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Leptonic Decay of Vector Mesons in an Algebraic Model for Hadrons*

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The decay rates for $V \rightarrow e\bar{e}$ are calculated using an octet of transition operators that has been introduced previously in the framework of an algebraic model. The results are compared with experimental data, and restrictions on the symmetry-breaking relations for these transition operators are obtained which are shown to be the analog of other well-known symmetry-breaking relations.

Within the framework of an algebraic model, an octet of vector operators and an octet of axial-vector operators have been suggested as transition operators between one-hadron states.[1] These transition operators are given by[2,3]

$$\begin{aligned} V_\mu^\alpha &= \{P_\mu + \rho\Gamma_\mu, E_\alpha\} = V_\mu^{(0)\alpha} + V_\mu^{(1)\alpha} \;, \\ A_\mu^\alpha &= \{P_\mu + \rho\Gamma_\mu, F_\alpha\} = A_\mu^{(0)\alpha} + A_\mu^{(1)\alpha} \;, \\ V_\mu^{(0)\alpha} &= \{P_\mu, E_\alpha\}\,, \quad V_\mu^{(1)\alpha} = \Gamma_\mu\{\rho, E_\alpha\}\,, \qquad (1) \\ A_\mu^{(0)\alpha} &= \{P_\mu, F_\alpha\}\,, \quad A_\mu^{(1)\alpha} = \Gamma_\mu\{\rho, F_\alpha\}\,, \\ & \alpha = \pm1, \pm2, \pm3, 0, 8 \end{aligned}$$

where E_α (scalar) and F_α (pseudoscalar) are the generators of[4] an $\mathrm{SL}(3,C)_{E_\alpha F_\alpha}$ or noncompact $\mathrm{SU}(3)_{E_\alpha+F_\alpha} \otimes \mathrm{SU}(3)_{E_\alpha - F_\alpha}$, P_μ ($\mu = 0, 1, 2, 3$) are the momentum operators, and Γ_μ are infinite-dimensional generalizations of the Dirac gamma matrices. ρ must have the dimensions of mass in order that both terms in V_μ^α and A_μ^α have the same dimensions (MeV). The simplest assumption one could make for ρ is that ρ is a constant of dimension MeV and is somehow connected with the elementary length that had been introduced in the framework of these algebraic models.[5] However, this possibility can already be ruled out[1] by requiring that V_μ and A_μ have definite transformation properties under time reversal and charge conjugation. It could, however, still be possible that ρ commutes with the "charges" E_α and "axial charges" F_α.

To obtain some information on ρ from experimental data, and to further test the applicability of our algebraic model, we want to investigate the electromagnetic transitions of mesons. In an earlier application of these ideas, the weak leptonic and semileptonic decays of pseudoscalar mesons P have been calculated.[6] In these calculations, only $V_\mu^{(0)\alpha}$ and $A_\mu^{(0)\alpha}$ appear because the matrix elements of $V_\mu^{(1)\alpha}$ and $A_\mu^{(1)\alpha}$ between spin-zero states are zero.

In the present work, we will calculate the electromagnetic transitions

$$\text{vector meson} \rightarrow e\bar{e}$$

using the same method as employed in Ref. 6 for the weak decays $P \rightarrow P'l\nu$ and $P \rightarrow l\nu$. Only the term $V_\mu^{(1)\alpha}$ gives nonzero contributions to the decay rates in the present calculation. We will calculate the decay rates first under the assumption that $\rho = \text{const}$ and show that this will lead to disagreement with the experimental data. The next simple and – because of its position in the transition operators (1) – more natural assumption for ρ is that it is an operator proportional to the mass operator. We will show that a calculation with this assumption will lead to agreement with the experimental data. We will then give a theoretical argument for this assumption.

In Ref. 6, we had assumed that the dynamics of the transition is described by the matrix elements of the operators V_μ^α and A_μ^α between the hadron states and that the function of the leptons is purely kinematical, i.e., the lepton pair is treated as two "free" noninteracting particles (this is in analogy to the γ in the quantum theory of radiation for the transition of an excited atom into the ground state). In order to calculate $V \rightarrow \bar{e}e$, we will just replace the lepton matrix elements of $e\bar{\nu}$ by the corresponding ones for $e\bar{e}$ and perform the calculation without any assumption other than those regarding the operator ρ, which we want to test.

In Ref. 6, we made the further simplifying assumption that the $\mathrm{SU}(3)_{E_\alpha}$ [and $\mathrm{SL}(3,C)_{E_\alpha F_\alpha}$] is that SU(3) which classifies the particles.[7] This assumption permitted us to express the reduced matrix elements of the "charges" E_α and F_α by the numbers that characterize the representation of the particle-spectrum-generating $\mathrm{SL}(3,C)$. We will have to remove this simplification here and treat the reduced matrix element as the parameter that has to be fitted from the experimental data.

(1) The decay rate for $V \rightarrow e\bar{e}$ is calculated here in complete analogy to the calculation of the leptonic weak decays $K^+ \rightarrow l\nu$ and $\pi^+ \rightarrow l\nu$ in Ref. 6:

$$P=\sum_{\text{Pol}}\int\int\frac{d^3p_+}{2E_+}\frac{d^3p_-}{2E_-}\,\delta(E_V-E_+-E_-)|\langle e\bar{e}\vec{p}_+\vec{p}_-|\,T\,|\,\phi_V\rangle|^2$$

$$=\sum_{\text{Pol}}\int\int\frac{d^3p_+}{2E_+}\frac{d^3p_-}{2E_-}\,\delta(E_V-E_+-E_-)\left|\int\frac{d^3p_V}{2E_Vc_V}\,\phi_V(\vec{p}_V)\langle e\bar{e}\,\vec{p}_+\vec{p}_-|\,T\,|\,\vec{p}_V ss_3V\rangle\right|^2, \tag{2}$$

where $E_\pm$ and $\vec{p}_\pm$ are the energy and momentum of $e^\pm$, E_V and $\vec{p}_V$ are the energy and momentum of the vector meson V, and $|\,\phi_V(\vec{p}_V)|^2$ is the expectation value for the momentum $\vec{p}_V$ in the ensemble of decaying vector mesons. The number c_α is due to our normalization of the generalized hadron states $|\,\vec{p}_\alpha\alpha\rangle$ (Appendix B, Ref. 6).

The transition operator T ("interaction Hamiltonian") is split into a leptonic part L^ν and a hadronic part H_ν which is an operator in the space of the hadron:

$$T=L^\nu H_\nu\ . \tag{3}$$

L^ν acts only on the lepton quantum numbers and H_ν is given by the hadronic transition operators (1) of our algebraic model for hadrons. The leptonic part L_ν for the electron-positron pair is immediately obtained from (3.11b) of Ref. 6 by replacing $(1+\gamma_5)u^{(\nu)}$ with the positron spinor $v_+(p_+)$:

$$\langle e^+e^-\vec{p}_+\vec{p}_-|\,L^\nu|\,(m_\sigma/m_V)\vec{p}_V\,\sigma\rangle=\frac{a\sqrt{c_V}}{2m_V}\,\delta^3(\vec{p}_++\vec{p}_--\vec{p}_V)\bar{u}(p_-)\gamma^\nu v_+(p_+)\ . \tag{4}$$

The hadronic part H_μ will be proportional to the charge components[8] of $V_\mu^\alpha+A_\mu^\alpha$:

$$\mathfrak{J}_\mu^{\text{el}}=\left\{(P_\mu+\rho\Gamma_\mu),\left(H_1+\frac{1}{\sqrt{3}}H_2\right)\right\}+\left\{(P_\mu+\rho\Gamma_\mu),\left(G_1+\frac{1}{\sqrt{3}}G_2\right)\right\}\ . \tag{5$'$}$$

Whether a term like

$$A_\mu^Q=\left\{(P_\mu+\rho\Gamma_\mu),\left(G_1+\frac{1}{\sqrt{3}}G_2\right)\right\}$$

is present in the electromagnetic interaction needs much further discussion. Fortunately it will not contribute to the decay of $V\to e\bar{e}$, as will become apparent below; therefore, in the case under investigation we take for $\mathfrak{J}_\mu^{\text{el}}$

$$\begin{aligned}\mathfrak{J}_\mu^{\text{el}}=V_\mu^Q\overset{\text{def}}{=}\left\{(P_\mu+\rho\Gamma_\mu),\left(H_1+\frac{1}{\sqrt{3}}H_2\right)\right\}\overset{\text{def}}{=}\{P_\mu+\rho\Gamma_\mu,\tilde{Q}\}\ ,\\ \tilde{Q}=H_1+\frac{1}{\sqrt{3}}H_2\ .\end{aligned} \tag{5}$$

We will now investigate two different assumptions for H_μ:

$$H_\mu=G\mathfrak{J}_\mu^{\text{el}}\ , \tag{6a}$$

where G is a constant that specifies the strength of the interaction, and

$$\rho^{-2}[P_\lambda,[P^\lambda,H_\mu]]=G\mathfrak{J}_\mu^{\text{el}}\ , \tag{6b}$$

where ρ and G are quantities of the type described above. For the calculation of $V\to e\bar{e}$ the magnitude of these quantities is inessential, because further unknown over-all constants enter through the reduced matrix elements of E_α and Γ_μ. [These constants are the ones that characterize the representations of the SL(3, C) and SO(3, 2)$_{\Gamma_\mu s_{\mu\nu}}$.] With assumption (6a), our model implies a complete analogy between weak and electromagnetic interaction (although it does not allow a comparison of the strength G with the strength for the leptonic decay of pseudoscalar mesons because different reduced matrix elements enter into the two types of calculation; see below). This of course goes against our feeling that the electromagnetic interactions are mediated by the photon, whereas for the weak interactions there is not such an intermediate boson. Assumption (6b) is constructed so that our algebraic model reproduces the photon propagator term and can therefore be accepted as at least a "first approximation" neglecting two-photon exchange.

For the process under consideration, $V\to\bar{e}e$, assumptions (6a) and (6b) lead to results that differ by a factor $m_V{}^2$ if we assume that ρ is just a constant with dimension MeV. These are the effects which we want to investigate. We want to see what choices for ρ the experimental data will permit; therefore, we want to keep both cases (b) and (a).[9] Assumption (6b), when applied to baryons, leads to a relation between the electromagnetic and the weak coupling constant and ρ, which could suggest that the weak and electromagnetic interaction are of the "same strength" G.

(2) The calculation of

$$A=\langle \bar{e}e\vec{\mathrm{p}}_+\vec{\mathrm{p}}_-|\, T\,|\,\vec{\mathrm{p}}_V s s_3 V\rangle \tag{7}$$

proceeds now along the same lines as in Ref. 6:

$$A=\sum_{\alpha}\int \frac{d^3p_\alpha}{2E_\alpha(\vec{\mathrm{p}}_\alpha)c_\alpha}\langle \bar{e}e\vec{\mathrm{p}}_+\vec{\mathrm{p}}_-|\,L^\lambda|\,\vec{\mathrm{p}}_\alpha\alpha\rangle \times\langle \alpha\vec{\mathrm{p}}_\alpha|\,H_\lambda|\,\vec{\mathrm{p}}_V V\rangle\,,$$

where $|\,\vec{\mathrm{p}}_\alpha\alpha\rangle$ is a basis in the representation space of the hadron tower. As L^λ does not change the hadron quantum numbers, we obtain

$$A=\int \frac{d^3p_\sigma}{2E_\sigma(\vec{\mathrm{p}}_\sigma)c_\sigma}\langle \bar{e}e\vec{\mathrm{p}}_+\vec{\mathrm{p}}_-|\,L^\lambda|\,\vec{\mathrm{p}}_\sigma\sigma\rangle \times\langle \sigma\vec{\mathrm{p}}_\sigma|\,H_\lambda|\,\vec{\mathrm{p}}_V V\rangle\,, \tag{8}$$

where $|\,\vec{\mathrm{p}}_\sigma\sigma\rangle$ is the vector in the hadron tower with the hadron quantum numbers of the vacuum.[5,6] The vector mesons are assigned to the states with $n=+1$, $s=1$ of the particle tower described by $\mathfrak{S}^{(R,0)}$.[5] σ is assigned to the state $n=0$, $s=0$; consequently, only the term $\Gamma_\mu\{\rho,\tilde{Q}\}$ of (5) contributes to the matrix element $\langle\sigma|\,H_\lambda|\,V\rangle$, since

$$\langle n=0,\ s=0^+|\,P_\mu|\,n=1,\ s=1^-\rangle=0\ .$$

Thus we obtain

$$A=G\int\frac{d^3p_\sigma}{2E_\sigma c_\sigma}\langle e\bar{e}|L^\lambda|\,\vec{\mathrm{p}}_\sigma\sigma\rangle\langle\vec{\mathrm{p}}_\sigma\sigma|\,\Gamma_\lambda\{\rho,\tilde{Q}\}\,|\,\vec{\mathrm{p}}_V V\rangle \tag{9}$$

if we use the assumption (6a) for H_λ. Using assumption (6b) with a constant ρ we calculate

$$G\langle\sigma\vec{\mathrm{p}}_\sigma|\,\mathcal{J}^{\mathrm{el}}_\mu|\,\vec{\mathrm{p}}_V V\rangle=\rho^{-2}(p_{(\sigma)}-p_{(V)})^2\langle\sigma\vec{\mathrm{p}}_\sigma|\,H_\lambda|\,\vec{\mathrm{p}}_V V\rangle$$

so that

$$\langle\sigma\vec{\mathrm{p}}_\sigma|\,H_\lambda|\,\vec{\mathrm{p}}_V V\rangle=\frac{G\rho^2}{(p_{(V)}-p_{(\sigma)})^2}\times\langle\sigma\vec{\mathrm{p}}_\sigma|\,\{\rho\Gamma_\lambda,\tilde{Q}\}\,|\,\vec{\mathrm{p}}_V V\rangle\ . \tag{10}$$

The matrix elements $\langle\sigma\vec{\mathrm{p}}_\sigma|\{\Gamma_\lambda,\tilde{Q}\}|\vec{\mathrm{p}}_V V\rangle$ are calculated under the model assumption

$$[X,P_\mu M^{-1}]=0\quad\text{for } X=\Gamma_\mu, E_\alpha, F_\alpha\ . \tag{11}$$

The shortcomings of this assumption have been discussed before[1,6] (constancy of the form factors); however, for $V\to\bar{e}e$ these shortcomings are irrelevant. We make the further use of the reasonable assumption

$$[\Gamma_\lambda, E_\alpha]=[\Gamma_\lambda, F_\alpha]=0\quad\text{for all }\alpha\ . \tag{12}$$

Then we obtain

$$\langle\sigma\vec{\mathrm{p}}_\sigma|\,\{\Gamma_\lambda,\tilde{Q}\}\,|\,\vec{\mathrm{p}}_V V\rangle=2\langle(\sigma)|\,\tilde{Q}^{\mathrm{int}}|\,(V)\rangle\langle n=0,\ s=0,\ \vec{\mathrm{p}}_\sigma|\,\hat{\Gamma}_\lambda|\,n=1,\ s=1,\ (m_\sigma/m_V)\vec{\mathrm{p}}_V\rangle\ . \tag{13}$$

(α) indicates only the intrinsic quantum number $I, I_3, Y, \ldots$; thus $\langle(\sigma)|\,\tilde{Q}^{\mathrm{int}}|\,(V)\rangle$ is the matrix element of $\tilde{Q}$ reduced to the space of intrinsic quantum numbers (i.e., an SU(3) matrix element). The action of $\Gamma_\lambda\tilde{Q}$ on the mass has been taken care of in the matrix element

$$\langle n=0,\ s=0,\ \vec{\mathrm{p}}_\sigma|\,\hat{\Gamma}_\lambda|\,n=1,\ s=1,\ (m_\sigma/m_V)\vec{\mathrm{p}}_V\rangle\ .$$

$\hat{\Gamma}_\lambda$ indicates that $\hat{\Gamma}_\lambda$ no longer acts on the mass, but only on $n, s, s_3, (\vec{\mathrm{p}}/m)$. The possibility of splitting Γ_λ into such a direct product is of course a consequence of (11). The calculation of $\langle n', s', s_3', \vec{\mathrm{p}}'|\,\tilde{\Gamma}_\lambda|\,nss_3\vec{\mathrm{p}}\rangle$ can be done in various ways. We choose the following:

$$\begin{aligned}\langle n's's_3'\vec{\mathrm{p}}'|\,\Gamma_\lambda|\,nss_3p\rangle &=\langle n's's_3'\vec{\mathrm{p}}'=0|\,U(L(p'))\Gamma_\lambda U^{-1}(L(p'))\,U(L(p'))\,|\,nss_3\vec{\mathrm{p}}\rangle\\ &=L^{-1}{}_\lambda{}^\nu(p')\langle n's's_3'0|\,\Gamma_\nu U(L(p'))\,|\,nss_3\vec{\mathrm{p}}\rangle\\ &=L^{-1}{}_\lambda{}^\nu(p')\sum_{\sigma_3,\sigma,\gamma}\int d\mu(k)\langle n's's_3'0|\,\Gamma_\nu|\,\vec{\mathrm{k}}\gamma\sigma\sigma_3\rangle\langle\gamma\sigma_3\sigma\vec{\mathrm{k}}|\,U(L(p'))\,|\,nss_3\vec{\mathrm{p}}\rangle\\ &=L^{-1}{}_\lambda{}^\nu(p')\langle n's's_3'0|\,\Gamma_\nu|\,0nss_3\rangle\langle nss_30|\,U(L(p'))\,|\,nss_3\vec{\mathrm{p}}\rangle\\ &=L^{-1}{}_\lambda{}^\nu(p')\langle n's's_3'0|\,\Gamma_\nu|\,0nss_3\rangle\,2c_{(n,s)}E_{(n,s)}(\vec{\mathrm{p}})\delta^3(\vec{\mathrm{p}}'-\vec{\mathrm{p}})\ .\end{aligned} \tag{14}$$

Here $L^{-1}(p)$ is the boost and we have made use of (11) and the fact that $U(L)$ does not affect n, s, s_3. Therewith we obtain for (13)

$$\langle\sigma\vec{\mathrm{p}}_\sigma|\,\{\Gamma_\lambda,\tilde{Q}\}\,|\,\vec{\mathrm{p}}_V V\rangle=2\langle(\sigma)|\,\tilde{Q}^{\mathrm{int}}|\,(V)\rangle\langle n=0,\ s=0|\,\Gamma_\nu|\,n=1,\ s=1,\ s_3\rangle L^{-1}{}_\lambda{}^\nu(p_\sigma)2c_\sigma E_\sigma(\vec{\mathrm{p}}_\sigma)\delta^3(\vec{\mathrm{p}}_\sigma-(m_\sigma/m_V)p_V)\ . \tag{15}$$

Inserting this into (9) we obtain for assumption (6a)

$$A_{(a)}=\langle \bar{e}e|\, L^{\lambda}|\, (m_{\sigma}/m_{V})\vec{\mathrm{p}}_{V}\, \sigma\rangle 2\rho G\langle(\sigma)|\, \tilde{Q}^{\mathrm{int}}|\, (V)\rangle L^{-1}{}_{\lambda}{}^{\nu}((m_{\sigma}/m_{V})p_{V})\langle 0, 0|\, \hat{\Gamma}_{\nu}|\, 1, 1\rangle\ , \tag{16}$$

and using (4) for the leptonic matrix element we obtain

$$A_{(a)}=\frac{a\sqrt{c_{V}}}{2m_{V}}\, 2\rho G\bar{u}(p_{-})\gamma^{\lambda}v(p_{+})\delta^{3}(\vec{\mathrm{p}}_{+}+\vec{\mathrm{p}}_{-}-\vec{\mathrm{p}}_{V})\langle(\sigma)|\, \tilde{Q}^{\mathrm{int}}|\, (V)\rangle L^{-1}{}_{\lambda}{}^{\nu}(p_{V})\langle 0, 0|\, \Gamma_{\nu}|\, 1, 1\rangle\ . \tag{17a}$$

Here we have used $L^{-1}{}_{\lambda}{}^{\nu}(p_{V})=L^{-1}{}_{\lambda}{}^{\nu}(\alpha p_{V})$.

If we use assumption (6b), we see from (10) that an additional factor $\rho^{2}/[(m_{\sigma}/m_{V})-1]^{2}m_{V}{}^{2}$ appears.

$$A_{(b)}=\frac{a\sqrt{c_{V}}}{2m_{V}}\, 2\rho G\, \frac{\rho^{2}}{m_{V}{}^{2}}\, \bar{u}(p_{-})\gamma^{\lambda}v(p_{+})\delta^{3}(\vec{\mathrm{p}}_{+}+\vec{\mathrm{p}}_{-}-\vec{\mathrm{p}}_{V})\langle(\sigma)|\, \tilde{Q}^{\mathrm{int}}|\, (V)\rangle L^{-1}{}_{\lambda}{}^{\nu}(p_{V})\langle 0, 0|\, \Gamma_{\nu}|\, 1, 1\rangle\ . \tag{17b}$$

Here we have already used $m_{\sigma}\rightarrow 0$.[6,5]

In (17) $\langle(\sigma)|\, \tilde{Q}^{\mathrm{int}}|\, (V)\rangle$ is the SU(3) matrix element of the charge component of an octet operator and therewith a familiar quantity. To reach resemblance with the conventional calculations[10] we replace

$$L^{-1}{}_{\mu}{}^{\nu}(p_{V})\langle n=0,\ s=0|\, \Gamma_{\nu}|\, n=1,\ s=1,\ s_{3}=\lambda\rangle$$

by the polarization vector $e_{\mu}(p_{V}, \lambda)$.

(3) We define the quantity

$$\epsilon_{\mu}(p, \lambda)=L^{-1}{}_{\mu}{}^{\nu}(p)\langle n=0,\ s=0|\, \Gamma_{\nu}|\, n=1,\ s=1,\ s_{3}=\lambda\rangle \tag{18}$$

and show that this quantity possesses the properties that are required of the vector-meson polarization vector. For this we will have to use the properties of the matrix elements of Γ_{ν} in the generalized Dirac representation of SO(3, 2) (Ref. 11) to which the vector mesons are assigned. We calculate

$$\begin{aligned}\sum_{\lambda}\epsilon_{\mu}(p, \lambda)\epsilon_{\nu}(p, \lambda)&=L^{-1}{}_{\mu}{}^{\rho}(p)L^{-1}{}_{\nu}{}^{\sigma}(p)\sum_{\lambda}\langle 0, 0|\, \Gamma_{\rho}|\, 1, 1\lambda\rangle\langle 0, 0|\, \Gamma_{\sigma}|\, 1, 1, \lambda\rangle\\ &=L^{-1}{}_{\mu}{}^{r}(p)L^{-1}{}_{\nu}{}^{s}(p)\sum_{\lambda}\langle 0, 0|\, \Gamma_{r}|\, 1, 1, \lambda\rangle\langle 0, 0|\, \Gamma_{s}|\, 1, 1, \lambda\rangle\ ,\end{aligned} \tag{19}$$

where $r, s=1, 2, 3$ because $\langle 0, 0|\, \Gamma_{0}|\, 1, 1\rangle=0$.

From Ref. 11 we obtain the matrix elements of $\Gamma(r)$ defined by

$$\Gamma(0)=\Gamma_{3},\qquad \Gamma(\pm 1)=\mp\frac{1}{\sqrt{2}}\, (\Gamma\pm i\Gamma_{2})\ .$$

From Eqs. (III, 11), (III, 15), and (III, 18) of Ref. 11 one obtains (a phase factor has been chosen real)

$$\langle 0, 0|\, \Gamma(r)|\, n=1,\ s=1,\ s_{3}=\lambda\rangle=[\tfrac{1}{2}(1-c^{2})]^{1/2}C(1, 1, 0; r, \lambda, 0)\ , \tag{20}$$

where $C(1, 1, 0; r, \lambda, 0)$ is the SU(2) Clebsch-Gordan coefficient and c is the imaginary number that characterizes the irreducible representation of SO(3, 2). It is connected with the eigenvalue R of the second-order Casimir operator by $\frac{1}{2}(R-2)=ic$. The meson tower, among other things, is characterized by $(R, 0)$, and a favorable assignment for the mesons was the degenerate representation $(R=2, 0)$, i.e., $c=0$.

Using the properties of the Clebsch-Gordan coefficients, we obtain from (20)

$$\sum_{\lambda}\langle 0, 0|\, \Gamma(r)|\, n=1,\ s=1,\ \lambda\rangle\langle 0, 0|\, \Gamma(s)|\, 1, 1,\ \lambda\rangle=\tfrac{1}{2}(1-c^{2})\frac{1}{\sqrt{3}}\, \delta_{rs}\ ,$$

and therewith

$$\begin{aligned}\sum_{\lambda}\langle 0, 0|\, \Gamma_{r}|\, 1, 1,\ \lambda\rangle\langle 0, 0|\, \Gamma_{s}\, |\, 1, 1,\ \lambda\rangle&=\frac{1-c^{2}}{2\sqrt{3}}\, \delta_{rs}\\ &=-f^{2}g_{rs}\ ,\end{aligned} \tag{21}$$

where we have called

$$+\left(\frac{1-c^{2}}{2\sqrt{3}}\right)^{1/2}=f\ . \tag{21$'$}$$

Therewith (19) gives

$$\sum_{\lambda}\epsilon_{\mu}(p, \lambda)\epsilon_{\nu}(p, \lambda)=L^{-1}{}_{\mu}{}^{r}(p)L^{-1}{}_{\nu}{}^{s}(p)g_{rs}(-f^{2})\ . \tag{22}$$

As the boost is given by the matrix

$$L^{-1}{}_{\mu}{}^{\nu}(p)=\begin{array}{cc} \nu=0 & \nu=i \\ \left(\begin{array}{cc} \dfrac{p_{0}}{m} & -\dfrac{p^{i}}{m} \\ \dfrac{p_{j}}{m} & g^{i}_{j}-\dfrac{p_{j}p^{i}}{m(m+p_{0})} \end{array}\right) & \begin{array}{c} \mu=0 \\ \mu=j \end{array} \end{array}\ , \tag{23}$$

one calculates

$$\sum_{\lambda} \epsilon_\mu(p,\lambda)\epsilon_\nu(p,\lambda) = -\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right) f^2 . \tag{24}$$

Consequently, if one defines

$$e_\mu(p,\lambda) = \left(\frac{2\sqrt{3}}{1-c^2}\right)^{1/2} L^{-1}{}_\mu{}^\nu(p)\langle 0,0|\,\Gamma_\nu|\,n=1,\ s=1,\ s_3=\lambda\rangle , \tag{25}$$

then $e_\mu(p,\lambda)$ fulfills all the properties that are needed for the vector-meson polarization vector.

(4) With (25), (17) becomes

$$A_{(a)} = \frac{a\sqrt{c_V}}{2m_V}\, 2\rho G \left(\frac{1-c^2}{2\sqrt{3}}\right)^{1/2} \langle(\sigma)|\,\tilde{Q}^{\text{int}}|\,(V)\rangle\, \delta^3(\vec{p}_+ + \vec{p}_- - \vec{p}_V)\bar{u}(p_-)\gamma^\mu v(p_+)e_\mu(p_V,\lambda) . \tag{26a}$$

We insert this into (2) and calculate the transition rate under the usual assumption:

$$|\,\phi_V(\vec{p}_V)|^2 = 2E_V(\vec{p}_V)c_V\delta^3(\vec{p}_V - \vec{q}), \qquad \vec{q}=0 \tag{27}$$

i.e., that the ensemble of decaying vector mesons is exactly at rest.

$$\Gamma_{(a)} = \int\!\!\int \frac{d^2p_+}{2E_+}\frac{d^2p_-}{2E_-}\,\delta(E_V - E_+ - E_-)\delta^3(\vec{p}_+ + \vec{p}_- - \vec{q})\frac{1}{2E_V(\vec{p}_+ + \vec{p}_-)}$$

$$\times \frac{a^2}{m_V{}^2}\rho^2G^2\frac{1-c^2}{2\sqrt{3}}\,|\,\langle(\sigma)|\,\tilde{Q}^{\text{int}}|\,(V)\rangle|^2 \sum_{\text{Pol}} |\bar{u}(p)\gamma^\mu v(p_+)e_\mu(q,\lambda)|^2 . \tag{28}$$

This is integrated in the usual way to give

$$\Gamma_{(a)} = \tfrac{1}{3}\pi m_V\left(\frac{a\rho Gf}{m_V}\right)^2 |\,\langle(\sigma)|\,\tilde{Q}^{\text{int}}|\,(V)|^2\left(1-\frac{2m^2}{m_V{}^2}\right)^{1/2}\left(1+\frac{2m^2}{m_V{}^2}\right)$$

$$\approx \tfrac{1}{3}\pi m_V\left(\frac{a\rho Gf}{m_V}\right)^2 |\,\langle(\sigma)|\,\tilde{Q}^{\text{int}}|\,(V)\rangle\,|^2 , \tag{29a}$$

where

$$f = \left(\frac{1-c^2}{2\sqrt{3}}\right)^{1/2} .$$

If we use the assumption (6b), we obtain with (17b)

$$\Gamma_{(b)} = \tfrac{1}{3}\pi m_V\left(\frac{a\rho Gf}{m_V}\right)^2 \frac{\rho^4}{m_V{}^4}\,|\,\langle(\sigma)|\,\tilde{Q}^{\text{int}}|\,(V)\rangle\,|^2 . \tag{29b}$$

Comparing this with the usual expression [e.g., Eq. (8) of Ref. 10] we obtain for case (a)

$$\frac{m_V}{f_V^{(a)}} = \frac{aG\rho f}{2\alpha}\,\langle(\sigma)|\,\tilde{Q}^{\text{int}}|\,(V)\rangle , \tag{30a}$$

and for case (b)

$$\frac{m_V{}^3}{f_\nu^{(b)}} = \frac{aG\rho^3 f}{2\alpha}\,\langle(\sigma)|\,\tilde{Q}^{\text{int}}|\,(V)\rangle . \tag{30b}$$

If we assume that ρ is constant (more precisely, that ρ is the same for all the vector mesons ϕ, ω, ρ) then assumption (a) leads to the same results as model III of Ref. 10, i.e., to the prediction that m_V/f_V obey the SU(3) relations, and $\Gamma(V)$ obey the Das-Mathur-Okubo sum rule.

(5) Since the present experimental data differ from those used in Ref. 10, we shall make a comparison with the latest experimental results[12]:

$$\begin{aligned}\Gamma^{\text{ex}}(\rho\to e^+e^-)&=6.11\pm0.53\text{ keV}\quad\text{(Orsay)}\\&=5.2\pm0.5\text{ keV}\quad\text{(Novosibirsk)},\\\Gamma^{\text{ex}}(\omega\to e^+e^-)&=0.76\pm0.08\text{ keV}\quad\text{(Orsay)},\\\Gamma^{\text{ex}}(\phi\to e^+e^-)&=1.36\pm0.10\text{ keV}\quad\text{(Orsay, Novosibirsk)},\\\frac{\Gamma^{\text{ex}}(\omega\to e\bar{e})}{\Gamma^{\text{ex}}(\phi\to e\bar{e})}&=0.56\pm13\%\ .\end{aligned}\tag{31}$$

For the above three cases, the matrix elements of the SU(3) vector operator $\tilde{Q}$ are as follows:

$$\begin{aligned}\langle(\sigma)|\,\tilde{Q}^{\text{int}}|\,(\rho^0)\rangle&=\langle I=0,\ Y=0\{1\}|\,[H_1+(1/\sqrt{3})H_2]\|I=1,I_3=0,\ Y=0\{8\}\rangle\\&=(-1)(1/\sqrt{8})\langle\{1\}\|\,\mathfrak{N}\|\,\{8\}\rangle\ ,\\\langle(\sigma)|\,\tilde{Q}^{\text{int}}|\,(\omega)\rangle&=\langle I=0,\ Y=0\{1\}|\,[H_1+(1/\sqrt{3})H_2]\|\,I=0,\ Y=0\{8\}\rangle(-\sin\theta)\\&=(+1)(1/\sqrt{8})\langle\{1\}\|\,\mathfrak{N}\|\,\{8\}\rangle(1/\sqrt{3})(-\sin\theta)\ ,\\\langle(\sigma)|\,\tilde{Q}^{\text{int}}|\,(\phi)\rangle&=\langle I=0,\ Y=0\{1\}|\,[H_1+(1/\sqrt{3})H_2]\|\,I=0,\ Y=0\{8\}\rangle\cos\theta\\&=(+1)(1/\sqrt{8})\langle\{1\}\|\,\mathfrak{N}\|\,\{8\}\rangle(1/\sqrt{3})\cos\theta\ .\end{aligned}\tag{32}$$

Here $\langle\{1\}\|\,\mathfrak{N}\|\,\{8\}\rangle$ is the reduced matrix element[13] of the E_α between the SU(3) singlet and octet, and θ is the ϕ-ω mixing angle. For θ we take two values: $\theta=35.2°$, the value that is obtained under the assumption that the mesons belong to SU(4) multiplets [SU(6) value], and $\theta=40.2°$, the value that follows from the experimental masses of ϕ and ω and the Gell-Mann–Okubo mass formula.

We introduce the new constant

$$g=(-1)\frac{\langle\{1\}\|\,\mathfrak{N}\|\,\{8\}\rangle}{\sqrt{8}}\,\tfrac{1}{3}\pi aG\left(\frac{1-c^2}{2\sqrt{3}}\right)^{1/2}\ .\tag{33}$$

One sees that it is just the unknown reduced matrix element $\langle\{1\}\|\,\mathfrak{N}\|\,\{8\}\rangle$ that prevents us from obtaining an estimate on all three decay rates and a check on the values of a, G, and c obtained from different processes.

The decay rates (29) are now written

$$\Gamma_{(a)}(V\to e\bar{e})=g^2m_V(\rho/m_V)^2K_V{}^2\ ,\tag{34a}$$

with

$$K_\rho=1\ ,\ K_\omega=(1/\sqrt{3})\sin\theta\ ,$$

and

$$K_\phi=(1/\sqrt{3})\cos\theta\ ,$$

and

$$\Gamma_{(b)}(V\to e\bar{e})=g^2m_V(\rho/m_V)^6K_V{}^2\ .\tag{34b}$$

We first try to fit the experimental data with $\rho=\text{const}$. Using the Orsay value for $\Gamma^{\text{ex}}(\rho\to e\bar{e})$ as an input, one calculates[14] from (34a) with $\theta=35.2°$

$$\Gamma_{(a)}(\omega\to e\bar{e})=0.66\text{ keV}\pm20\%\ ,$$

$$\Gamma_{(a)}(\phi\to e\bar{e})=1.02\text{ keV}\pm20\%\ ,$$

$$\frac{\Gamma_{(a)}(\omega\to e\bar{e})}{\Gamma_{(a)}(\phi\to e\bar{e})}=0.65\pm1.3\%$$

There is not too much discrepancy between these values and the experimental data (31), although the values for $\Gamma_{(a)}(\phi\to e\bar{e})$ and $\Gamma_{(a)}(\omega\to e\bar{e})/\Gamma_a(\phi\to e\bar{e})$ do not quite agree with the experimental values; using the Novosibirsk value instead of the Orsay value for $\Gamma^{\text{ex}}(\rho\to e\bar{e})$ increases the disagreement. With $\theta=40.2°$, $\Gamma_{(a)}(\phi\to e\bar{e})$ is still smaller and $\Gamma_{(a)}(\omega)/\Gamma_{(a)}(\phi)=0.93$, so that this value is certainly excluded for case (a). With a smaller value for θ ($\theta\approx29°$), $\Gamma_{(a)}(\phi\to e\bar{e})$ can be brought into agreement with $\Gamma^{\text{ex}}(\phi\to e\bar{e})$. However, then $\Gamma_{(a)}(\omega\to e\bar{e})$ and $\Gamma^{\text{ex}}(\omega\to e\bar{e})$ will disagree. So we conclude that agreement of case (a) with experiments is not very likely, but cannot be ruled out completely.

With $\Gamma^{\text{ex}}(\rho\to e\bar{e})$ as input, we calculate from (34b)

$$\Gamma_{(b)}(\omega\to e\bar{e})=0.602\text{ keV}\pm80\%\ ,$$

$$\Gamma_{(b)}(\phi\to e\bar{e})=0.322\text{ keV}\pm80\%\ ,$$

and there is no way this can be brought into agreement with experimental data.

Thus constant ρ is definitely ruled out if we use the theoretically favored assumption (b) for H_λ.

(6) Let us now assume that ρ is not a constant, but that the matrix element of ρ is proportional to m_V, where V is one of the nonstrange vector mesons:

$$\langle\sigma|\,\rho|\,V\rangle=\zeta m_V\ .\tag{35}$$

In this case $\Gamma_{(a)}$ as well as $\Gamma_{(b)}$ will give

$$\Gamma(V)=g'^2m_VK_V{}^2\ ,\tag{36}$$

with $g'=\zeta g$ in case (a) and $g=\zeta^3g$ in case (b).

The result (36) is the same as that obtained from the mass-mixing model (model II of Ref. 10). We will see that this result is in agreement with the latest experimental data. With $\Gamma^{ex}(\rho)$ of (31) as input one calculates

	for $\theta = 35.2°$	for $\theta = 40.2°$
$\Gamma(\omega \to e\bar{e})$	$= 0.70$ keV ± 10%	$= 0.87$ keV ± 10%
$\Gamma(\phi \to e\bar{e})$	$= 1.81$ keV ± 10%	$= 1.58$ keV ± 10%
$\frac{\Gamma(\omega \to e\bar{e})}{\Gamma(\phi \to e\bar{e})}$	$= 0.385$	$= 0.549$.

Although the SU(6) value for θ does not quite fit the present experimental data, $\theta = 40.2°$ fits very well. Thus (35) is definitely suggested by the experimental data.

One way of obtaining (35) is to assume that ρ is an operator and that

$$\rho = \zeta M , \tag{37}$$

where ζ is a number and M is the mass operator. M does not commute with all E_α and F_α (or with Γ_i, which is however, irrelevant for the present case). When ρ is an operator, one has to replace (6b) by the Hermiticized expression, e.g.,

$$[P_\lambda, [P^\lambda, H_\mu]] = G^{1/2}\{\mathcal{J}_\mu^{el}, \rho^2\},$$

from which one obtains the result (36).

It is clear that (37) is not the only possibility for obtaining (35); e.g., if ζ were strongly dependent upon the hypercharge we could not have noticed it here.

(7) If one observes the place that ρ takes in the transition operators (1), one cannot be surprised at all about a relation like (37), which means that $\rho\Gamma_\mu$ has the same SU(3) properties that P_μ has, and it is worth discussing further theoretical arguments for (37).

The charges and axial charges E_α and F_α do not commute with the mass operator M or the momentum P_μ but fulfill certain algebraic relations as a consequence of which SU(3) or SL(3, C) [SU(3)⊗SU(3)] is "broken in an orderly fashion." Relations of this kind have been proposed for certain models and applied to the mass-spectrum problem.[15] These algebraic symmetry-breaking relations imply certain symmetry-breaking relations on the transition operators V_μ^α and A_μ^α of (1). For example, for the spin-zero meson tower one obtains

$$[E_\alpha, [P^\mu, V_\mu^{(0)\alpha}]] = 0 \tag{38}$$

for all α, with no summation over α. We will generalize these relations and assume that the following symmetry-breaking relations are fulfilled:

$$\text{(a)}\ \ [E_\alpha, [P^\mu, V_\mu^\alpha]] = 0, \qquad \text{(c)}\ \ [F_\alpha, [P^\mu, V_\mu^\alpha]] = 0,$$
$$\text{(b)}\ \ [E_\alpha, [P^\mu, A_\mu^\alpha]] = 0, \qquad \text{(d)}\ \ [F_\alpha, [P^\mu, A_\mu^\alpha]] = 0, \tag{39}$$

not summed over α. These relations are the algebraic analog of relations between charges and current divergences which have been suggested by Nishijima for the exclusion of strangeness-2-changing transitions in his model of CP violation and which have been used extensively in the literature.[16-19]

This brings us to the question of some possible connections between this algebraic approach and the existing approaches. The expressions (1) for V_μ^α and A_μ^α and relations (39) show that this algebraic model has many of the structures that current algebra also has [in particular the $SU(3)_{E_\alpha + iF_\alpha} \otimes SU(3)_{E_\alpha - iF_\alpha}$ as a "broken", "dynamic" symmetry]. The approach followed here is however rather different. We started with a very simple model of relativistic one-particle quantum mechanics which describes a very narrow domain of particle physics, and gradually enlarged this model by enlarging the algebra in such a way that an increasing number of experimental data were incorporated. Our model is therefore much simpler. But it also has some additional structures which are not specified in current algebra and which have physical consequences (e.g., a mass spectrum of states of different angular momenta and parity). Therefore it could well be that our empirically constructed expressions are in a certain way approximations of a representation of the continuously infinite algebra of currents. Another possibility could be that our algebraic expressions have properties that are not contained in the currents and that the usual currents and our transition operators have only certain structures in common.

Returning to our particular problem we will show now that as a consequence of (39) ρ cannot be an SU(3) scalar. As there is experimental evidence for (38) because it is in agreement with the experimentally correct mass relations for pseudoscalar mesons, (39a) will have to be valid for both parts independently.

$$[E_\alpha, [P^\mu, V_\mu^{(0)\alpha}]] = 0, \tag{40}$$

$$[E_\alpha, [P^\mu, V_\mu^{(1)\alpha}]] = 0 . \tag{41}$$

Generalizing (40), we will assume that all the symmetry-breaking relations (39) are valid for both parts $V_\mu^{(0)\alpha}$, $A_\mu^{(0)\alpha}$ and $V_\mu^{(1)\alpha}$, $A_\mu^{(1)\alpha}$ independently.

In the following, however, we will use only (40) to show that ρ cannot be an SU(3) scalar. After a short calculation, it follows from (40) that

$$[E_\alpha, [M^2, E_\alpha]] = 0 . \tag{42}$$

From this it follows that

$$\begin{aligned} 0 &= [E_\alpha, \{M, [M, E_\alpha]\}] \\ &= \{M, [E_\alpha, [M, E_\alpha]]\} + \{[M, E_\alpha], [E_\alpha, M]\}, \end{aligned}$$

from which we see that

$$[E_\alpha, [M, E_\alpha]] = 0 \text{ only if } [M, E_\alpha] = 0 . \tag{43}$$

(This is the case for $\alpha = 0, \pm 1, 8$ but not for $\alpha = \pm 2, \pm 3$.) As we are here only interested in the relation of ρ to E_α and F_α, we make the assumption that ρ fulfills the relations

$$[P^\mu, \rho] = 0, \quad [\Gamma^\mu, \rho] = 0 . \tag{44}$$

Under these conditions, and with $[\Gamma^\mu, E_\alpha] = 0$, one calculates

$$\begin{aligned} [P^\mu, \{\rho\Gamma_\mu, E_\alpha\}] &= [P^\mu \Gamma_\mu, \{\rho, E_\alpha\}] \\ &= [M, \{\rho, E_\alpha\}] \hat{P}_\mu \Gamma^\mu , \end{aligned} \tag{45}$$

where $\hat{P}_\mu = M^{-1} P_\mu$ and where we have used $[\hat{P}_\mu \Gamma^\mu, E_\alpha] = 0$. Using (45) in (41) yields

$$[E_\alpha, [M, \{\rho, E_\alpha\}]] = 0 . \tag{41$'$}$$

Let us now assume that ρ is a SU(3) scalar [i.e., $\rho = \text{const}$ if one also take (44) into account]. Then we would obtain from (41′) $[E_\alpha, [M, E_\alpha]] = 0$, which is impossible according to (43). Thus (41) cannot be fulfilled with $\rho = \text{const}$.

Hence we have shown that $\rho = \text{const}$, which is ruled out by experimental data for $\Gamma(V \to e\bar{e})$, is ruled out by theoretical arguments using reasonable assumptions concerning the transition operators.

Let us now investigate the assumption (37) for ρ, which yields agreement with the experimental data for $\Gamma(V \to e\bar{e})$. With $\rho = \zeta M$, we calculate

$$\begin{aligned} [E_\alpha, [M, \{\rho, E_\alpha\}]] &= \zeta [E_\alpha, \{M, [M, E_\alpha]\}] \\ &= \zeta [E_\alpha, [M^2, E_\alpha]] = 0 \end{aligned}$$

according to (42). Consequently (41′), and also (41), is a consequence of (40) for ρ given by (37). It is easily seen by analogous arguments that the other symmetry-breaking relations of (39) are also in agreement with (37). However, we wish to mention that the relations (39) are not very restrictive relations and do not contain all information for the symmetry breaking. Furthermore, (37) is not the only possibility for ρ that fulfills the symmetry-breaking relations. ρ can very well be an operator that reduces to (37) when applied to the process $V \to e\bar{e}$. In order to check how general (37), is we have to investigate additional processes.

In conclusion, we have shown that the transition operators (1) provide a description of the process $V \to e\bar{e}$ in agreement with the known experimental data. The experimental data as well as some theoretical considerations suggest that ρ has nontrivial SU(3)-transformation properties, and the assumption that ρ transforms like the mass operator is in good agreement with the experimental situation.

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[2] $\{A, B\} \overset{\text{def}}{=} AB + BA$.

[3] It is always assumed that $[E_\alpha, \Gamma_\mu] = 0$; however, in general if ρ is an operator, $[\rho, \Gamma_\mu] \neq 0$, so $V_\mu^{(1)\alpha}$ should read more precisely $V_\mu^{(1)\alpha} = \frac{1}{2}\{\{\rho, \Gamma_\mu\}, E^\alpha\}$, and similarly for $A_\mu^{(1)\alpha}$, F_α. This has been taken into account in the calculation.

[4] G_{X_i} indicates that the generators of the group G are the observables X_i.

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[7] The final results in Ref. 5 for the ratios of the rates for strangeness-changing and strangeness-nonchanging decays do not depend upon this assumption.

[8] In our notation $\sqrt{3}\,[H_1 + (1/\sqrt{3})\, H_2] = (\tilde{I}_3 + \frac{1}{2}\tilde{Y})$, where $\hat{I}_3$ and $\hat{Y}$ are the I_3 and Y components of octet operators.

[9] The $1/\rho^2$ factor could, of course, also have been included in the leptonic part if we would have chosen instead of (3) a suitably modified expression which would differ from the leptonic part in Ref. 5.

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[14] Wherever m_ρ enters in the calculation we have also included the error on this mass. As it is not clear what to take for this error, the total width of the ρ or the

error on the central value of the mass distribution Δm_ρ, we have decided to do the following: Where a disagreement has to be established (as in the calculated $\Gamma_{(a)}$ and $\Gamma_{(b)}$) we take the largest possible error (20%); where an agreement has to be established (as in the calculated Γ) we take the smallest possible error (10%).

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Optimal Bounds on Cross-Section Moments in π-π and π-N Scattering

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Froissart-like bounds are derived for a selection of moments of the total cross section in π-π and π-N scattering. These bounds stem from an economical set of constraints – unitarity and the Froissart-Gribov representation of the D-wave scattering length in the crossed channel α_2^t – which are rigorous to within the value assigned to this scattering length. The results are optimally tight for the given constraints and represent a substantial improvement on the bounds of Yndurain and Common, from whose work this paper has been developed. In the case of π-N scattering, it is possible to invert the argument by using experimental data on total cross sections to impose a lower bound on α_2^t.

I. INTRODUCTION

Yndurain and Common[1-3] have originated a method of clarifying the role of the Froissart bound in inhibiting the growth of total cross sections with energy. They define moments of the total cross section by

$$\bar{\sigma}_T(s_1, s_2) \equiv \int_{s_1}^{s_2} q(s')\sigma_T(s')ds', \tag{1}$$

where $q(s')$ is a positive weight function normalized over the energy range (s_1, s_2), and bound this quantity. Their results are explicit for finite s_2 (in their work, s_1 is invariably fixed at threshold), and the energy scale in the $\ln^2(s/s_0)$ is defined.

These bounds are rigorous to within a value assigned to a D-wave scattering length and have been evaluated both here and in the work cited above for π-π and π-N scattering.

In this paper, we employ a variational technique described by Einhorn and Blankenbecler[4] to improve and generalize the results obtained by Yndurain and Common. The main aspect of improvement is that general theorems of optimization theory show that the bounds derived here are maximally tight for the given set of constraints. The principal sources of generalization are that we consider a wider class of weight functions $q(s)$, and that in the method applied here it is not necessary to keep s_1 at threshold: Any energy interval in the physical region can be considered.

The rest of the paper is organized as follows: Section II contains a resumé of earlier work in the field. In Sec. III, we give an account of the method employed here. The numerical outcome of this analysis is presented in Sec. IV. Finally, conclusions and further applications are summarized in Sec. V.

II. CASE HISTORY

From an input of partial-wave unitarity and sufficient analyticity to guarantee the Froissart-Gribov representation of the t-channel, D-wave scattering length α_2^t, Yndurain manipulated the following bound from the partial-wave expansion of α_2^t, for π^0-π^0 scattering:

$$\bar{\sigma}_T^{(m)}(s) \leqslant \left(\frac{n+1}{n}\right)^2 \left(\frac{m+1}{m}\right) \frac{\pi}{\mu^2} \ln^2\left(\frac{s}{\mu^2}\right) + \left(\frac{n+1}{n}\right)\left(\frac{m+1}{m-\frac{1}{2}}\right) \frac{4\pi}{\mu\sqrt{s}} \ln\left(\frac{s}{\mu^2}\right) + (m+1)30\pi^2 M_n^{-1}\alpha_2^t, \tag{2}$$