Comments and Addenda

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Unitarity Bounds on Elastic Cross Sections

C. Eftimiu
Department of Physics, University of Missouri–St. Louis, St. Louis, Missouri 63121
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Necessary conditions to be satisfied by any elastic differential cross section are derived directly from the unitarity condition and compared with the Martin-Newton condition. Possible practical uses of these conditions are briefly indicated.

It has been pointed out, first by Sakmar[1] for a special case which will be further discussed below, and by Goldberg[2] for a number of elastic π-p angular distributions, that the Martin-Newton[3] condition for the existence of a solution of the elastic-unitarity nonlinear integral equation may not be satisfied in cases of physical interest.

In terms of the absolute magnitude of the elastic amplitude, which is related to the elastic differential cross section through $G(x)=k(d\sigma/d\Omega)^{1/2}$ ($x=\cos\theta$, $k=$relative momentum), the nonlinear integral equation for the phase $\phi(x)$ reads

$$G(x)\sin\phi(x)=\iint\frac{G(y)G(z)}{2\pi(1-x^2-y^2-z^2+2xyz)^{1/2}}\cos[\phi(y)-\phi(z)]dydz\,, \tag{1}$$

where the domain of integration is the region in the yz plane where the radicand is non-negative, and a solution $\phi(x)$ is sought such that $0\leq\phi(x)\leq\pi/2$. The Martin-Newton condition which guarantees the existence (and quite possibly also the uniqueness) of a solution of (1) is

$$Q(x)=\frac{1}{G(x)}\iint\frac{G(y)G(z)dydz}{2\pi(1-x^2-y^2-z^2+2xyz)^{1/2}}<1\,. \tag{2}$$

It should be stressed that

$$Q(1)<1 \tag{3}$$

is a necessary condition, but max $Q(x)<1$ can only be a sufficient condition of existence of a solution since it is not satisfied[4] by a number of experimental elastic angular distributions. The purpose of this short note is to indicate that *necessary* conditions on $G(x)$, which appear to be quite close to being optimum, can be easily derived and practically used.

Namely, if one multiplies Eq. (1) by the Legendre polynomial $P_n(x)$ and integrates over x, after changing the order of integration on the right-hand side so as to integrate first with respect to x and after using the addition formula for Legendre polynomials, one easily obtains

$$\mathrm{Im}\left(\tfrac{1}{2}\int_{-1}^{1}G(x)P_n(x)\exp[i\phi(x)]dx\right)^{-1}=-1 \tag{4}$$

or

$$\tfrac{1}{2}\int_{-1}^{1} G(x)P_n(x)\exp[i\phi(x)]dx = \sin\delta_n \exp(i\delta_n), \quad (5)$$

which is, of course, equivalent to the usual partial-wave expansion and where the real numbers δ_n are the ordinary phase shifts. If one rewrites (5) as

$$\tfrac{1}{2}\int_{-1}^{1} G(x)P_n(x)[\cos\phi(x)+\sin\phi(x)]dx = \sin\delta_n(\cos\delta_n+\sin\delta_n),$$

one sees immediately that

$$\frac{1-\sqrt{2}}{2} \leq \tfrac{1}{2}\int_{-1}^{1} G(x)P_n(x)[\cos\phi(x)+\sin\phi(x)]dx \leq \frac{1+\sqrt{2}}{2}. \quad (6)$$

Since $0 \leq \phi(x) \leq \pi/2$, and therefore

$$1 \leq \cos\phi(x)+\sin\phi(x) \leq \sqrt{2},$$

we obviously must have

$$0 \leq \tfrac{1}{2}\int_{-1}^{1} G(x)dx \leq \frac{1+\sqrt{2}}{2} = 1.207 \quad (7)$$

and hence, for $n \geq 1$ also

$$c_n \equiv \tfrac{1}{2}\int_{-1}^{1} G(x)P_n(x)dx \leq \tfrac{1}{2}\int_{-1}^{1} G(x)dx \leq \frac{1+\sqrt{2}}{2}. \quad (8)$$

Before proceeding further, let us note that from the Martin-Newton condition (2), it follows in a quite straightforward manner that

$$0 \leq \tfrac{1}{2}\int_{-1}^{1} G(x)dx \leq 1, \quad (9)$$

which is to be compared with condition (7). It should be emphasized though that while (9) is part of a sufficient condition, (7) expresses a necessary bound.

A lower bound for the numbers c_n $(n \geq 1)$ can also be found by writing

$$\frac{1+\sqrt{2}}{2}c_n = \tfrac{1}{2}\int_{-1}^{1} G(x)P_n(x)[\cos\phi(x)+\sin\phi(x)]dx - \tfrac{1}{2}\int_{-1}^{1} G(x)P_n(x)[\cos\phi(x)+\sin\phi(x)-(1+\sqrt{2})/2]dx.$$

From (6) and the fact that

$$\max|\cos\phi(x)+\sin\phi(x)-(1+\sqrt{2})/2| = (\sqrt{2}-1)/2,$$

one gets

$$c_n \geq \frac{5-3\sqrt{2}}{2} = -0.379 \quad (n=1,2,\ldots). \quad (10)$$

At first sight, conditions (8) and (10) seem impractical because their number is infinite, but it is easy to see that in practice one has to check only a finite (and, probably, small) number of such inequalities. This is simply because of Schwarz's inequality:

$$\left(\tfrac{1}{2}\int_{-1}^{1} G(x)P_n(x)dx\right)^2 \leq \frac{1}{2(2n+1)}\int_{-1}^{1} G^2(x)dx \equiv \frac{\sigma}{2(2n+1)}. \quad (11)$$

Hence, the conditions (8) and (10) are satisfied if the interval $(-[\sigma/2(2n+1)]^{1/2}, [\sigma/2(2n+1)]^{1/2})$ is entirely contained in $((5-3\sqrt{2})/2, (1+\sqrt{2})/2)$, but it is obvious that beginning with a value N of n large enough this will be the case for all other $n > N$.

Let us now indicate briefly two circumstances under which the bounds derived in this paper could be useful. Suppose that $G(x)$ is given as a function which interpolates experimental data for some elastic process. In such a case a solution of (1) should naturally exist and, indeed, conditions (7), (8), and (10) are fulfilled for a number of experimental angular distributions we checked. If, however, one deals with a function like

$$G(x) = \lambda(a+bx+cx^2)^{1/2}$$

which for $\lambda = 2.018$, $a = 0.227$, $b = 0.152$, $c = 0.621$ represents[5] the angular distribution of π^+p elastic scattering at 220 MeV, one finds, e.g., that $c_0 = 1.295$. Of course, the experimental errors which affect the determination of the numbers λ, a, b, and c in this case are sufficiently large (of the order of 10%) to account for this discrepancy. But this only shows that the experimental errors cannot concur in producing a violation of the condition (7), which therefore imposes bounds on the magnitude and relative sign of such errors.

As an example of another possible use of conditions (7), consider the distribution indicated in Ref. 1:

$$G(x)=\lambda(1+3x^2)^{1/2},$$

whose analytic form has been determined on the basis of a model to represent π^+p elastic scattering at 1236 MeV. In this case, only the coefficient λ must be determined from experimental data. As one can easily check, $c_0=1.3801\lambda$, which makes the value $\lambda=0.988$ indicated in Ref. 1 unacceptable.

On the other hand, for a lower value like $\lambda=\sqrt{3}/2$, which is still compatible with experimental data, $c_0=1.195$, but this is hardly surprising since for this value of λ, Eq. (1) is known[6] to have an exact solution. Note also that $c_n=0$ for odd n, while $c_n>0$ for n even. Since $[\sigma/2(2n+1)]^{1/2}=1/\sqrt{2}$ for $\lambda=\sqrt{3}/2$ and $n=1$, it follows that out of all conditions only the one for $n=0$ provides an effective bound.

From a theoretical viewpoint, one might express the hope that the bounds (7), (8), and (10) could provide an indication as to the direction and range of an extension of the Martin-Newton condition covering all cases of physical interest.

I am deeply indebted to Professor A. Martin for suggesting the proof which leads to the lower bound (10), and for correcting a number of errors in an earlier version of the manuscript.

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[1] I. A. Sakmar, Lett. Nuovo Cimento 2, 256 (1969).

[2] H. Goldberg, Phys. Rev. D 1, 1242 (1970).

[3] R. G. Newton, J. Math. Phys. 9, 2050 (1968); A. Martin, Nuovo Cimento 59A, 131 (1969).

[4] The values shown in the columns labeled $Q(1)$ and $\alpha\lambda Q(1)$ for Group I in Table I of Ref. 2 are incorrect. The correct values are still such that $Q(1)<1$, and the values for max$Q(x)$ shown in this reference are correct, so that its main conclusion is valid. It should also be kept in mind that comparisons with experimental data, like those made in the present paper, as well as in Ref. 2, assume that spin effects at high enough energies are completely negligible.

[5] J. Ashkin *et al.*, Phys. Rev. 105, 724 (1957); this is case I5 in the Table I of Ref. 2.

[6] C. Eftimiu, Lett. Nuovo Cimento 4, 475 (1970).

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Coupled Angular Momentum Channels and Axial-Vector Resonance in $\pi\omega$ Scattering*

Pu Shen and Kyungsik Kang

Department of Physics, Brown University, Providence, Rhode Island 02912

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The possibility of binding an axial-vector meson in the $\pi\omega$ system is considered. A substantial admixture of s and d waves is required to bootstrap this particle, and the decay angular distribution calculated is in good agreement with the recent experimental information.

In this note, we would like to report on a simple S-matrix dynamical calculation which gives a resonant state in the $J^P=1^+$ state of the $\pi\omega$ channel. In this state, the $\pi\omega$ system can take two angular momentum values $l=0$ and 2, and the calculation is carried out by taking into account these two angular momentum channels.

All previous attempts[1,2] based on the self-consistency idea have preferred a bound state in the 2^- state. They are usually carried out either in the static approximations or in the relativistic dispersion methods. But none of them have seriously considered the possible admixture of the two angular momentum channels in the unnatural-spin-parity state. In fact, the recent experiments[3,4] not only favor 1^+ over 2^- for this $\pi\omega$ resonance but also give the angular distributions in detail, indicating the presence of both s and d waves.

We can explain these experiments from a *two-channel N/D calculation,* which assumes the driving forces coming from the u-channel exchanges only. The matrix N/D equations are solved alge-