one will replace this by

$$R(m') \propto \int dm P(m) \times \int d^3p \exp[-(m'^2 + \vec{p}^{\,2})^{1/2}/kT_{\text{eff}}(m, p)] , \qquad (32)$$

where the "effective temperature" T_{eff} depends on m and p in a way recently discussed by Frautschi and Hamer.[7] The momentum spectra of the emitted particles will be similarly affected.

B. Multiplicity Distribution

The parameter Δ also controls the multiplicity distribution of pions produced at high energy. From Eq. (18) it follows that

$$\frac{d\sigma_n}{dM} \propto s^{2\alpha_R - 2} \frac{(g^2 \ln s)^{n-2}}{(n-2)!} \times \prod_{i=1}^{n} \int_{m_0^2} dm_i^2 (m_i^2)^{\Delta-1} \delta\Big(M - \sum_i m_i\Big), \quad (33)$$

where M is the *total* of all the cluster masses. Now for M large, the mass integrals are dominated by the configuration in which one cluster mass is large and all the rest are small, so that

$$\prod_{i=1}^{n} \int_{m_0^2} dm_i^2 (m_i^2)^{\Delta-1} \delta\Big(M - \sum_i m_i\Big) \underset{M \gg n m_0}{\sim} 2nM^{2\Delta-1}\left(\frac{m_0^{2\Delta}}{-\Delta}\right)^{n-1}. \quad (34)$$

Therefore,

$$\frac{d\sigma_n}{dM} \underset{M \gg n m_0}{\sim} n\sigma_n M^{2\Delta-1}\left(\frac{-2\Delta}{m_0^{2\Delta}}\right) \quad (n \geq 2). \quad (35)$$

But the total number of pions produced is proportional to M:

$$n_\pi = c\left(\frac{M}{m_\pi}\right) + b. \quad (36)$$

(We neglect the narrow Gaussian distribution of n_π about this average value.[6]) Therefore,

$$\frac{d\sigma_n}{dn_\pi} \underset{n_\pi \gg n}{\sim} n\sigma_n \left[\frac{-2\Delta}{(cm_0/m_\pi)^{2\Delta}}\right] n_\pi^{2\Delta-1} \quad (n \geq 2). \quad (37)$$

This behavior is illustrated in Fig. 4, where we have chosen the parameters as follows: $\alpha_R = \frac{1}{2}$, $\omega = -0.7$, $g'^2 = 1$, $m_0 = m_\pi$, $c = 0.3$, $b = 0.7$.

The multiplicity distribution for the total cross section is therefore

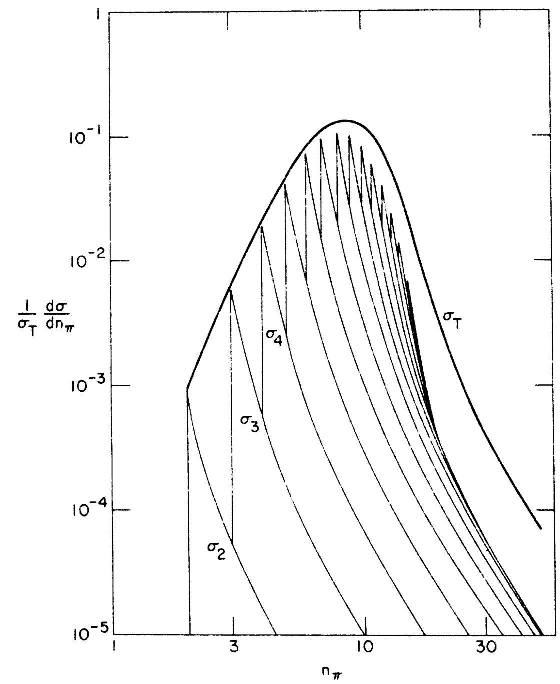

FIG. 4. Multiplicity distribution at a fixed energy, for the example discussed in the text.

$$\frac{1}{\sigma_t}\frac{d\sigma_t}{dn_\pi} = \frac{1}{\sigma_t}\sum_n \frac{d\sigma_n}{dn_\pi} \underset{\text{large } n_\pi}{\sim} \left[\frac{-2\Delta}{(cm_0/m_\pi)^{2\Delta}}\right] g'^2 \ln s \, n_\pi^{2\Delta-1}, \quad (38)$$

so that at large multiplicities the distribution exhibits a "tail," which drops off like a power of n_π (Fig. 4). The bound (25) implies that this power is such that

$$\frac{1}{\sigma_t}\frac{d\sigma_t}{dn_\pi} = o(n_\pi^{-2}). \quad (39)$$

At smaller multiplicities, the distribution is Poisson-like, as illustrated in Fig. 4. Let Q_k be the relative probability of producing k pions from each cluster, which behaves asymptotically like $k^{2\Delta-1}$. Then the average number of pions produced in all inelastic collisions is

$$\langle n_\pi \rangle \underset{s\to\infty}{\sim} \sum_{n=2}^{\infty} \frac{(g'^2 \ln s)^{n-2}}{(n-2)!} e^{-g'^2 \ln s} \sum_{\{k_i\}, i=1\cdots n} Q_{k_1} Q_{k_2} \cdots Q_{k_n}(k_1 + k_2 + \cdots + k_n)$$
$$= \sum_{n=2}^{\infty} \frac{(g'^2 \ln s)^{n-2}}{(n-2)!} e^{-g'^2 \ln s} n\langle k \rangle \sim g'^2 \ln s \langle k \rangle, \quad (40)$$

where $\langle k \rangle$ is the average number of pions produced per cluster. According to Eq. (26)

$$\langle k\rangle \simeq c m_0\left(\frac{\Delta}{\Delta+\frac{1}{2}}\right) . \tag{41}$$

So the average number of pions produced grows logarithmically in the same fashion as in the ordinary multiperipheral model. This was already shown in Eq. (26). But we can now go on to deduce the correlation parameter f_2, which characterizes the width of the multiplicity distribution:

$$f_2=\langle n_\pi(n_\pi-1)\rangle-\langle n_\pi\rangle^2 . \tag{42}$$

Asymptotically, we have

$$\langle n_\pi(n_\pi-1)\rangle \sim \sum_{n=2}^{\infty} \frac{(g'^2 \ln s)^{n-2}}{(n-2)!} e^{-g'^2 \ln s} \sum_{\{k_i\}, i=1\cdots n} Q_{k_1} Q_{k_2} \cdots Q_{k_n}[(k_1+k_2+\ldots+k_n)(k_1+k_2+\ldots+k_n-1)]$$

$$=\sum_{n=2}^{\infty} \frac{(g'^2 \ln s)^{n-2}}{(n-2)!} e^{-g'^2 \ln s}[n\langle k(k-1)\rangle+n(n-1)\langle k\rangle^2] \sim g'^2 \ln s\langle k(k-1)\rangle+(g'^2 \ln s)^2\langle k\rangle^2 . \tag{43}$$

Therefore,

$$f_2 \sim g'^2 \ln s\langle k(k-1)\rangle , \tag{44}$$

so that f_2 increases logarithmically, in accordance with the analysis of Mueller,[31] provided that $\langle k(k-1)\rangle$ is finite (i.e., $\Delta<-1$).

This multiplicity distribution is to be contrasted with those of simplified multiperipheral models,[25] on the one hand, and the diffractive excitation model, on the other. In simplified multiperipheral models the distribution is of the Poisson type, with a peak which moves outward proportionally to $\ln s$; whereas in the diffractive excitation model the multiplicity distribution peaks at a small, constant value, and drops off like n_π^{-2} at large multiplicities. The present model is intermediate between these two extremes. In the limit $[\Delta \to -\infty$, $g^2/(-\Delta)=\text{const}]$ it reduces to a simplified multiperipheral model; and in the opposite limit $[\Delta \to -\frac{1}{2}$, $g^2/-(\Delta+\frac{1}{2})=\text{const}]$ it becomes equivalent to the diffractive excitation model.

The experimental situation is not yet clear, but there are some indications of the clustering effect. A preliminary experiment at the National Accelerator Laboratory[32] (NAL) appears to confirm the logarithmic increase of $\langle n_\pi\rangle$ with energy. It also indicates that the width parameter f_2^- for the π^- distribution has begun to rise, a feature which is compatible with Eq. (44), but which is not allowed in ordinary multiperipheral models.[33] The correlation found at lower energies between the average number $\langle n_{\pi^0}\rangle$ of *neutral* pions produced in inelastic collisions and the number n_{ch} of *charged* particles may also be evidence of resonance production,[34] or "clustering." No evidence for a "tail" has yet been found. But at low energies this tail is cut off by the purely kinematic restriction that the total cluster mass M must be less than $(s)^{1/2}$. At NAL and ISR energies there should be enough room for the tail to develop, and precise experiments will detect whether it is there or not.

C. Central Collisions

Finally, we consider the behavior of the partial cross section σ_1, which was neglected in the preceding considerations. This term is peculiar to the present model, and describes the small fraction of events which are "central collisions," giving rise to a cloud of decay products which are all emitted with small momenta in the center-of-mass system, and whose multiplicity rises proportionally to the center-of-mass energy. The model gives

$$\sigma_1 \propto s^{\omega-1} , \tag{45}$$

and since $\alpha_R \leq 1$, one finds from condition (25) that

$$\sigma_1=o(s^{-1/2}) . \tag{46}$$

Let us now compare the expected contribution from this term with Frautschi's treatment of fluctuations in KN elastic scattering.[8] Equations (7) and (45) imply that the total fluctuation cross section in the elastic channel at center-of-mass energy E is given by

$$\sigma_{\text{el}}^F(E) \propto \sigma_1(E) \frac{\phi_{\text{el}}(E)}{\phi_{\text{tot}}(E)} \tag{47}$$

$$\propto (E^2)^{\omega-1} \frac{E^2}{E^{-3} e^{E/kT_0}}$$

$$\propto E^{2\omega+3} e^{-E/kT_0} . \tag{48}$$

The differential cross section $d\sigma^F/d\Omega$ is isotropic in the present crude treatment where angular momentum conservation is neglected[35]; hence the amplitude behaves like

$$F^F(E, \theta) \propto (E^{2\omega+5} e^{-E/kT_0})^{1/2} . \tag{49}$$

Now using the relationship $\sigma^F/\sigma^c \propto 1/N$ at $\theta=0°$,[8] where all the resonances couple with positive sign, and assuming that the average resonance width Γ is approximately constant, one finds that the imaginary part of the coherent resonance amplitude at 0° is

$$\operatorname{Im} F^C_{\text{res}}(E, 0°) \propto |F^F(E, 0°)| (E^{-3} e^{E/kT_0})^{1/2} \propto E^{\omega+1} . \tag{50}$$

Frautschi, on the other hand, determined $\operatorname{Im} F^C_{\text{res}}$ from the non-Pomeranchuk part of the total cross section, using the optical theorem and the duality hypothesis.[12] This implies that

$$\operatorname{Im} F^C_{\text{res}}(E, 0°) \propto E . \tag{51}$$

Taken at face value,[36] a comparison of Eqs. (50) and (51) requires that $\omega = 0$. Such a value is allowed under condition (25) (i.e., $\Delta = \omega - 2\alpha_R + 1 < -\frac{1}{2}$) if $\alpha_R \simeq 1$, but is *forbidden* if $\alpha_R = \frac{1}{2}$. This would imply the interesting conclusion that the multiperipheral cluster model with $\alpha_R = \frac{1}{2}$, plus the scaling hypothesis, is *inconsistent* with the usual form of duality in which the imaginary amplitudes due to non-Pomeranchuk Regge exchanges are taken to be *identical* to the coherent direct-channel resonance contributions. This does not exclude the possibility that direct-channel resonances may "generate" the exchange terms via unitarity.

There remains the question of how σ_1 contributes to the multiplicity distribution. The statistical bootstrap model gives[6]

$$\frac{1}{\sigma_1} \frac{d\sigma_1}{dn_\pi} = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left\{-\frac{[n_\pi - c(s)^{1/2}]^2}{2\sigma^2}\right\} , \tag{52}$$

with σ proportional to the square root of the mass of the cluster, i.e.,

$$\sigma = ds^{1/4} . \tag{53}$$

So the term σ_1 gives rise to a narrow Gaussian peak in the multiplicity distribution. The question now is whether this peak can be experimentally distinguished from the tails of the other terms, $d\sigma_n/dn_\pi$ ($n \geq 2$). To test this, we compare the energy dependence of the peak

$$\frac{d\sigma_1}{dn_\pi}(n_\pi = c(s)^{1/2}) \propto s^{\omega-5/4} \tag{54}$$

with the behavior of the high multiplicity tail from Eq. (38):

$$\frac{d\sigma_t}{dn_\pi}(n_\pi = c(s)^{1/2}) \propto s^{2\alpha_R - 2 + g'^2} s^{\Delta - 1/2} \propto s^{\omega + g'^2 - 3/2} . \tag{55}$$

It can be seen that the answer is model-dependent. If we were to assume a constant inelastic cross section resulting from multiple exchange of the ρ family of trajectories ($\alpha_R \simeq \frac{1}{2}$), then $g'^2 \simeq 1$ and the central collision peak will be submerged in the tail of the multiplicity distribution at high energies. If we were to assume multiple Pomeranchuk exchanges, on the other hand, then g'^2 would be small, and central collisions would be experimentally distinguishable at high energies as a separate bump in the multiplicity distribution – a rather surprising possibility.

VI. SUMMARY AND CONCLUSIONS

A multiperipheral cluster model of high-energy inelastic collisions has been constructed, as a way of accounting for Ericson fluctuation terms in the inelastic cross sections. These fluctuation cross sections are due to random variations of the resonance coupling strengths and spacings about their average values, and are ignored in the ordinary multiperipheral model. The major assumption was made that the ratio of the average fluctuation cross sections between different channels is governed by phase space. Under this assumption, the fluctuation terms can be regarded as resulting from the formation of "clusters," or resonances whose decay proceeds according to the statistical bootstrap theory of Hagedorn[4] and Frautschi.[5] As in the statistical bootstrap, such a theory has the virtue of treating the production and decay of all particles, whether stable or unstable, on an equal footing.

It should perhaps be emphasized that this model involves no "double counting." Any coherent resonance contributions are indeed included in the Regge exchange terms (σ^C of Sec. II). The terms we are adding are the *incoherent* resonance contributions, σ^F of Sec. II.

It was argued that this model is essentially equivalent to Hagedorn's thermodynamic model, but offers several theoretical advantages, chiefly: (i) It incorporates a more fundamental explanation for the longitudinal-momentum correlations, namely, the multiperipheral-exchange mechanism, which is a logical extension of the well-studied dynamical processes of two-body scattering. (ii) It incorporates energy and momentum conservation. (iii) It "predicts" (in a statistical sense) exclusive as well as inclusive cross sections. On the other hand, it may prove more difficult to make realistic calculations with the cluster model.

The present scheme will be subject to many of the same theoretical difficulties and ambiguities as the ordinary multiperipheral model. It will improve matters in two respects, however, in the following ways.

(i) By including fluctuation terms, in a statistical way it takes account of resonance production, which was missing in earlier models.[37] It will thus tend to increase the multiplicity of final-state pions, and raise the effective coupling constant

[g'^2 of Eq. (21)]. This should facilitate better agreement with experiment in phenomenological fits.

(ii) The error made by ignoring interference from "crossed" diagrams in estimating the cross sections should be reduced. These diagrams are most important at low subenergies of the produced particles; but at low subenergies the dominant term in the cluster model will be that in which the particles in question emerge from a *single* cluster, for which the interference problem does not arise.

The question as to which exchanges are most important will remain as difficult as ever. It has not been settled whether the Pomeranchuk, the ρ family, or the pion provides the dominant links in the multiperipheral chain.[1] Lower-lying trajectories are usually ignored. Their influence will be qualitatively similar to that of the fluctuation terms, in that it will mainly be felt at low subenergies; so it will be phenomenologically difficult to separate the two effects. Nevertheless, they are theoretically distinct: One is a coherent term, and drops off like a power of the energy in any given channel; the other is an incoherent term, and dies away exponentially.

In Secs. IV and V a simplified mathematical model was studied which incorporates the cluster concept. Spins, internal quantum numbers, and transverse momenta were ignored, and the exchanges were parametrized by an average Regge trajectory α_R. The square of the vertex coupling constant was assumed to be $g^2 m^{2\omega}$, for a cluster of mass m. It was then shown that, provided the parameters ω and α_R obeyed the bound

$$\Delta \equiv \omega - 2\alpha_R + 1 < -\tfrac{1}{2}, \qquad (25')$$

then the asymptotic properties of the model included

(i) Feynman scaling,[26] a logarithmic increase in the average multiplicity of produced particles, and a flat, constant plateau in the single-particle inclusive rapidity distribution;

(ii) a probability distribution for producing clusters of mass m,

$$P(m) \propto m^{2\Delta-1} = o(m^{-2}), \qquad (29')$$

which determines the predicted production rates for heavy particles, and their momentum distributions;

(iii) a multiplicity distribution which peaks at a value

$$n_\pi \propto \ln s \qquad (56)$$

and drops off like

$$P(n_\pi) \sim n_\pi^{\;2\Delta-1} = o(n_\pi^{\;-2}) \qquad (38')$$

at large multiplicities; and a correlation parameter f_2, describing the width of the multiplicity distribution, which rises proportionally to $\ln s$.

The model is thus intermediate between the ordinary multiperipheral model (e.g., Ref. 22), which is obtained in the limit [$\Delta \to -\infty$, $g^2/(-\Delta) = \text{const}$], and the diffractive excitation model,[29,30] which corresponds to the limit [$\Delta \to -\frac{1}{2}$, $g^2/-(\Delta+\frac{1}{2}) = \text{const}$]. Our model is certainly oversimplified. The average cluster mass is likely to be low (unless $\Delta = -\frac{1}{2}$), and at low masses there is no reason to suppose that the vertex coupling will behave like a simple polynomial as assumed in Eq. (10), or that a statistical treatment will provide a very precise description of the resonance decay modes. Nevertheless, the model should give a qualitatively correct idea of the effects of clustering.

The question of the manner in which the asymptotic limiting behavior (scaling) is *approached* is also of interest. Ordinary multiperipheral models usually predict that the limiting logarithmic curve in $\langle n_\pi \rangle$ versus s will be approached from below, and that the single-particle rapidity distribution in the central (plateau) region will be approached from above.[23] But experimental results from NAL and ISR show the opposite behavior.[33] The diffractive excitation model, on the other hand, predicts the correct behavior in this regard: This is one of the main reasons for the model's phenomenological success. We expect the cluster model to share in this success.

The "central-collision" cross section, σ_1, was discussed, together with the relationship between our model and Frautschi's treatment[8] of Ericson fluctuations. The possibility of observing a bump in the multiplicity distribution corresponding to "central collisions" was observed. But these effects are too model-dependent for firm conclusions to be drawn.

An important test of the concept of cluster production will consist of a precise experimental measurement of the behavior of the tail of the multiplicity distribution at NAL and ISR energies, which is predicted by Eq. (38′) [which results, in turn, from the *ad hoc* assumption of a power law for the mass dependence of the cluster couplings in Eq. (10)]. If this power-law decay is observed to occur, then the parameter Δ can be measured directly and determines the behavior of $P(m)$ by Eq. (29′) – this connection seems to be independent of any detailed model of the way in which the clusters are produced. Several refinements to the predictions of Hagedorn's thermodynamic model[4,18,19] could then be made immediately (see Sec. V). If, on the other hand, the tail of the multiplicity distribution falls faster than a power, then either our present model of cluster production is incorrect, or else the probability of producing high-mass clusters is negligible.

Finally, it should be pointed out that although we have here chosen to discuss cluster production in association with the multiperipheral model, the idea can be applied in much more general situations. Incoherent resonance cross sections will occur in any process to which the resonances themselves contribute.

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