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Scaling in Hadronic Collisions and the New Kinematic Variable n^2 †

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We present the underlying theoretical motivation for the recently introduced kinematic variable n^2. Scattering amplitudes are considered as functions of s and n^2, and asymptotic formulas for $s \to \infty$ at fixed n^2 are derived. It is shown empirically that many diffractive reactions appear to scale, i.e., they show little or no energy dependence when considered as functions of n^2. The shrinkage of the forward diffraction peak in $d\sigma/dt$, seen in many reactions at present energies, is predicted to die out with increasing energy. It is shown that $\bar{p}p \to \bar{p}p$ scattering should exhibit antishrinkage. Several reactions are studied: $pp \to pp$, $K^+p \to K^+p$, $\pi^\pm p \to \pi^\pm p$, $\gamma p \to \rho p$, $\gamma p \to \phi p$, $pp \to pN^*$. Speculative predictions are made for a kind of "superscaling" in inclusive reactions and in deep-inelastic electron scattering.

I. INTRODUCTION AND MAIN RESULTS

In a recent letter [1] we introduced a new kinematic variable n^2 (whose origin lies in certain group-theoretical considerations) in terms of which high-energy scattering data show remarkable regularities. In the above-mentioned letter we were content to discuss some of the phenomenological consequences, mainly in connection with the energy dependence of the slope in diffraction scattering, and to point out the connection in pp scattering between n^2 and the variable $\beta^2 p_\perp{}^2$ introduced long ago by Krisch,[2] on totally different grounds.

The aims of this present paper are

(i) to describe in detail the basic theoretical reasons for suggesting that n^2 is a more suitable and natural variable than t for the description of scattering at high energies, and

(ii) to make a more detailed, though still somewhat qualitative, comparison between theory and experiment for several high-energy reactions. It will be shown as an empirical fact that diffractive processes exhibit a type of scaling, i.e., they are essentially energy-independent when considered at fixed n^2. The empirical results go far beyond the predictions of the theory and suggest that n^2 is not only a preferred variable from the point of view of kinematics, but also perhaps is singled out for some underlying dynamical reason.

Unfortunately, the starting point for the derivation of the variable n^2 is a rather technical one, based on considerations of the theory of conspiracies and Toller poles. Nevertheless we believe that the principles involved and the results obtained are of great interest. We shall therefore attempt, in this introduction, to give a qualitative and nontechnical discussion of the main principles involved, and also to summarize the essential results of our analysis. A discussion of phenomenological applications is to be found in Sec. III. Our conclusions, as well as some speculative predictions about the behavior of inclusive reactions, about deep-inelastic electron scattering, and about nondiffractive $2 \to 2$ scattering, are located in Sec. IV. All technical details are contained in Sec. II and can be skipped by the reader who is primarily interested in the phenomenological implications of our results.

The scattering amplitude f for any $2 \to 2$ process is basically a function of two independent continuous variables – say k and θ or s and t – and very often one expands f in terms of well-defined functions of one of the variables with coefficients which depend on the other variable, for example, the usual partial-wave expansion,

$$f(k,\theta)=\sum_{l=0}^{\infty}(2l+1)f_l(k)P_l(\cos\theta). \qquad (1.1)$$

The question we wish to discuss is "What determines *which expansion* we choose? Why use the Legendre expansion and not a Fourier integral or any other type of expansion?" The answer is always that *simplicity* is a vital criterion. It makes sense, for example, to use the Legendre expansion, Eq. (1.1), for $\pi p \to \pi p$ scattering at 200 MeV because we know, from other considerations, that only a *few terms* in the series are necessary to obtain an excellent approximation to $f(k,\theta)$ in this energy region. It does not make sense to use Eq. (1.1) at CERN Intersecting Storage Rings (ISR) energies.[1] *We shall thus take as an axiom that an expansion is only useful or worthwhile if there is at least a chance that the function under study can be reasonably approximated by one or a very small number of terms in the series.*

Now it sometimes happens that we can tell, on general grounds, that there is no hope at all of approximating a function by one or just a few terms of a certain expansion. Suppose, just for the purposes of illustration, that $f(k, \theta)$ possesses a peculiar symmetry such that

$$f(k, \theta) = f(k, \theta + \psi), \tag{1.2}$$

where ψ is some given angle, say 23°. Then it is clear that the expansion, Eq. (1.1), is not a useful one since to satisfy Eq. (1.2) we would have to calculate $P_l(\cos(\theta + \psi))$ in terms of $P_{l'}(\cos\theta)$, involving a range of values of l', and the condition (1.2) would then impose a host of linear relations among the $f_l(k)$.

In general terms, if $f(k, \theta)$ possesses some exact symmetry under $\theta \to g(\theta)$, say

$$f(k, g(\theta)) = \lambda f(k, \theta),$$

then, if simplicity is our aim, we should use expansion functions that *themselves possess this symmetry*, i.e., we should write

$$f(k, \theta) = \sum_n a_n(k) d_n(\theta),$$

where each $d_n(\theta)$ has the symmetry property

$$d_n(g(\theta)) = \lambda d_n(\theta).$$

In this way it is clear that there is no inconsistency in approximating $f(k, \theta)$ by just one term in the expansion.

It is precisely this type of difficulty that appears in Regge theory. One considers a process

$$A + B \to C + D \tag{1.3}$$

and uses as variables t and θ_t the energy squared and the scattering angle, respectively, in the center-of-mass system of the t channel $A\bar{C} \to \bar{B}D$. One then performs a Legendre expansion based on the t-channel scattering angle and obtains, with $z_t = \cos\theta_t$,

$$f(k, \theta) \equiv F(t, \theta_t) = \sum_l (2l+1) f_l(t) P_l(z_t). \tag{1.4}$$

It is very well known how one replaces the sum over l by the Sommerfeld-Watson integral, etc., and eventually obtains the Regge asymptotic behavior

$$F(t, \theta_t) \underset{z_t \to \infty}{\sim} \beta(t) P_{\alpha(t)}(-z_t). \tag{1.5}$$

For almost all values of t, the above procedure is perfectly all right. The trouble arises at certain special values of t where $F(t, \theta_t)$ suddenly develops additional symmetry properties, symmetry properties which are not possessed by the $P_l(z_t)$ themselves. Thus one obtains complicated equations among the $f_l(t)$ for different l values. To avoid this one should not use the Legendre expansion at these special t values, but rather an expansion based on representation functions which themselves possess the necessary symmetry property.

The most important special t value, from the point of view of describing physical scattering processes, is $t = 0$. For example, in *elastic scattering*, it can be shown[3,4] that $F(t = 0, \theta_t)$ has an extra symmetry as compared with $F(t \neq 0, \theta_t)$. This necessitates, according to our criterion of simplicity, that we restrict our use of the $P_l(\theta)$, i.e., representation functions of the rotation group O(3), to $t \neq 0$ and that for $t = 0$ we use the representation functions of the homogeneous Lorentz group, O(3, 1).

We ask now, "What happens at $t = 0$ in *inelastic scattering*?" Again it can be shown[4,5] that $F(t = 0, \theta_t)$ has a special symmetry, but – and this is the crux of our entire investigation – the special symmetry is *not* the same as in the elastic case at $t = 0$, and to satisfy our criterion of simplicity we should now employ representation functions of the group of rotations and translations in a plane, i.e., of $T_2 \times O(2)$.

Now this is a very peculiar state of affairs. The representation functions of O(3, 1) and $T_2 \times O(2)$ are quite different in character, yet on physical grounds it is almost impossible to believe that a reaction in which the masses of the final particles equal those of the initial particles and a very slightly inelastic reaction are totally different from each other as $s \to \infty$. For example, if we compare the case of reaction (1.3) in which $m_A = m_C$, $m_B = m_D$ with a reaction in which $m_A = m_C$ but $m_B = m_D + \Delta m$ we would expect the reactions to look more and more alike as $\Delta m \to 0$, and we would expect some kind of smooth transition from one case to the other. However, the mathematical structure changes discontinuously – no matter how small Δm is one must use $T_2 \times O(2)$ and not O(3, 1). We have a situation, therefore, in which a small continuous change in the physics is being described by a major discontinuous change in the mathematical structure. It is not difficult to show[5] that the cause of this peculiar behavior lies in a bad choice of kinematic variables, namely in the use of t, or more precisely in the use of the momentum-transfer vector

$$K = p_A - p_C, \quad K^2 = t \tag{1.6}$$

as a fundamental variable in the description of the scattering amplitude. We have therefore looked for a vector to replace K, in terms of which the mathematical structure would be insensitive to whether the reaction was elastic or inelastic. In other words, if the scattering amplitude has a certain symmetry at some fixed value of the new vector, then we require this symmetry to remain unchanged as we vary the external masses of the reaction. In this way the unnatural distinction between processes with different external masses is eliminated and it becomes possible to use the same expansion functions in all types of reactions. For example, if we wish to compare the diffractive production $pp \to pN^*$ with $pp \to pp$ then in terms of the new vector the mathematical structure of the two reactions is similar and any differences would presumably reflect genuine dynamical effects.

The only vector we have been able to find with the above properties is the 4-dimensional normal to the scattering plane:

$$N_\mu = 2\epsilon_{\mu\nu\rho\sigma} p_A^\nu p_B^\rho p_C^\sigma \ . \tag{1.7}$$

Because of momentum conservation it is essentially irrelevant which three of the vectors p_A, p_B, p_C, p_D we choose for the definition of N_μ.

In the c.m. system of the reaction (1.3), with the y axis taken perpendicular to the scattering plane as usual, one has

$$N_0 = 0, \quad \vec{\mathrm{N}} = 2s^{1/2} p_i p_f (0, \sin\theta, 0), \tag{1.8}$$

where $p_i = |\vec{\mathrm{p}}_i|$ and $p_f = |\vec{\mathrm{p}}_f|$ are the magnitudes of the initial and final c.m. momenta.

As is discussed in detail in Sec. II it is the invariance properties of N_μ that determine the symmetry group[6] and thus the expansion functions. It is clear that for $\theta \neq 0$ or π, N_μ is unchanged by the pure Lorentz transformations in the x or z directions and by rotations about the y axis, and this will lead to the use of representation functions of the group O(2, 1) made up of these three operations. On the other hand, for $\theta = 0$ or π, N_μ becomes a null vector and is unchanged by *any* Lorentz transformation; so we will be led to use the representations of O(3, 1) for forward or backward scattering. The most important point is that these statements hold regardless of the values of the external masses and therefore apply uniformly to all $2 \to 2$ reactions. However, it will turn out that we have to pay a price for the use of n^2 – it will be necessary to make some continuation off the mass shell, and to assume that in small excursions from it no serious mass dependence is introduced.

We therefore consider the scattering amplitudes as a function of the vector N_μ together with

$$p = p_B + p_D \quad \text{and} \quad q = p_A + p_C,$$

i.e., (1.9)

$$f = f(N; p, q) .$$

The general property of Lorentz covariance, as usual, allows f to depend only on two independent scalars, which in this case are N^2 and $p \cdot q$; $p \cdot q$ is essentially s and

$$N^2 = -\phi(s, t, u), \tag{1.10}$$

where ϕ is the Kibble function,[7] whose vanishing defines the boundaries of the physical regions in the Mandelstam plane for all of the related channels

$$AB \to CD, \quad A\overline{C} \to \overline{B}D, \quad A\overline{D} \to C\overline{B}.$$

While specifying s and t defines a unique kinematic point in the Mandelstam plane, it should be noted that specifying s (>0) and N^2 (≤0) defines two points in the s channel, related by having scattering angles θ and $\pi - \theta$, respectively. This implies that a function of the variables (s, t) is actually a function of (s, N^2, σ), where $\sigma = \mathrm{sgn}(\frac{1}{2}\pi - \theta)$. The introduction of such a sign is a common feature of coordinate transformations. This ± sign will be taken as *implicitly present* in all that follows even when not explicitly written. Of course, $f(s, N^2, +)$ and $f(s, N^2, -)$ will, in general, be completely different. However, in special cases like pp scattering

$$f = f(s, N^2, +) = f(s, N^2, -) = f(s, N^2) .$$

In general, for any scattering amplitude, one can write

$$f(s, \theta) = f_S(s, N^2) + \cos\theta \, f_A(s, N^2), \tag{1.11}$$

from which it follows that only the symmetrized and antisymmetrized scattering amplitudes

$$\begin{aligned} f_S(s, N^2) &\equiv \tfrac{1}{2}[f(s, \theta) + f(s, \pi - \theta)], \\ f_A(s, N^2) &\equiv \frac{1}{2\cos\theta}[f(s, \theta) - f(s, \pi - \theta)] \end{aligned} \tag{1.12}$$

can be *analytic* functions of s and N^2.

For the purposes of making the little-group expansion, to be discussed in Sec. II, we define from N_μ a vector, n_μ, which has the dimensions of a 4-momentum. This little vector is defined by

$$n_\mu = \frac{N_\mu}{R(s, t, m_i^2)} , \tag{1.13}$$

where $R(s, t, m_i^2)$ is a Lorentz scalar with mass-squared dimensions. This vector n_μ then has exactly the same group properties as N_μ. We then consider the symmetrized and antisymmetrized scattering amplitudes of Eq. (1.12) as functions of s and n^2. A Regge-type analysis leads to the asymptotic behavior

$$f(s, n^2) \underset{s \to \infty;\, n^2 \text{ fixed}}{\sim} b(n^2) s^{a(n^2)}, \tag{1.14}$$

where $a(n^2)$ is the position of a pole in an angular-momentum-like plane conjugate to the momentum vector n_μ. Since N^2 is completely crossing-symmetric between all three channels, it might seem aesthetically appealing to choose R = constant (GeV)^2 so that n^2 still has the same crossing properties as N^2 – for the purposes of this analysis the normalization R could be any scalar. However, a difficulty arises when we study the high-energy behavior of $f(s, n^2)$ at fixed n^2 if R = constant. Since as $s \to \infty$

$$N^2 \sim s^2 t \tag{1.15}$$

and for small n^2 we have

$$a(n^2) = a(0) + a'(0) n^2 , \tag{1.16}$$

Eq. (1.14) gives

$$\frac{d\sigma}{dt} \sim \left| b\left(\frac{s^2 t}{R^2}\right)\right|^2 s^{2[a(0)+a' s^2 t/R^2 - 1]} . \tag{1.17}$$

However, for small t and asymptotic energies we expect

$$\frac{d\sigma}{dt} \sim g(t) s^{2[\alpha(0)+\alpha' t - 1]} . \tag{1.18}$$

We see that Eq. (1.17) is in complete contradiction with this expected behavior if R = constant. Indeed with R = constant and $a' \neq 0$ the amplitude given by Eq. (1.14) is not polynomially bounded for $0 < t < 4m_\pi^2$. Thus we are forced to choose the function R such that

$$R \sim s$$

as $s \to \infty$ for small t. We shall find that the most convenient choice is just $R = s$ so we define

$$n_\mu = \frac{2}{(p_A + p_B)^2} \epsilon_{\mu\nu\rho\sigma} p_A^\nu p_B^\rho p_C^\sigma = \frac{1}{s} N_\mu . \tag{1.19}$$

A more detailed discussion of this problem, given in Sec. III, shows that it is not advisable to attempt to make R a crossing-symmetric function.

We shall therefore describe the scattering amplitude as a function of s and n^2, not forgetting $\sigma = \mathrm{sgn}[t_0(s - s_1)]$, where

$$t_0 = t - u + (m_A^2 - m_B^2)(m_C^2 - m_D^2)/s \tag{1.20}$$

and the line $t = t_0$ intersects the curve n^2 = constant at $s = s_1, s_2$ with $s_1 < s_2$, which means $\sigma = \mathrm{sgn}(\frac{1}{2}\pi - \theta)$ in the s channel. We note that

$$n^2 = -4 \frac{p_i^2 p_f^2}{s} \sin^2\theta = -\frac{4 p_i^2}{s} p_\perp^2 \tag{1.21}$$

and that $n^2 \to -p_\perp^2$ as $s \to \infty$. However, as will be discussed in Sec. III, one cannot neglect the s dependence of the factor p_i^2/s even at Serpukhov energies.

From Eqs. (1.10) and (1.19) one has, in general,

$$n^2 = t\left(1 - \frac{\Sigma - t}{s}\right) + \frac{1}{s}(m_A^2 - m_C^2)(m_B^2 - m_D^2) + \frac{1}{s^2}[t(m_A^2 - m_B^2)(m_C^2 - m_D^2) + (m_A^2 m_D^2 - m_B^2 m_C^2)(m_A^2 - m_B^2 - m_C^2 + m_D^2)] , \tag{1.22}$$

with

$$\Sigma \equiv s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2 .$$

In most $2 \to 2$ reactions either $m_A = m_C$ or $m_B = m_D$. In these cases

$$n^2 = t\left(1 - \frac{\Sigma - t}{s}\right) + O\left(\frac{1}{s^2}\right) . \tag{1.23}$$

While by definition, Eq. (1.19), $n^2 \to t$ at fixed t as $s \to \infty$, n^2 is *very different* from t for nonasymptotic energies and for large scattering angles. This is illustrated in Fig. 1 where a typical curve of n^2 = constant is shown in the Mandelstam plane.

The main results of our theoretical analysis using the variables s and n^2 are as follows:

(i) Scattering amplitudes considered as functions of s and n^2 possess a mathematical structure that is independent of the values of the external

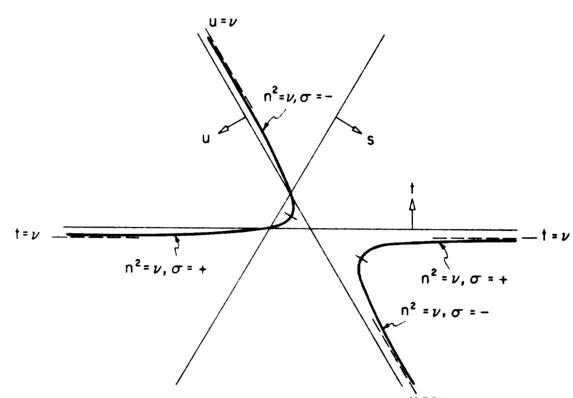

FIG. 1. A typical curve of $n^2 = \nu$ (<0) plotted in the Mandelstam plane – for simplicity we have chosen the masses of all the external particles to be equal. This hyperbola is asymptotic to the lines $t = \nu$ and $u = \nu$ which are shown. The branches of the curves with their signs, σ, defined by Eq. (1.20), are also given.

masses.

(ii) Comparison of experimental data as a function of s and n^2 for similar reactions but with differing external masses should give a direct indication of dynamical effects.

(iii) A "Toller-like" treatment shows that for *forward scattering* in *any* reaction

$$f_{\lambda_C\lambda_D,\lambda_A\lambda_B}(s,\theta=0) \underset{s\to\infty}{\sim} \delta_{\lambda_A-\lambda_C,\lambda_B-\lambda_D}\, s^{\alpha_+(0)-1-|M-|\lambda_A-\lambda_C||} \tag{1.24}$$

for a pole with Lorentz quantum number M and trajectory intercept $\alpha_+(0)$ at $n^2=0$. We see that the dominant pole for a given helicity amplitude has $M=\pm(\lambda_A-\lambda_C)$. It should also be noted that this result is identical to Toller's result for *elastic* scattering,[3] but that Eq. (1.24) now holds for all processes.

(iv) For *backward scattering* we obtain

$$f_{\lambda_C\lambda_D,\lambda_A\lambda_B}(s,\theta=\pi) \underset{s\to\infty}{\sim} \delta_{\lambda_A-\lambda_C,\lambda_D-\lambda_B}\, s^{\alpha_-(0)-1\,|M-|\lambda_A-\lambda_C||}\,, \tag{1.25}$$

where $\alpha_-(0)$ is now the intercept of a pole which dominates in the backward region.

(v) For the region *inside* the physical boundaries one has

$$f_{\lambda_C\lambda_D,\lambda_A\lambda_B}(s,n^2)\sim b^{\pm}_{\lambda_A-\lambda_C,\lambda_B-\lambda_D}(n^2)s^{\alpha_\pm(n^2)}\,, \tag{1.26}$$

where the + (−) sign refers to poles which dominate scattering in the region $\theta<\pi/2$ ($\theta>\pi/2$). These formulas are expected to hold only for $|n^2|\ll s$.

As emphasized earlier the only point in using such expansions of the scattering amplitude is if they simplify the description of the physics. We have therefore studied several reactions, plotting the data as a function of n^2 to see what emerges. The results of this *phenomenological* study are quite dramatic. We find:

(vi) Differential cross sections for reactions that are mainly diffractive, such as $pp\to pp$, $K^+p\to K^+p$, $\gamma p\to\phi p$, $\gamma p\to\rho^0 p$, and $pp\to pN^*$, show a remarkable s independence, or scaling, as a function of n^2. Data from a very large range of energies and angles all fall close to a universal curve which is a function of n^2 only, i.e., one has *phenomenologically* that

$$\frac{d\sigma}{dt}\equiv\frac{d\sigma}{dt}(s,n^2)\simeq G(n^2)\text{ only.} \tag{1.27}$$

This means, Eq. (1.26), that the leading pole in diffractive scattering has

$$\alpha(n^2)\simeq 1 \tag{1.28}$$

in the physical scattering region and all the shrinkage in t for these reactions comes not from a moving pole but merely from the s-dependent relation between n^2 and t.

It will be shown in Sec. III that each pole in our angular-momentum-like plane, which produces the asymptotic behavior given by Eq. (1.26), induces an infinite set of poles in the usual Regge j plane. Thus our leading trajectory, Eq. (1.28), corresponds to a model for the Pomeranchukon which would look like an infinite sequence of "fixed" poles at $j=1, 0, -1, -2, \ldots,$ in the Regge j plane. There is an important lesson to be learned from this result. Normally one associates shrinkage of the forward diffraction peak with the nonzero slope of a Regge trajectory, and one always neglects very low-lying trajectories, say those with $\alpha\le 0$, or equivalently terms of order $1/s$ compared to the leading ones. Yet in the above a sequence of essentially *flat* poles, corresponding to terms of order $1/s$, $1/s^2, \ldots,$ etc., compared to the leading one, add up to give a differential cross section in which the forward diffraction peak shrinks, even at Serpukhov energies.

(vii) For those diffractive reactions which are exotic in the s channel, the slope parameter

$$b(s,m_i^2)\equiv\left[\frac{d}{dt}\ln\left(\frac{d\sigma}{dt}\right)\right]_{t=0} =\beta_0\left(1-\frac{\Sigma}{s}\right), \tag{1.29}$$

where $\Sigma=\sum_i m_i^2$ and β_0 is a reaction-dependent constant.

For reactions reached from these by s-u crossing, one predicts that ultimately their slope parameters will be given by

$$\bar{b}(s,m_i^2)=\beta_0\left(1+\frac{\Sigma}{s}\right), \tag{1.30}$$

where β_0 is the same constant as in the crossed reaction. In particular since $pp\to pp$ forward scattering obeys Eq. (1.29) we find that $\bar{p}p\to\bar{p}p$ scattering should obey Eq. (1.30) and so exhibit antishrinkage (at least at energies where secondary trajectories cease to be important).

(viii) For reactions that are not purely diffractive, e.g., $\pi^\pm p\to\pi^\pm p$, we find that the data cluster on an n^2 plot and seem to oscillate around a universal n^2 function as the energy is varied. There is some indication that the magnitude of the oscillations is dying out as the energy increases and we are tempted to conjecture that the data will ultimately collapse onto the universal n^2 curve. A more detailed analysis of the s dependence of the data at fixed n^2 is under current investigation.

II. PARTIAL-WAVE ANALYSIS; THE NEW LITTLE-GROUP EXPANSION

A. General Discussion

We have seen that the little group of n_μ within the physical region is O(2, 1) and on the physical region boundary is O(3, 1) regardless of the masses of the external particles. It is therefore our aim, in this section, to expand the scattering amplitude for arbitrary masses and spins in terms of unitary irreducible representations of the appropriate little group of the normal vector n_μ. In order to achieve this we must express the helicity amplitudes as a function of the Lorentz transformation leaving n_μ invariant. We do this in the spirit of Toller's approach[3] but using essentially the formalism of Delbourgo, Salam, and Strathdee.[8]

We begin by defining a type of M function from the s-channel helicity amplitude

$$\langle p_C S_C \lambda_C;\, p_D S_D \lambda_D | T | p_A S_A \lambda_A;\, p_B S_B \lambda_B\rangle$$

which describes the process $A+B\to C+D$:

$$\begin{aligned} M_{cd,ab}(p_C,p_D;p_A,p_B) &\equiv \langle p_C S_C c;\, p_D S_D d|T|p_A S_A a;\, p_B S_B b\rangle \\ &= \sum_{\lambda_i} D^{S_C 0}_{c\lambda_C}(L_{p_C}) D^{S_D 0}_{d\lambda_D}(L_{p_D}) \langle p_C S_C \lambda_C;\, p_D S_D \lambda_D|T|p_A S_A \lambda_A;\, p_B S_B \lambda_B\rangle D^{S_A 0}_{\lambda_A a}(L_{p_A}^{-1}) D^{S_B 0}_{\lambda_B b}(L_{p_B}^{-1}) . \end{aligned} \tag{2.1}$$

The states $|p_A S_A a\rangle$ and $\langle p_C S_C c|$ transform according to the finite-dimensional representations of the Lorentz group as follows:

$$|p_A S_A a\rangle = \sum_{\lambda_A} |p S_A \lambda_A\rangle D^{S_A 0}_{\lambda_A a}(L_p^{-1}), \tag{2.2}$$

$$\langle p_C S_C c| = \sum_{\lambda_C} D^{S_C 0}_{c \lambda_C} \langle p S_C \lambda_C| . \tag{2.3}$$

Using these equations we see that the M function satisfies the covariance condition

$$\begin{aligned} M_{cd,ab}(p_C,p_D;p_A,p_B) = \sum_{c'd'a'b'} & D^{S_C 0}_{cc'}(\Lambda^{-1}) D^{S_D 0}_{dd'}(\Lambda^{-1}) \\ & \times M_{c'd',a'b'}(\Lambda_{p_C},\Lambda_{p_D};\Lambda_{p_A},\Lambda_{p_B}) \\ & \times D^{S_A 0}_{a'a}(\Lambda) D^{S_B 0}_{b'b}(\Lambda) . \end{aligned} \tag{2.4}$$

We rewrite $M_{cd,ab}$ as a function of the three four-vectors n, p, and q defined in Eqs. (1.9) and (1.19):

$$M_{cd,ab}(n;p,q) = M_{cd,ab}(p_C,p_D;p_A,p_B) . \tag{2.5}$$

Just as in conventional Regge theory we will find separate asymptotic formulas for the forward and backward regions. In the following analysis we shall assume that we are in the forward hemisphere. The results for the backward hemisphere then follow by analogy.

We next couple spins S_A, S_C to give J and S_B, S_D to give[9] J' and define formally

$$\mathfrak{M}_{J'm',Jm}(n;p,q) \equiv \langle pJ'm'|T(n)|qJm\rangle \tag{2.6}$$

$$\begin{aligned} = \sum_{cdab} & C(S_B b;\, J'm',\, S_D d) \\ & \times C(S_C c;\, Jm,\, S_A a) \\ & \times M_{cd,ab}(n;p,q) . \end{aligned} \tag{2.7}$$

This new matrix element, $\mathfrak{M}_{J'm',Jm}(n;p,q)$, satisfies the following covariance condition:

$$\begin{aligned} \mathfrak{M}_{J'm',Jm}(n;p,q) = \sum_{kk'} & D^{J'0}_{m'k'}(\Lambda^{-1}) \\ & \times \mathfrak{M}_{J'k',Jk}(\Lambda n;\Lambda p,\Lambda q) \\ & \times D^{J0}_{km}(\Lambda) . \end{aligned} \tag{2.8}$$

Recalling Eq. (2.6) it can be seen that we can formally write this covariance requirement as

$$U(\Lambda)|qJm\rangle = \sum_k |\Lambda qJk\rangle D^{J0}_{km}(\Lambda), \tag{2.9}$$

$$\langle pJ'm'|U^{-1}(\Lambda) = \sum_{k'} D^{J0}_{m'k'}(\Lambda^{-1}) \langle \Lambda pJ'k'| , \tag{2.10}$$

$$U(\Lambda)T(n)U^{-1}(\Lambda) = T(\Lambda n) . \tag{2.11}$$

Having described some of the formalism we now go on to discuss our partial-wave expansion for the forward scattering amplitude.

B. Expansions on the Physical Region Boundary

In any reaction, for either forward or backward scattering we have $n=(0,0,0,0)$ so that for any $\Lambda \in$ O(3, 1)

$$T(\Lambda n) = T(n) = T(0). \tag{2.12}$$

Now in the s channel both the vectors p and q are timelike so that we can define the standard vectors $p^{(0)}, q^{(0)}$ such that Λ_p, Λ_q are real transformations:

$$p = \Lambda_p p^{(0)} = \Lambda_p |p| (1, 0, 0, 0), \tag{2.13}$$

$$q = \Lambda_q q^{(0)} = \Lambda_q |q| (1, 0, 0, 0). \tag{2.14}$$

We can now rewrite the covariance condition, Eq. (2.8), with $\Lambda = \Lambda_p^{-1}$ and obtain

$$\mathfrak{M}_{J'm', Jm}(0; p, q) = \sum_{kk'} D^{J'0}_{m'k'}(\Lambda_p) \times \mathfrak{M}_{J'k', Jk}(0; p^{(0)}, \Lambda_p^{-1}\Lambda_q q^{(0)}) \times D^{J0}_{km}(\Lambda_p^{-1}). \tag{2.15}$$

We lastly define the function

$$F_{J'm', Jm}(\Lambda) = \sum_{m''} \mathfrak{M}_{J'm', Jm''}(0; p^{(0)}, \Lambda q^{(0)}) D^{J0}_{m''m}(\Lambda). \tag{2.16}$$

With $\Lambda = \Lambda_p^{-1}\Lambda_q$ we see that

$$\begin{aligned} F_{J'm', Jm}(\Lambda) &= \sum_{m''} \sum_{kk'} D^{J'0}_{m'k'}(\Lambda_p^{-1}) \mathfrak{M}_{J'k', Jk}(0; p, q) \times D^{J0}_{km''}(\Lambda_p) D^{J0}_{m''m}(\Lambda_p^{-1}\Lambda_q) \\ &= \sum_{kk'} D^{J'0}_{m'k'}(\Lambda_p^{-1}) \langle pJ'k' | T | qJk \rangle D^{J0}_{km}(\Lambda_q) \\ &= \langle p^{(0)}J'm' | U^{-1}(\Lambda_p) T U(\Lambda_q) | q^{(0)}Jm \rangle \\ &= \langle p^{(0)}J'm' | T U(\Lambda_p^{-1}\Lambda_q) | q^{(0)}Jm \rangle , \end{aligned}$$

i.e.,

$$F_{J'm', Jm}(\Lambda) = \langle p^{(0)}J'm' | T U(\Lambda) | q^{(0)}Jm \rangle , \tag{2.17}$$

where we have used the fact that with $n = 0$, $T(n) = T(\Lambda n)$. We have at last succeeded in defining a function F which depends only on the little-group transformation of the vector n_μ, and this is the function we shall expand. However, before discussing the actual expansion we must first consider not just the transformations $\Lambda = \Lambda_p^{-1}\Lambda_q$ but also $\Lambda = (\Lambda_p h_p)^{-1}(\Lambda_q h_q)$, where $\Lambda_p h_p, \Lambda_q h_q$ belong to the so-called left and right covariance groups. These are the intersections of the groups of transformations which leave both n and $p^{(0)}$ and both n and $q^{(0)}$, respectively, invariant. As $p^{(0)}, q^{(0)}$ are timelike, the transformations h_p and h_q can be any rotations:

$$\begin{aligned} h_p p^{(0)} &= R p^{(0)} = p^{(0)}; \\ h_q q^{(0)} &= R q^{(0)} = q^{(0)} . \end{aligned} \tag{2.18}$$

These invariances give rise to the right and left covariance conditions, as follows:

$$F_{J'm', Jm}(\Lambda R) = \langle p^{(0)}J'm' | T U(\Lambda) U(R) | q^{(0)}Jm \rangle = \sum_{m''} F_{J'm', Jm''}(\Lambda) D^{J}_{m''m}(R) \tag{2.19}$$

and similarly

$$F_{J'm', Jm}(R\Lambda) = \sum_{m''} D^{J'}_{m'm''}(R) F_{J'm'', Jm}(\Lambda). \tag{2.20}$$

We are now ready to consider the expansion of the function $F_{J'm', Jm}(\Lambda)$. Any function which is square-integrable over the noncompact group manifold may be expanded in terms of the representations of the principal series. We parametrize $\Lambda \in O(3,1)$ in the following way[3]:

$$\Lambda(\mu_1, \theta_1, \nu_1, \zeta, \theta_2, \nu_2) = R_1(\mu_1, \theta_1, \nu_1) \times a_z(\zeta) R_2(0, \theta_2, \nu_2), \tag{2.21}$$

where

$$\begin{aligned} &0 \le \mu_1 < 4\pi, \quad 0 \le \theta_1 \le \pi, \quad 0 \le \nu_1 \le 2\pi, \\ &0 \le \zeta < \infty, \quad 0 \le \theta_2 \le \pi, \quad 0 \le \nu_2 < 2\pi . \end{aligned} \tag{2.22}$$

R_1, R_2 are rotations and a_z a z-direction boost. We can then expand a square-integrable function $F(\Lambda)$, in the following way[3]:

$$F(\Lambda) = \sum_M \int_{-i\infty}^{i\infty} d\lambda (M^2 - \lambda^2) \times \sum_{j\mu; j'\mu'} F^{M\lambda}_{j\mu, j'\mu'} D^{M\lambda}_{j\mu, j'\mu'}(\Lambda), \tag{2.23}$$

where the parameters M, λ are such that

$$\begin{aligned} &\mathrm{Re}\lambda = 0, \quad -\infty < \mathrm{Im}\lambda < \infty, \\ &|M| = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots, \\ &j, j' = |M|, |M| + 1, \ldots . \end{aligned} \tag{2.24}$$

Then

$$F^{M\lambda}_{j'\mu', j\mu} = \int d\Lambda \, D^{M\lambda}_{j'\mu', j\mu}(\Lambda)^* F(\Lambda), \tag{2.25}$$

where the invariant measure on the group $d\Lambda$ is given by

$$d\Lambda = d\mu_1 d\nu_1 d\nu_2 \sin\theta_1 d\theta_1 \sin\theta_2 d\theta_2 \sinh^2\zeta \, d\zeta \tag{2.26}$$

and the integration is over the ranges given in Eq. (2.22). We refer to the work of Toller[3] for the properties of the representations of the homogeneous Lorentz group, $D^{M\lambda}_{j\mu, j'\mu'}(\Lambda)$.

It should be stressed that in order to carry out the integration in Eq. (2.25), one has to know $F(\Lambda)$ for all $\Lambda \in O(3,1)$. However, the definition of $F(\Lambda)$ in Eq. (2.17) is only physical for a restricted range of Λ. We thus require a continuation of Eq.

(2.17) off the mass shell at fixed n^2. We *assume* that for small excursions off the mass shell at fixed n^2, it is permissible to ignore the mass dependence. The same remark will also apply to the O(2, 1) analysis in Sec. II D.

Before applying the expansion formula, Eq. (2.23), to $F_{J'm',Jm}(\Lambda)$, which we shall assume to be square-integrable over the O(3, 1) group manifold,[10] we use the covariance conditions to eliminate some of the indices. We note from Eq. (2.21) that any $\Lambda \in$ O(3, 1) can be written as

$$\Lambda = R_1 a_z(\zeta) R_2 ,$$

where R_1, R_2 are rotations and a_z is a z-direction boost. Applying Eq. (2.25) to the function $F(\Lambda) \equiv F_{J'm',Jm}(\Lambda)$ and using the right and left covariance conditions, Eqs. (2.19), (2.20), together with the properties of the representation functions[3] $D^{M\lambda}$ we obtain

$$\int dR_1 da_z dR_2 D^{M\lambda *}_{j'\mu',j\mu}(R_1 a_z R_2) F_{J'm',Jm}(R_1 a_z R_2) = \delta_{j'J'}\delta_{\mu'm'}\delta_{jJ}\delta_{\mu m} F^{M\lambda}_{J'J} \tag{2.27}$$

which defines the generalized partial-wave amplitudes $F^{M\lambda}_{J'J}$. Now using the inverse of this last equation, i.e., Eq. (2.23), we obtain

$$F_{J'm',Jm}(\Lambda) = \sum_M \int_{-i\infty}^{+i\infty} d\lambda (M^2-\lambda^2) \times F^{M\lambda}_{J'J}(n^2=0) D^{M\lambda}_{J'm',Jm}(\Lambda). \tag{2.28}$$

This integral corresponds to the generalization of the background integral in the usual complex angular momentum plane.

In order to pick up the contributions of poles in the complex λ plane, we must move the contour in Eq. (2.28) to the left, which requires replacing the representation functions D by representation functions of the second kind, A. These are analogous to the $Q_j(z)$'s and like them have more suitable asymptotic behavior than the functions of the first kind. The $A^{M\lambda}_{j\mu,j'\mu'}(\Lambda)$ are defined by

$$D^{M\lambda}_{j\mu,j'\mu'}(\Lambda) = A^{M\lambda}_{j\mu,j'\mu'}(\Lambda) + U^{M\lambda}_j A^{-M-\lambda}_{j\mu,j'\mu'}(U^{M\lambda}_{j'})^{-1} , \tag{2.29}$$

where

$$U^{M\lambda}_j = U^{-M\lambda}_j = \sum_{s=|M|}^{j} \frac{s-\lambda}{s+\lambda} \tag{2.30}$$

and the D's have the property that

$$D^{-M-\lambda}_{j\mu,j'\mu'}(\Lambda) = (U^{M\lambda}_j)^{-1} D^{M\lambda}_{j\mu,j'\mu'} U^{M\lambda}_{j'} . \tag{2.31}$$

With $\Lambda = a_z(\zeta)$ the functions of the second kind have large ζ behavior given by

$$A^{M\lambda}_{j\mu,j'\mu'}(a_z) \underset{\zeta\to\infty}{\sim} \delta_{\mu\mu'} \exp[\zeta(-\lambda - 1 - M + \mu)] \times [1 + O(e^{-2\zeta})] , \tag{2.32}$$

with $M \geq \mu$. Now from the properties of the D's, Eq. (2.31), and the definition of $F^{M\lambda}_{J'J}$ of Eq. (2.27) we have that

$$F^{M\lambda}_{J'J} = (U^{M\lambda}_{J'})^{-1} F^{-M-\lambda}_{J'J} U^{M\lambda}_J . \tag{2.33}$$

This property can be used in Eq. (2.28) to restrict the sum over M to positive values; however, instead we shall substitute Eq. (2.29) into Eq. (2.28) and using the property Eq. (2.33) obtain

$$F_{J'm',Jm}(\Lambda) = 2\sum_{M=-\infty}^{\infty} \int_{-i\infty}^{+i\infty} d\lambda (M^2-\lambda^2) \times F^{M-\lambda}_{J'J} A^{M-\lambda}_{J'm',Jm}(\Lambda) . \tag{2.34}$$

The purpose of replacing the D functions by the A's is exactly the same as that of replacing P_j's by the Q_j's in the spinless Regge expansion; i.e., they have suitable behavior at infinity in the complex λ plane enabling us to neglect the contributions from infinity when we move the contour into the left half-plane. This we do now, assuming the $F^{M-\lambda}_{J',J}$ have poles in the left half in the complex λ plane at $\lambda = a_i(0)$, whose positions are independent of J, J'. We define the residues of these poles – called Lorentz poles – in the following way:

$$\lim_{\lambda\to a_+(0)} \left(\frac{1}{\pi i} [\lambda - a_+(0)] F^{M-\lambda}_{J'J}(0) \right) = g^{+Ma_+}_{J'J}(0), \tag{2.35}$$

where we have introduced a plus sign to remind us that the poles and residues are those which dominate scattering in the forward hemisphere. Then assuming $F^{M\lambda}$ is meromorphic in a strip $|\mathrm{Re}\lambda| < \lambda'$ we move the contour in Eq. (2.34) to some λ_0, $|\mathrm{Re}\lambda| = \lambda_0 < \lambda'$, to give

$$F_{J'm',Jm}(\Lambda) = \sum_M \int_{-\lambda_0 - i\infty}^{-\lambda_0 + i\infty} d\lambda (M^2-\lambda^2) F^{M-\lambda}_{J'J}(0) A^{M-\lambda}_{J'm,Jm}(\Lambda) + \sum_{i,M} (M^2 - a_{+i}{}^2) g^{+Ma}_{i\,J'J}(0) A^{M-a_i}_{J'm',Jm}(\Lambda) + \text{fixed poles}. \tag{2.36}$$

The fixed poles occur because the functions $A^{M\lambda}(\Lambda)$ can have poles at integral values of $\lambda - M$. However, it can be shown[3] that their contributions either actually cancel out or at least are asymptotically negligible.

Because of Eq. (2.33) we see that if $F^{M-\lambda}$ has a pole at $\lambda = a(0)$, then there is also a pole in the right half of the complex λ plane at $\lambda = -a(0)$; this is called a mirror pole. These mirror poles, as we shall see [Eq. (2.43)], have the same asymptotic behavior as their corresponding Lorentz poles.

Before we can consider the asymptotic behavior of the amplitude $F_{J'm',Jm}(\Lambda_p{}^{-1}\Lambda_q)$, Eq. (2.17), and hence of the s-channel helicity amplitudes, we must define some kinematics. We choose to work in the s-channel center-of-mass frame. With $n_\mu = 0$ we can take

$$p_I^\mu = (E_I, 0, 0, p_I) \quad (I = A, B, C, D), \qquad (2.37)$$

where the z components of the momenta satisfy $p_B = -p_A$ and $p_D = -p_C$ and of course $E_A + E_B = E_C + E_D$. We then have, Eq. (1.9),

$$\begin{aligned} p^\mu &= (E_B + E_D, 0, 0, -p_A - p_C), \\ q^\mu &= (E_A + E_C, 0, 0, p_A + p_C) \end{aligned} \qquad (2.38)$$

which we can rewrite as

$$\begin{aligned} p^\mu &= |p|(\cosh(\zeta - \zeta_0), 0, 0, -\sinh(\zeta - \zeta_0)), \\ q^\mu &= |q|(\cosh\zeta_0, 0, 0, \sinh\zeta_0), \end{aligned} \qquad (2.39)$$

where

$$|p|^2 = 2(m_B{}^2 + m_D{}^2) - t,$$

$$|q|^2 = 2(m_A{}^2 + m_C{}^2) - t,$$

and

$$|p| \sinh(\zeta - \zeta_0) = |q| \sinh\zeta_0 .$$

Since

$$p \cdot q = s - u = |p|\,|q| \cosh\zeta ,$$

we have therefore

$$\cosh\zeta = \frac{s - u}{[2(m_A{}^2 + m_C{}^2) - t]^{1/2}[2(m_B{}^2 + m_D{}^2) - t]^{1/2}} \qquad (2.40)$$

which is only large for $s \to \infty$ and n^2 fixed if we are considering scattering in the forward hemisphere. We then have

$$\begin{aligned} U(\Lambda_p) &= e^{i(\zeta - \zeta_0)J_{03}}, \\ U(\Lambda_q) &= e^{-i\zeta_0 J_{03}} \end{aligned} \qquad (2.41)$$

and so

$$U(\Lambda_p{}^{-1}\Lambda_q) = e^{-i\zeta J_{03}}. \qquad (2.42)$$

Putting $\Lambda = \Lambda_p{}^{-1}\Lambda_q = a_z(\zeta)$ in the expansion formula, Eq. (2.36), and letting $\zeta \to \infty$ we obtain the high-energy behavior of the forward scattering amplitude. Then the rightmost Lorentz pole at $\lambda = a_+(0)$ will dominate, giving

$$F_{J'm',Jm}(n=0, \zeta) \sim \delta_{m'm}(M^2 - a_+{}^2) g_{J'J}^{+Ma_+}(0) \times e^{\zeta(a_+ - 1 - |M - |m||)} . \qquad (2.43)$$

Recalling Eqs. (2.1) and (2.7) and noting that

$$U(L_p) = e^{-i\phi J_{12}} e^{-i\theta J_{31}} e^{i\phi J_{12}} e^{-i\zeta J_{03}},$$

$$D_{\mu\nu}^{S0}(L_p) = e^{-im\phi} d_{\mu\nu}^{S}(\theta) e^{i\nu(\phi + i\zeta)} ,$$

we have that as $s \to \infty$

$$\langle p_C S_C \lambda_C ; p_D S_D \lambda_D | T | p_A S_A \lambda_A ; p_B S_B \lambda_B \rangle|_{n^2=0,\theta=0} \sim \delta_{\lambda_A - \lambda_C, \lambda_B - \lambda_D} s^{a_+(0) - 1 - |M - |\lambda_A - \lambda_C||} . \qquad (2.44)$$

The Kronecker δ ensures angular momentum conservation in the forward direction and we see that the dominant Lorentz pole has $M = \pm(\lambda_A - \lambda_C)$.

It should be stressed that this behavior of the forward scattering amplitude, Eq. (2.44), holds regardless of the external mass configuration. To leading order in s, Eq. (2.44) is in complete agreement with both analyticity and other group-theoretic results. Of course, Eq. (2.44) is absolutely identical to the result of Toller[3,11] in the EE mass configuration [we denote a process of the type (equal-mass pair) → (equal-mass pair) in the t channel as an EE process] at $t = 0$ to all orders in s, since in this case $t = 0$ corresponds to $n^2 = 0$. However, in other mass configurations our result has been derived without the need for assuming either analyticity in the external masses[12] or that expansions should be made with respect to the classification group of Regge trajectories rather than the appropriate little group.[13] Indeed previous impositions of O(3, 1) symmetry at $t = 0$ may be regarded as a first approximation to the exact Lorentz symmetry when $n^2 = 0$, since, asymptotically in s, $t = 0$ means forward scattering and so coincides with $n^2 = 0$ there.

Up to now we have considered $p = p_B + p_D$, $q = p_A + p_C$ and accordingly coupled spins S_B, S_D and S_A, S_C. This corresponds to the usual t-channel analysis. As we have seen, $(p \cdot q / |p|\,|q|) \to \infty$, as $s \to \infty$, at fixed n^2, only in the forward hemisphere. We could equally well have chosen $p = p_B + p_C$, $q = p_A + p_D$ and have coupled spins S_A, S_D to give J and S_B, S_C to give J'. Then

$$\cosh\zeta = \frac{p\cdot q}{|p|\,|q|} = \frac{s-t}{[2(m_B{}^2+m_C{}^2)-u]^{1/2}[2(m_A{}^2+m_D{}^2)-u]^{1/2}} \tag{2.45}$$

and $s\to\infty$ gives $\zeta\to\infty$ at fixed n^2 for scattering in the backward hemisphere. Thus in analogy with the forward case, we now obtain at $n^2=0$ for backward scattering at asymptotic energies:

$$\langle\lambda_C,\lambda_D|T|\lambda_A,\lambda_B\rangle|_{n^2=0,\theta=\pi} \sim \delta_{\lambda_A-\lambda_C,\lambda_D-\lambda_B} s^{a_-(0)-1-|M-|\lambda_A-\lambda_C||} . \tag{2.46}$$

Here $\lambda=a_-(0)$ refers to the Lorentz pole which dominates scattering in the backward hemisphere. In a similar way we could choose $p=p_C-p_D$, $q=p_A-p_B$, but with n^2 fixed $p\cdot q/|p|\,|q|$ is never asymptotic in the s-channel physical region.

C. Relationship to the Theory of Cosenza, Sciarrino, and Toller

We have already mentioned in Sec. I that the O(3, 1) analysis of Toller only applies at $t=0$ to elastic reactions of the type $A+B\to A+B$. In an attempt to extend their analysis to processes of the type $A+B\to A+C$, referred to as UE reactions [(unequal-mass pair)→ (equal-mass pair) in the t channel],[5] and processes like $A+B\to C+D$ (UU type), Cosenza, Sciarrino, and Toller[11] studied expansion based on the complex Lorentz group and predicted the asymptotic behavior

$$\langle\lambda_C,\lambda_D|T|\lambda_A,\lambda_B\rangle|_{\theta=0} \sim \delta_{\lambda_A-\lambda_C,\lambda_B-\lambda_D} s^{\lambda(0)-1-|M-|\lambda_A-\lambda_C||-k} , \tag{2.47}$$

where k is a non-negative integer which can only be nonzero in UE-type reactions. The value of k is specified in terms of the internal quantum numbers (I, G, B, Y) of the exchanged pole. On the contrary, in our n^2 analysis, the result with $k\equiv 0$ holds for *all* reactions.

Another difference between this n^2 approach and the complex Lorentz group result is in the allowed range of M in $F_{J'm',Jm}$ [see (2.24)]. In our analysis we always have

$$M \le \min\{S_A+S_C,\ S_B+S_D\} \tag{2.48}$$

whereas in Ref. 11 M may take any non-negative value in UE- and UU-type reactions. This difference arises because we exploit the exact O(3, 1) symmetry at $\theta=0$ in *all* reactions and not just in EE scattering.

D. Expansions Inside the Physical Region

We now consider the case of scattering in the physical region when $n^2<0$ and derive high-energy expansions at fixed negative n^2. We shall see that the dominance of a single term in our expansion gives results very different from the usual Regge case. As before we have different asymptotic expansions in the forward and backward hemispheres. The analysis presented refers to scattering in the forward hemisphere and the backward case then follows trivially.

We return to Eq. (2.17) and define

$$F_{J'm',Jm}(n,\Lambda) = \langle p^{(0)}J'm'|T(\Lambda_p{}^{-1}n)U(\Lambda)|q^{(0)}Jm\rangle , \tag{2.49}$$

where $\Lambda=\Lambda_p{}^{-1}\Lambda_q$. The four-vector, n_μ, is always spacelike inside the physical scattering region, so we can choose $n_\mu=(0,0,0,(-n^2)^{1/2})$. Clearly if $\Lambda_p,\Lambda_q\in O(2,1)$ then $\Lambda n=n$. Now the left and right covariance groups are the intersections of the groups of transformations which leave both $n, p^{(0)}$ and $n, q^{(0)}$, respectively, invariant. Both covariance groups are $O(2,1)\cap O(3)$, i.e., O(2), the group of rotations about the z axis. The covariance conditions are

$$F_{J'm',Jm}(n;\Lambda R_z) = \sum_k F_{J'm',Jk}(n;\Lambda) D^J_{km}(R_z), \tag{2.50}$$

$$F_{J'm',Jm}(n;R_z\Lambda) = \sum_{k'} F_{J'k',Jm}(n;\Lambda) D^{J'}_{m'k'}(R_z), \tag{2.51}$$

where $D^J_{mm'}(R_z(\phi)) = \delta_{mm'}e^{-im\phi}$.

If $F_{J'm',Jm}(n,\Lambda)$ is square-integrable over the O(2, 1) group manifold, we can perform an O(2, 1) expansion as follows.[10] We note that any $\Lambda\in O(2,1)$ can be written as

$$\Lambda = R_z(\phi_1)a_x(\zeta)R_z(\phi_2). \tag{2.52}$$

We use this fact together with the covariance conditions, Eqs. (2.50) and (2.51), to simplify the indices and obtain[14]

$$F_{J'm',Jm}(n,\Lambda) = \sum_{\epsilon=0,1/2}\int_{-1/2-i\infty}^{-1/2+i\infty} dl \frac{2l+1}{\tan\pi(l-\epsilon)} f^{l,\epsilon}_{J'm',Jm}(n^2) D^{l,\epsilon}_{m'm}(\Lambda) + \sum_{k=1-\epsilon}^{\infty}(2k-1) f^{k,\epsilon}_{J'm',Jm}(n^2) D^{k,\epsilon}_{m'm}(\Lambda) . \tag{2.53}$$

The first term on the right-hand side of this equation involves the principal series representations for which $m, m' = \epsilon, \epsilon \pm 1, \epsilon \pm 2, \ldots$, and the second term involves the discrete series for which $m, m' = \pm k, \pm(k+1), \ldots$. With $n_\mu = (0, 0, 0, (-n^2)^{1/2})$ we can choose

$$\Lambda_p = a_x(\zeta_0 - \zeta), \quad \Lambda_q = a_x(\zeta_0)$$

so that

$$\Lambda_p^{-1}\Lambda_q = a_x(\zeta), \tag{2.54}$$

where $|p| \sinh(\zeta - \zeta_0) = |q| \sinh\zeta_0$ gives us the s-channel center-of-mass frame. $\cosh\zeta$ is given by Eq. (2.40) just as before. This is because with n_μ in the z direction $p_3 = q_3 = 0$ and so the Lorentz scalar $p\cdot q = p_0 q_0 - p_1 q_1 - p_2 q_2$—the O(2, 1) product.

In order to move the integral contour of Eq. (2.53) to the left and to pick up contributions to $F_{J'm', Jm}(n^2, \Lambda)$ from poles in the complex l plane we replace the continuous class of representations by representation functions of the second kind, A, which are defined by

$$D^{l,\epsilon}_{m'm}(\Lambda) = A^l_{m'm}(\Lambda) + U^l_{m'} A^{-l-1}_{m'm}(\Lambda)(U^l_m)^{-1}, \tag{2.55}$$

where

$$U^l_m = \Gamma(l + m + 1)/\Gamma(m - l)$$

and (2.56)

$$D^{-l-1,\epsilon}_{m'm} = (U^l_{m'})^{-1} D^{l,\epsilon}_{m'm} U^l_m .$$

From this last property of the D's we have that

$$f^{l,\epsilon}_{J'm', Jm} = (U^l_{m'})^{-1} f^{-l-1,\epsilon}_{J'm', Jm} U^l_m . \tag{2.57}$$

We use this relation together with Eq. (2.55) to write the integral of Eq. (2.53) as

$$2\sum_\epsilon \int_{-1/2-i\infty}^{-1/2+i\infty} dl \frac{2l+1}{\tan\pi(l-\epsilon)} f^{-l-1,\epsilon}_{J'm', Jm}(n^2) A^{-l-1}_{m'm}(\Lambda) . \tag{2.58}$$

If we assume that the function f^l is meromorphic for $|\mathrm{Re}(l+\frac{1}{2})| \leq L$ we can move the contour from $\mathrm{Re}\,l = \frac{1}{2}$ to $\mathrm{Re}\,l = -L_0$ $(L \geq L_0)$. The functions A are such that the contributions to the integral at infinity are negligible. We assume that the partial wave f^{-l-1} have a pole at $l = a(n^2)$ with residue defined by

$$\lim_{l \to a(n^2)} \left(\frac{1}{\pi i}[l - a(n^2)] f^{-l-1,\epsilon}_{J'm', Jm}(n^2)\right) = b^\epsilon_{J'm', Jm}(n^2) . \tag{2.59}$$

We can then rewrite the expression, Eq. (2.58), as

$$2\sum_\epsilon \int_{-L_0-i\infty}^{-L_0+i\infty} dl \frac{2l+1}{\tan\pi(l-\epsilon)} f^{-l-1,\epsilon}_{J'm', Jm}(n^2) A^{-l-1}_{m'm}(\Lambda) + \sum_{\epsilon, i} \frac{2a_i+1}{\tan\pi(a_i-\epsilon)} b^\epsilon_{iJ'm', Jm}(n^2) A^{-a_i-1}_{m'm}(\Lambda) . \tag{2.60}$$

From Eq. (2.57) we see that a pole in f^{-l-1} at $l = a(n^2)$ implies the existence of a mirror pole at $l = -a(n^2) - 1$, both having the same asymptotic behavior.

Recalling that $\Lambda = a_x(\zeta)$, Eq. (2.54), we shall consider the behavior of $F_{J'm', Jm}(s, n^2)$ as $s \to \infty$. We note that

$$A^l_{m'm}(a_x(\zeta)) \underset{\zeta\to\infty}{\sim} e^{-(l+1)\zeta}\left[1 - \frac{2m'm}{l+1} e^{-\zeta} + O(e^{-2\zeta})\right] \tag{2.61}$$

and that the contribution of the discrete class decreases faster than $e^{-\zeta}$ and so is neglected. If we are considering scattering in the forward hemisphere $\cosh\zeta$ is given by Eq. (2.40) and in the backward hemisphere by Eq. (2.45). The leading high-energy behavior is given by the rightmost pole in the complex l plane [15]

$$F_{J'm', Jm}(s, n^2) \underset{s\to\infty}{\sim} \frac{2a_\pm(n^2)+1}{\tan\pi[a_\pm(n^2) - \epsilon]} \times b^{\pm\epsilon}_{J'm', Jm} s^{a_\pm(n^2)} , \tag{2.62}$$

where we have again introduced the $\pm$ signs to refer to the poles which dominate forward or backward hemisphere scattering, respectively.

We note that when we take the limit $n^2 \to 0$ of a single pole contribution

$$F_{J'm', Jm}(s, n^2 = 0) \sim \frac{2a+1}{\tan\pi(a-\epsilon)} \times b_{J'm', Jm}(0) A^{-a-1}_{m'm}(\tilde{\Lambda}) , \tag{2.63}$$

where $\tilde{\Lambda} \in \mathrm{O}(2, 1)$, we find that the covariance conditions for $\Lambda \in \mathrm{O}(3, 1)$ which apply when $n = 0$ cannot be satisfied by such l-plane contributions individually. However, of course, a single Lorentz pole does satisfy such covariance conditions and as shown by Sciarrino and Toller [16] such a pole at $\lambda = \lambda(0)$ corresponds to an infinite family of l-plane poles at $l_\nu(0) = \lambda(0) - \nu - 1$, $\nu = 0, 1, 2, \ldots$. We see therefore that as $n^2 \to 0$ an infinity of single l-plane pole contributions, Eq. (2.63), must conspire to produce a single Lorentz pole term so that the scattering amplitude may satisfy the correct covariance conditions, Eqs. (2.19) and (2.20), at $n = 0$.

E. Connection Between the Complex l_{n^2} and l_t Planes

In the previous sections we have performed a Regge-type analysis which was based on assuming the existence of poles in an angular-momentum-like plane, l_{n^2}, conjugate to the vector n_μ. It is not at all clear whether it is in fact meaningful to postulate poles in this plane. However, any singularity in the complex l_{n^2} plane gives rise to a sequence of singularities in the more familiar complex angular momentum plane l_t, and it is therefore of interest to ask what singularity structure is induced in the l_t plane by a simple pole in the l_{n^2} plane.

To see this, we consider the usual Froissart-Gribov integral defining $f_l(t)$ at fixed t:

$$f_l(t)=\int_{z_0}^{\infty} dz_t Q_l(z_t)F(t,z_t), \tag{2.64}$$

where l here is short for l_t. We now feed in for F the asymptotic form induced by a pole in the l_{n^2} plane at $l_{n^2}=a(n^2)$, i.e.,

$$F\sim b(n^2)\left(\frac{s}{s_0}\right)^{a(n^2)} . \tag{2.65}$$

If the residue $b(n^2)$ is a polynomial in n^2 we can rewrite it in the following form:

$$b(n^2)=\sum_{m=0}^{\infty} b_m(t)\left(\frac{s_0}{s}\right)^m , \tag{2.66}$$

and we write $Q_l(z_t)$ as

$$Q_l(z_t)=\sum_{p=0}^{\infty} q_p(t)\left(\frac{s}{s_0}\right)^{-l-2p-1} . \tag{2.67}$$

We shall further assume that the "trajectory function," $a(n^2)$, is linear in n^2 so that

$$a(n^2)=a(0)+ta'(0)+\frac{A(t)}{s}a'(0)+\frac{B(t)}{s^2}a'(0), \tag{2.68}$$

where

$$A(t)=t^2-t\sum_i m_i^2+(m_A^2-m_C^2)(m_B^2-m_D^2),$$

$$B(t)=t(m_A^2-m_B^2)(m_C^2-m_D^2)+(m_A^2m_D^2-m_B^2m_C^2)(m_A^2+m_D^2-m_B^2-m_C^2).$$

Then using the shorthand

$$\alpha(t)=a(0)+ta'(0) \tag{2.69}$$

and

$$\beta_{ikmp}(t)=\frac{b_m(t)q_p(t)}{\Gamma(i+1)\Gamma(k+1)}\left(\frac{Aa'}{s_0}\right)^i\left(\frac{Ba'}{s_0^2}\right)^k , \tag{2.70}$$

we have on putting Eqs. (2.65)–(2.70) into Eq. (2.64) and integrating at fixed t that

$$f_l(t)=\sum_{ikmp}\frac{\beta_{ikmp}(t)\Gamma(i+k+1)}{[l-\alpha(t)+i+2k+m+2p]^{i+k+1}} . \tag{2.71}$$

We see from this last equation that a $b(n^2)s^{a(n^2)}$ asymptotics corresponds to a very complicated family of Regge poles and multipoles. However, the leading pole at $l=\alpha(t)$ is a simple Regge pole.

In a similar way a single Lorentz pole in the λ_{n^2} plane gives a series of Toller multipoles at $t=0$, except in the case of EE scattering, where there is a one-to-one correspondence between our Lorentz poles and those of Toller.[3]

It should be noted from Eq. (2.71) that if the leading Regge pole has a factorizable residue, then $b(n^2)$ will not in general be completely factorizable. However, in the expansion of $b(n^2)$ in inverse powers of s, Eq. (2.65), the leading term will factorize.

It has been argued that the use of the variable n^2 simplifies and unifies the group-theoretical structure of the expansions used for the scattering amplitude. However, there is no guarantee that the amplitude is dominated by a simple set of singularities in the complex l_{n^2} plane. It is a dynamical question as to whether the singularity structure will be simpler in the l_t or l_{n^2} planes. Thus it might be that a few l_{n^2} poles suffice, implying the need for an infinity of Regge poles, or vice versa. In our present state of ignorance it is impossible to answer questions of this kind by means of dynamical calculations and therefore the only way to test for simplicity is by means of a phenomenological study of scattering data. It turns out, as will be discussed in the next section, that the data do indicate quite remarkably simple properties in the n^2 description.

III. ANALYSIS OF SCATTERING DATA AS A FUNCTION OF n^2

The main result of the above analysis is embodied in the suggestion that the description of scattering amplitudes as functions of s and n^2, rather than s and t, could lead to simplifications, in the sense that all spurious kinematical effects are absent and that the behavior of the scattering amplitudes is a direct reflection of the underlying dynamics.

Thus the first issue to be settled is to see whether experimental data when plotted against s and n^2 do show any simplicity. It will be seen in what follows that *diffractive processes show a remarkable kind of "scaling" or universality and that their cross sections appear to be independent of s over a very large range of energies.*

The second issue relates to the specific Regge-like model based on the existence of poles in the complex l_{n^2} plane. Here we have the predictions that at fixed n^2, with $n^2 \ll s$,

$$\frac{d\sigma}{dt}\bigg|_{\text{forward hemisphere}} \underset{s\to\infty}{\sim} b_+(n^2)s^{2[\alpha_+(n^2)-1]}, \qquad (3.1)$$

$$\frac{d\sigma}{dt}\bigg|_{\text{backward hemisphere}} \underset{s\to\infty}{\sim} b_-(n^2)s^{2[\alpha_-(n^2)-1]}, \qquad (3.2)$$

where $\alpha_\pm(n^2)$ are "trajectory functions" associated with the quantum numbers of the exchanges which dominate forward and backward scattering respectively. To test these one must perform the same kind of analysis as is usually done to test the Regge model, except that here one works at fixed n^2 rather than fixed t.

It should be stressed that the above two issues are quite separate. There are good theoretical grounds for suggesting the use of the variable n^2. On the other hand, the analytic structure in the l_{n^2} plane is not well understood and the polelike model may be far too simple. Even if it is, it will still be of great interest to look at the structure of scattering amplitudes as functions of s and n^2, as indicated by the data themselves.

It should also be stressed that some care must be taken in analyzing the data as a function of n^2. As mentioned in the Introduction, one should, strictly speaking, plot the symmetric and antisymmetric combinations,

$$\left(\frac{d\sigma}{dt}\right)_S = \frac{1}{2}\left[\frac{d\sigma(\theta)}{dt} + \frac{d\sigma(\pi-\theta)}{dt}\right], \qquad (3.3)$$

$$\left(\frac{d\sigma}{dt}\right)_A = \frac{1}{2\cos\theta}\left[\frac{d\sigma(\theta)}{dt} - \frac{d\sigma(\pi-\theta)}{dt}\right] \qquad (3.4)$$

when looking at data which cover a very large angular range.

For small θ,

$$\frac{d\sigma(\theta)}{dt} \sim 100\frac{d\sigma(\pi-\theta)}{dt},$$

typically, so that near the forward or backward regions the above construction is of little importance. It is vital, however, if the data include the region near $\theta = \frac{1}{2}\pi$. The exceptions to the above are reactions like $pp \to pp$ which are symmetric around $\theta = \frac{1}{2}\pi$, so that $(d\sigma/dt)_A \equiv 0$, and one need only look at $d\sigma/dt$ itself. However, in general, if one is testing for scaling over a large range of angles and energies, it is necessary to use the combinations given in Eqs. (3.3) and (3.4). On the other hand, in testing the Regge-like predictions listed in Eqs. (3.1) and (3.2), it should be borne in mind that the formulas are only expected to be valid for values of θ close to 0° or 180°, respectively. Thus for testing Eqs. (3.1) and (3.2) it is not necessary to form the symmetric and antisymmetric combinations.

We shall now consider several $2 \to 2$ reactions and show that the s, n^2 behavior of the data possess quite dramatic features.

(i) $pp \to pp$. In Fig. 2 is shown the differential cross section for $pp \to pp$ scattering as a *function of* t for various momenta between $p_L = 1.7$ and 21.3 GeV/c.[17] It is seen that the curves show a

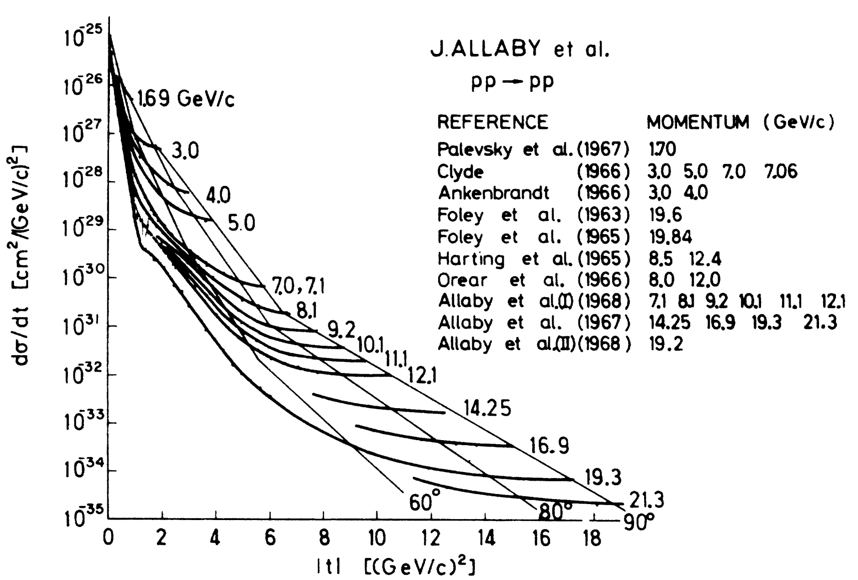

FIG. 2. pp elastic differential cross section for p_L from 1.7 to 21.3 GeV/c for $|t| < 20$ (GeV/c)^2. The graph is from Ref. 17 and references to the data are given there.

very strong energy dependence at fixed t. At small t one has the characteristic shrinkage of the diffraction peak, and at large t, $d\sigma/dt$ may vary by 2 or 3 orders of magnitude over the energy range plotted. The same data (only for $p_L \geq 5$ GeV/c) are shown in Figs. 3 and 4 plotted against n^2. It is seen that the shrinkage has disappeared and aside from the region around $n^2 = -1.2$ (GeV/c)^2 (enlarged in Fig. 4), which corresponds to the shoulder seen at a similar value of t (in Fig. 2), there is very little s dependence at fixed n^2 – in fact the data appear to "scale" at fixed n^2.[18]

That the pp data look universal is not really surprising since for pp elastic scattering n^2 becomes just equal to the Krisch variable $-\beta^2 p_\perp{}^2$.[19]

If we interpret this scaling in terms of the pole model, then from Eqs. (3.1) and (3.2) we see we must have

$$a(n^2) \approx 1 \tag{3.5}$$

for all n^2 in the scattering region. This suggests that the dominant term in high-energy diffractive processes, the analog of the Pomeranchukon, looks like a fixed pole in the l_{n^2} plane. We thus have

$$\frac{d\sigma}{dt} \simeq |b(n^2)|^2 \tag{3.6}$$

even at moderately large energies. Assuming that in the near forward direction the differential cross section is given by a single exponential in n^2, with an energy-independent slope, i.e.,

$$\frac{d\sigma}{dt} = \frac{d\sigma}{dt}\bigg|_{n^2=0} e^{\beta_0 n^2}, \tag{3.7}$$

with β_0 constant, we then have, as pointed out in Ref. 1, a definite formula for the s dependence of the near forward logarithmic slope on a t plot:

$$b(s) \equiv \left[\frac{d}{dt}\ln\left(\frac{d\sigma}{dt}\right)\right]_{t=0}$$

$$= \beta_0 \frac{dn^2}{dt}\bigg|_{t=0} . \tag{3.8}$$

For $pp \to pp$ scattering this gives

$$b(s) = \beta_0\left(1 - \frac{4m^2}{s}\right). \tag{3.9}$$

In this way the energy variation of the slope is completely determined. It was shown in Ref. 1 that Eq. (3.9) provides a reasonable interpolation for the pp slope data all the way from threshold up to ISR energies. It predicts that shrinkage will stop and that $b(s)$ will flatten out to an ultimate value β_0, which is just the slope on an n^2 plot at any energy.

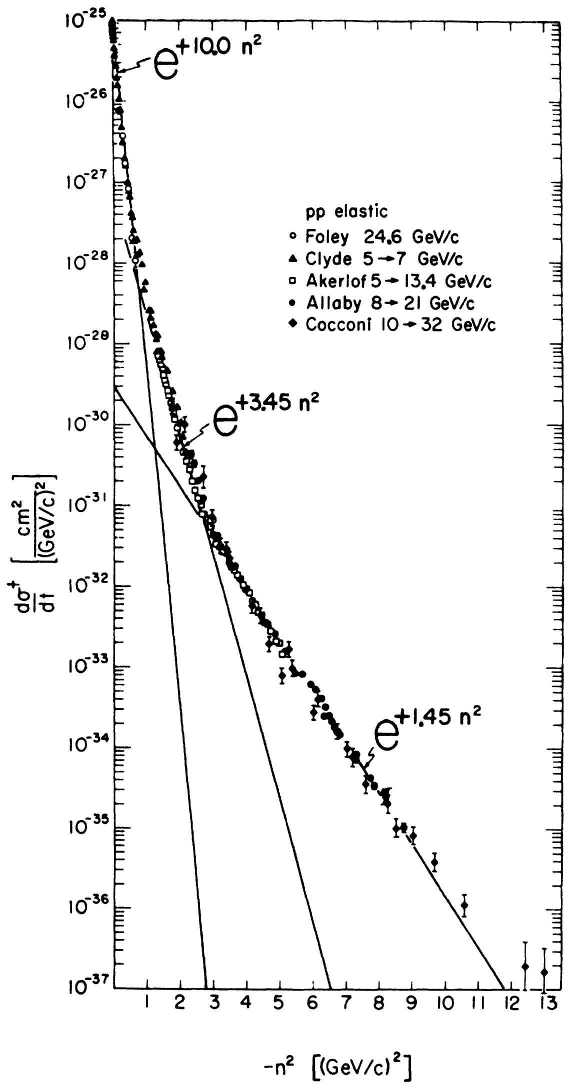

FIG. 3. pp elastic differential cross section plotted against $n^2 = -\beta^2 p_\perp{}^2$ taken from Krisch.[2] The lines indicate the three characteristic slopes in the differential cross-section data. The $d\sigma^+/dt$ is not actually the differential cross section, but rather $d\sigma/dt$ multiplied by a monotonically decreasing factor which equals 1 at $\theta = 0$ and 0.5 at $\theta = \frac{1}{2}\pi$. For an explanation of this factor and for references to the data plotted see Ref. 2.

In summary the pp data look remarkably simple as a function of n^2 and appear to show almost no s dependence at fixed n^2.

(ii) $K^+p \to K^+p$ and $\gamma p \to \phi p$. Both these reactions are similar to pp elastic scattering in that they are very largely diffractive processes with little structure in $d\sigma/dt$ as a function of t other than the usual shrinkage. Both should be dominated at large s by the Pomeranchukon and therefore, if

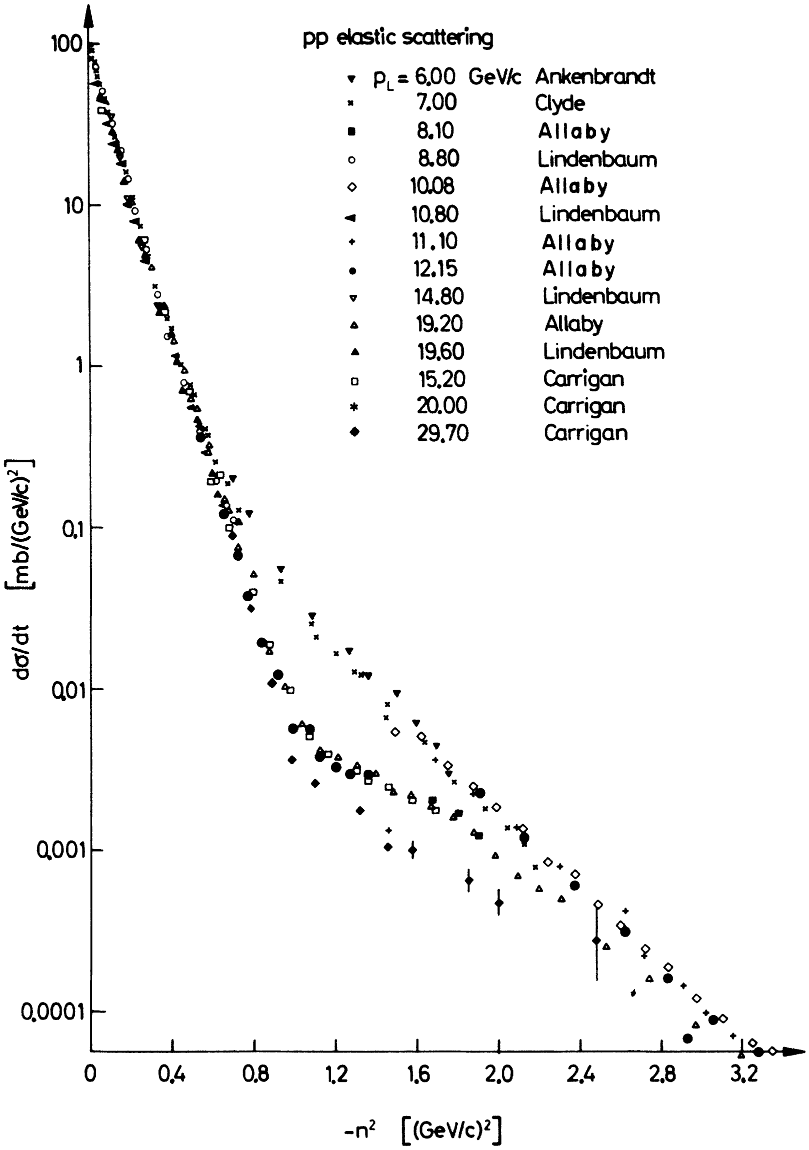

FIG. 4. pp elastic differential cross section for p_L between 6.0 and 29.7 GeV/c for $|n^2| < 3.3\,(\text{GeV}/c)^2$. This is essentially the top left-hand corner of Fig. 3 in more detail. The data are from Ref. 18.

there is any validity in our interpretation of the pp elastic reaction, we should expect both these reactions to show little energy dependence at fixed n^2. Figures 5 and 6 show that this is indeed the case.[20] Both reactions scale at fixed n^2 over a wide range of energies and there is no visible shrinkage of $d\sigma/dt$ as a function of n^2. That this happens is *nontrivial*, since the shrinkages in t for $pp\to pp$, $K^+p\to K^+p$, and $\gamma p\to\phi p$ are all different, and the mechanism for transforming these varied shrinkages in t into nonshrinkage against n^2 is completely contained in the mass dependence of the factor dn^2/dt in Eq. (3.8) [see also Eq. (1.22)].

(iii) $\gamma p\to\rho^0 p$. It is known that the cross section for this reaction is fairly constant above 2 GeV and that the natural-parity exchange dominates. Thus it has the main characteristics of a diffractive process and we might hope to find an n^2 universality similar to the cases studied above. A plot of $d\sigma/dt$ against n^2 for E_L ranging from 6 to 17.8 GeV is shown in Fig. 7. It is again seen that there is essentially no s dependence in the data[20] at fixed n^2.

It was remarked in the Introduction that the normalization used to define n_μ from N_μ is to some extent arbitrary from a group-theoretical point of

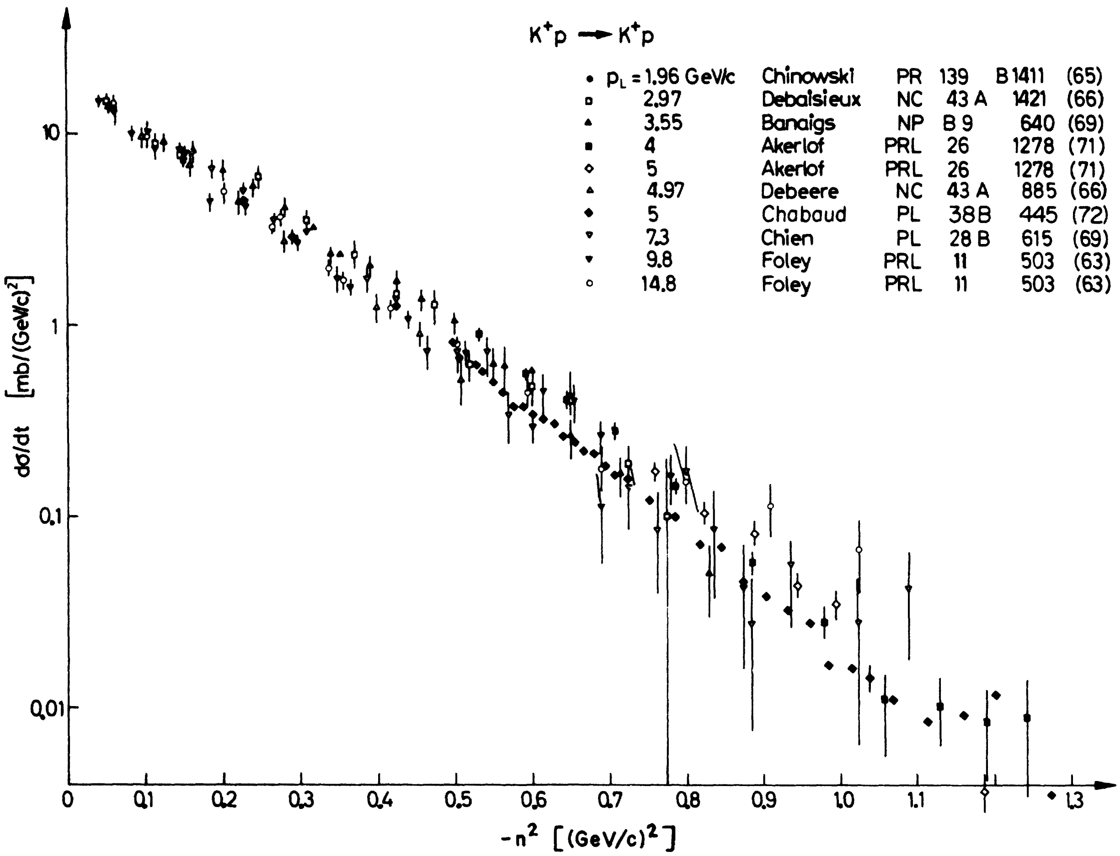

FIG. 5. K^+p elastic differential cross section for $p_L = 1.96$–14.8 GeV/c plotted against n^2. References for the data are shown on the figure.

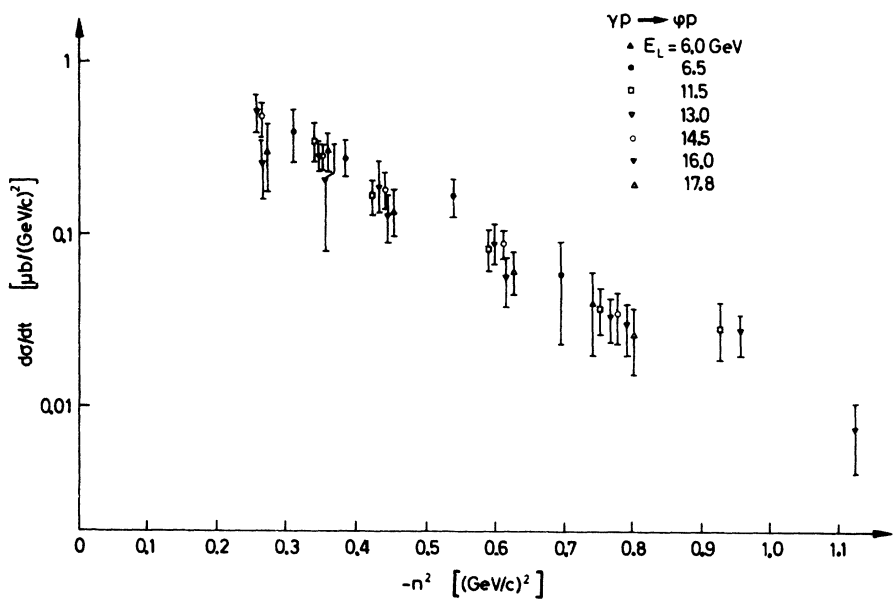

FIG. 6. $\gamma p \to \phi p$ differential cross section plotted against n^2 – data from Ref. 20.

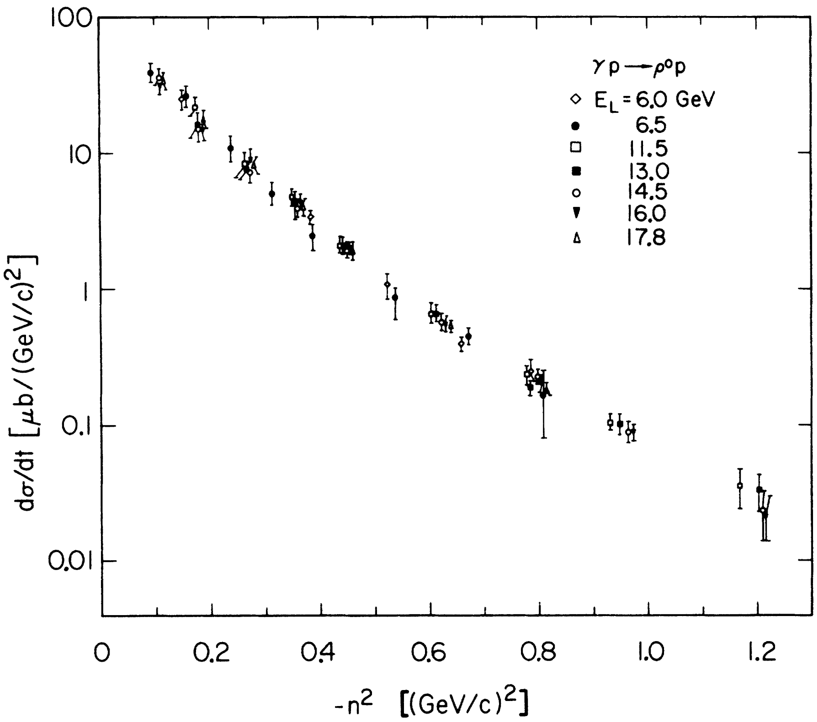

FIG. 7. $\gamma p \to \rho^0 p$ differential cross section plotted against n^2 – data from Ref. 20.

view. Any renormalization of the form $n_\mu = N_\mu / R(s,t,u)$, where

$$R(s,t,u) \underset{s\to\infty;\, t\,\text{fixed}}{\sim} s\,,$$

would provide an acceptable description of scattering at very small t. However, the choice $R(s,t,u) \equiv s$ is essentially unique in providing a description of the data in which the s dependence disappears, i.e., in which the shrinkage on a t plot becomes automatically accounted for as a kinematic effect. For example, one can show that no crossing-symmetric polynomial in s, t, u and the external masses exists which has this property. It is possible that there is some deep underlying dynamical reason for the particular choice n_μ singled out by the data.

(iv) $\pi^\pm p \to \pi^\pm p$. These reactions are not purely diffractive. Their cross sections are varying with energy and there is a considerable amount of structure in the t dependence of $d\sigma/dt$. Nevertheless, as is seen in Figs. 8–11 the large s variation of $d\sigma/dt$ at fixed t is very much reduced when considered at fixed n^2.[21,22] The n^2 plots are not nearly so universal as in the previous reactions, but this is in accordance with our knowledge that $\pi^\pm p \to \pi^\pm p$ are not completely dominated (in Regge language) by the Pomeranchukon, and that large contributions must be attributed to the secondary trajectories. In the n^2 description the secondary effects play a much smaller role, and we are at present trying to study them quantitatively.

(v) N^* *production*. There are not many detailed data on the energy variation of $d\sigma/dt$ for processes like $pp \to N^*p$, but we show one example in Fig. 12 for the $N^*(1690)$ with $I = \frac{1}{2}$, $J^P = \frac{5}{2}^+$.[23] It is seen that within the limited statistics the data points are compatible with little or no energy variation at fixed n^2.

(vi) $\bar{p}p \to \bar{p}p$. Since the dominant component of the pp elastic differential cross section is just a function of $-\phi/s^2 = -tu/s$, it follows from the crossing properties that the diffractive part of the $\bar{p}p$ elastic differential cross section must be the same function of $-\phi/u^2 = -st/u$. In particular this implies that the slope parameter for this diffractive component of $\bar{p}p$ scattering satisfies

$$b_{\bar{p}p}(s) = \beta_0\left(1 + \frac{4m^2}{s}\right)\,, \qquad (3.10)$$

where β_0 is the same constant that appears in Eq. (3.9). Thus the $\bar{p}p$ diffraction peak is predicted to *expand* until it ultimately has the same s-independent limiting slope β_0 as in pp scattering. Of

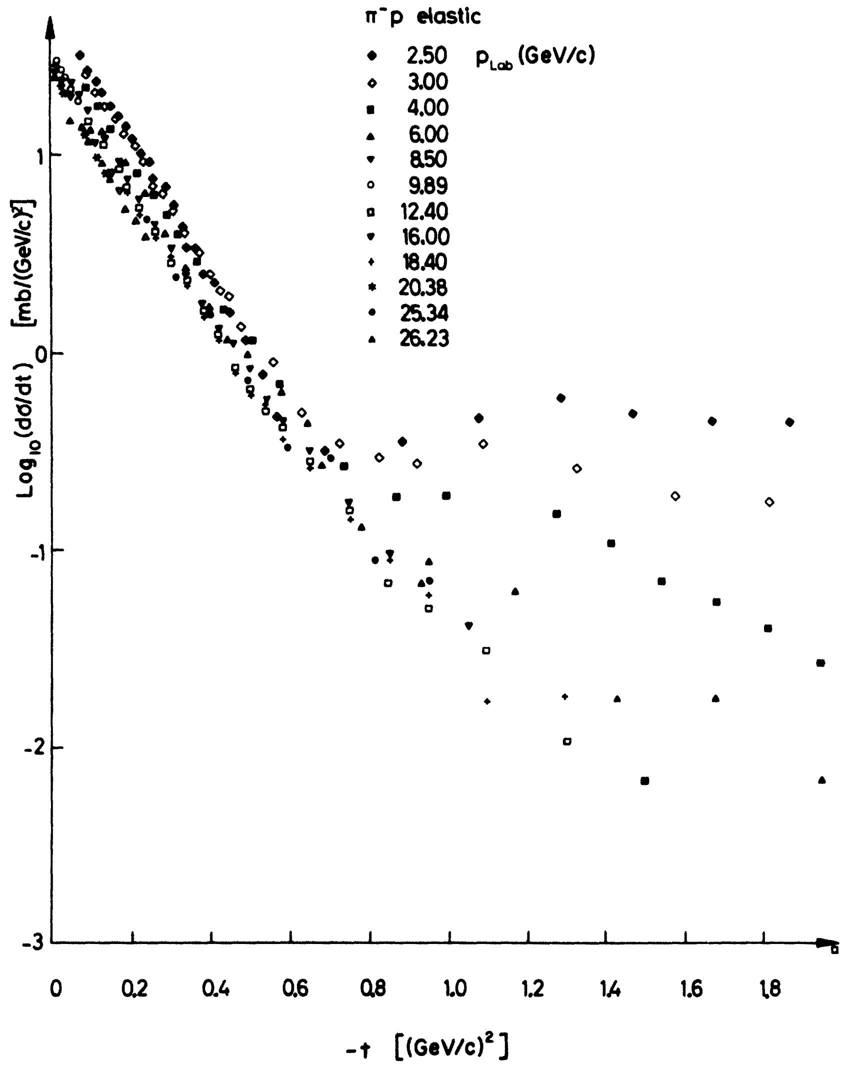

FIG. 8. Differential cross section for $\pi^- p$ elastic scattering plotted against t for $p_L = 2.50$–26.23 GeV/c. The data are from Ref. 21.

course secondary effects are very important in $\bar{p}p$ scattering at accelerator energies, so Eq. (3.10) cannot be expected to fit the $\bar{p}p$ slope at these energies, but one would expect Eq. (3.10) to hold at NAL energies. Nevertheless it is interesting to note that the $\bar{p}p$ differential cross section appears to show antishrinkage already at medium energies. It is important to note, as suggested by Odorico [24] and discussed in detail by Pinsky,[25] that the breaks in pp elastic scattering data and the dips in the crossed reaction $\bar{p}p \to \bar{p}p$ fall on the same n^2 = constant curves. The relation of this effect of crossing to the shrinkage of the pp diffraction peak and the antishrinking in $\bar{p}p$ scattering has already been discussed in Refs. 24 and 25.

IV. CONCLUSIONS

The variable n^2 has been introduced in order to unify the group-theoretical kinematic structure of elastic and inelastic reactions and thereby to provide the same high-energy expansions for these different types of processes. A Regge-type analysis has led to predictions of the form

$$\frac{d\sigma}{dt} \underset{s\to\infty;\ n^2 \text{ fixed}}{\sim} b(n^2) s^{2[\alpha(n^2)-1]} \tag{4.1}$$

valid for $|n^2| \ll s$. Plots of $d\sigma/dt$ against n^2 for many diffractive reactions show a remarkable lack of energy dependence at fixed n^2. This "scal-

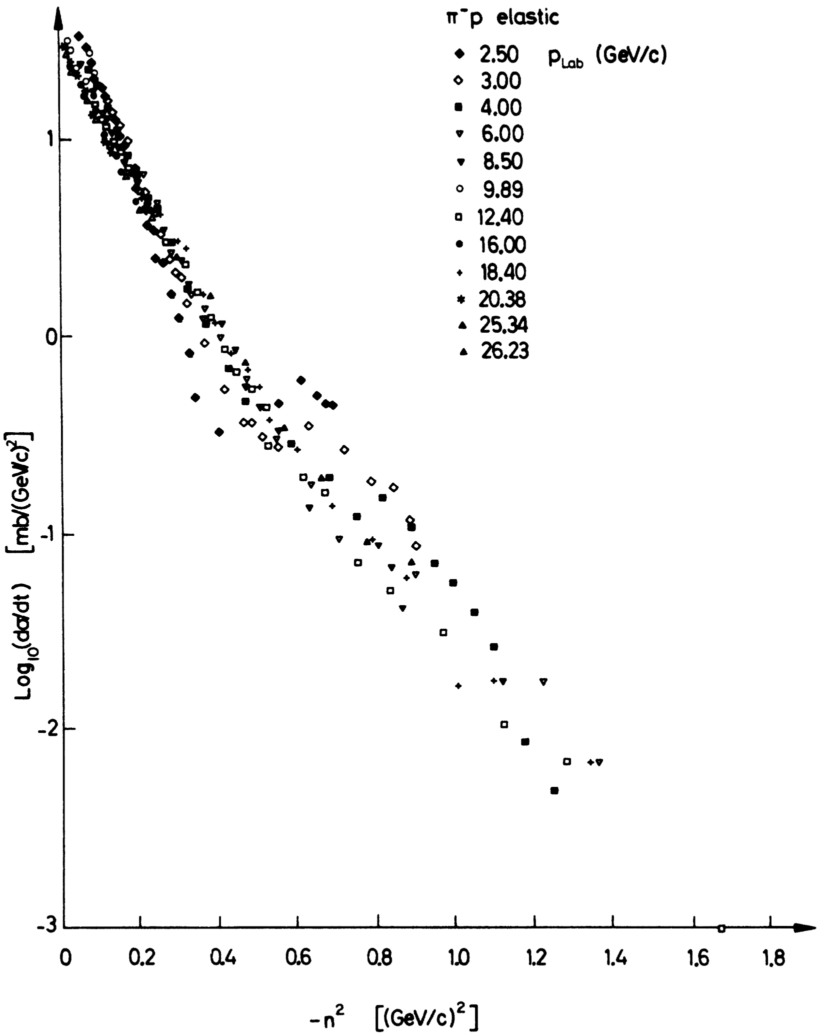

FIG. 9. Differential cross section for $\pi^- p$ elastic scattering plotted against n^2 for $p_L = 2.50$–26.23 GeV/c. The data are the same as those shown in Fig. 8.

ing" corresponds to having $a(n^2)=1$ for the dominant diffractive term at high energies. Such a pole is of course the analog of the Pomeranchukon. The shrinkage of $d\sigma/dt$ versus t appears here as a purely kinematic effect and is predicted to die out at very high energies. Thus our picture of the "Pomeranchukon" is quite different from the traditional Regge version. Since in any case one has never had a *clear* idea of the nature of the Regge Pomeranchukon, it is perhaps not too difficult to accept our new description of what the "Pomeranchukon" term is like.

The situation as regards nondiffractive processes is still not clear and awaits further study. However, Maor,[26] and independently Pond,[27] have plotted $d\sigma/dt$ versus n^2 for the two classic Regge reactions $\pi^- p \to \pi^0 n$ and $\pi^- p \to \eta n$, which are supposed to isolate ρ and A_2 exchange, respectively, and which indeed are the main sources of our knowledge of $\alpha_\rho(t)$ and $\alpha_{A_2}(t)$ for $t<0$. In both cases they find that the s variation at different fixed n^2 values is controlled by an n^2-independent power. Maor finds $a_\rho(n^2) \simeq 0.4$, $a_{A_2}(n^2) \simeq 0.3$ and Pond finds that any allowable slope in n^2 would have to be ≤ 0.2 for both a_ρ and a_{A_2}.[28] These results are very surprising and perhaps suggest that also in these nondiffractive reactions the shrinkage with increasing t is a kinematic effect and will die out at higher energies. This viewpoint would be quite different from the usual Regge one and it is of great importance to test it experimentally. The most direct method would be to perform a high-

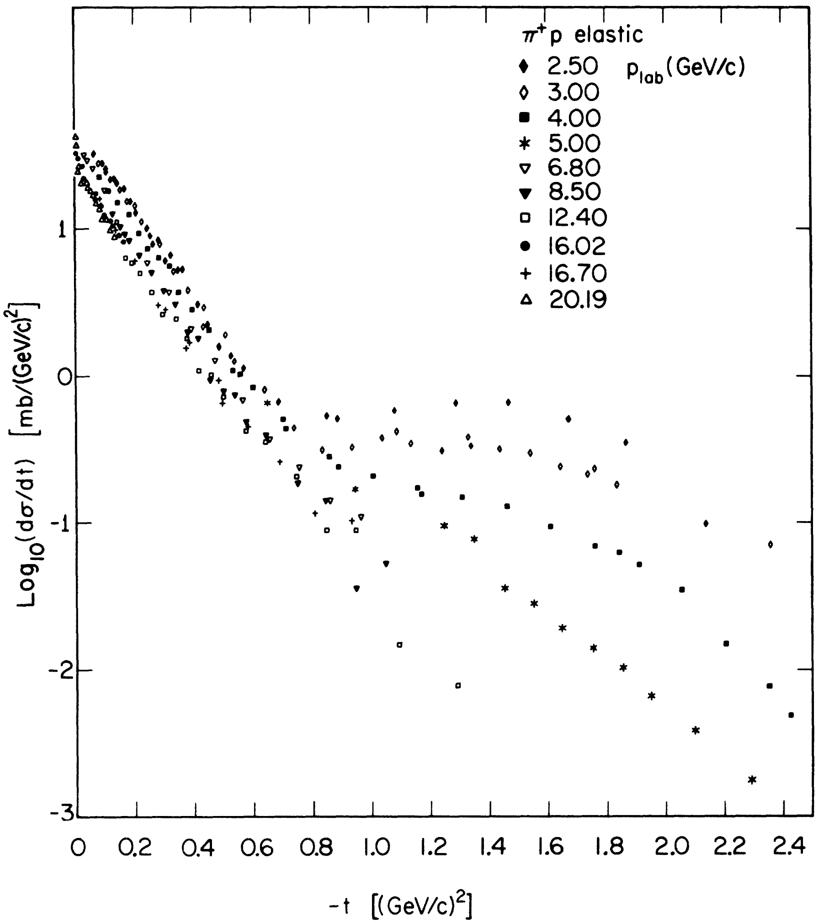

FIG. 10. π^+p elastic differential cross section plotted against t for $p_L = 2.50$–20.19 GeV/c. The data are from Refs. 21 and 22.

statistics measurement of the differential cross section for these reactions at Serpukhov energies and see if the s dependence of the data is at all compatible with the behavior

$$\frac{d\sigma}{dt}(\pi^-p\to\pi^0n)\simeq b(n^2)s^{2[\alpha_\rho(n^2)-1]}\ , \tag{4.2}$$

$$\frac{d\sigma}{dt}(\pi^-p\to\eta n)\simeq c(n^2)s^{2[\alpha_{A2}(n^2)-1]} \tag{4.3}$$

at fixed n^2 values, in which $a_\rho(n^2)$ and $a_{A_2}(n^2)$ are constants or weakly dependent on n^2.

The remarkable simplicity found in the data for many reactions, when considered as a function of s and n^2 (and σ), goes *far beyond the expectations of the original theory*. The theory suggested that describing amplitudes as $f(s,n^2)$ would simplify comparison of elastic and inelastic reactions by eliminating spurious kinematical effects. The empirical discovery that diffractive amplitudes are strongly dominated by s-independent terms, $f(n^2)$ only, is a surprise, and although nicely compatible with the theory is not really predicted by it. Thus it may be that there is some deep underlying *dynamical* significance to n^2 which is not yet understood.[29]

In view of our ignorance of dynamics it seems imperative to extend the empirical study of scattering data as functions of s and n^2, so as to learn as much as possible about the structure of the amplitudes $f(s,n^2,\sigma)$. To this end one should study the predictions that

$$\frac{d\sigma}{dt}\sim|b_\pm(n^2)|^2s^{2[\alpha_\pm(n^2)-1]} \tag{4.4}$$

for $s\to\infty$ and n^2 fixed such that $|n^2|\ll s$, in both the forward and backward hemispheres, and one should plot the symmetrized cross sections $(d\sigma/dt)_S$ and $(d\sigma/dt)_A$ [see Eqs. (3.3) and (3.4)] as functions of n^2 for the full range of n^2 for all

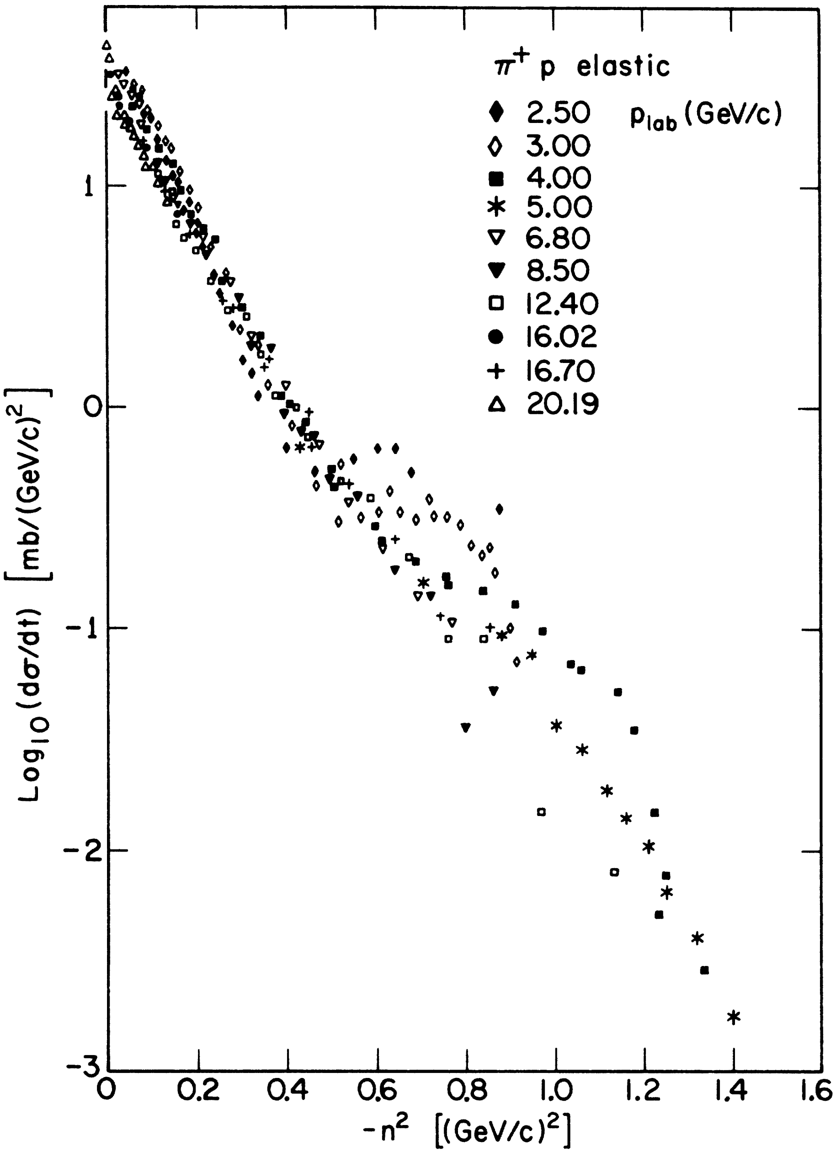

FIG. 11. π^+p elastic differential cross section plotted against n^2 for $p_L = 2.50$–20.19 GeV/c. The data are the same as those shown in Fig. 10.

known $2 \to 2$ reactions.

We have stressed in the Introduction that the use of n_μ makes the mathematical structure of the scattering amplitudes invariant under changes in the external masses. Thus it would be extremely interesting to look at a reaction in which we can vary smoothly the mass of one of the external particles while leaving unchanged all its other properties. Just such a possibility is provided by deep-inelastic reactions in which a final hadron is actually monitored. For example in $ep \to ep\rho^0$ we are essentially studying the photoproduction reaction

$$\gamma(q^2) + p \to \rho^0 + p$$

in which the mass of the γ can be continuously varied ($m_\gamma{}^2 = q^2 < 0$). Since for $q^2 = 0$, as mentioned in Sec. III C and shown in Fig. 7,

$$\frac{d\sigma}{dt}(\gamma p \to \rho^0 p) \simeq f(n^2) \text{ only},$$

one might hope that this holds for reasonably small $q^2 \neq 0$. If this is so, then defining $b(s, q^2)$ by

$$\frac{d\sigma}{dt}(\gamma(q^2)p \to \rho^0 p) = A \exp[b(s, q^2)t] \qquad (4.5)$$

for small t, one has

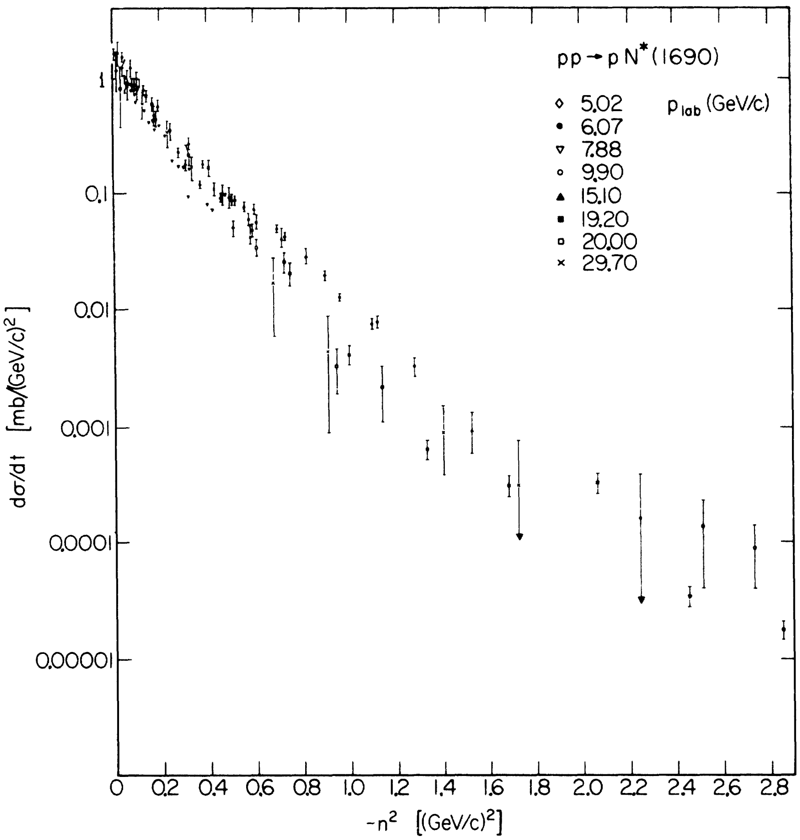

FIG. 12. Differential cross section for $pp \to pN^*(1690)$ plotted against n^2 for $p_L = 5.02$–29.70 GeV/c – data from Ref. 23.

$$b(s, q^2) \simeq b(s, 0)\left(1 - \frac{q^2}{s - 2m_N^2 - m_\rho^2}\right), \quad (4.6)$$

where $b(s, 0)$ is the logarithmic slope in true ρ^0 photoproduction. It should be noted that according to Eq. (4.6) the diffraction peak gets *narrower* as $|q^2|$ increases, at fixed s – the square of the γp c.m. energy. The data are at present somewhat self-contradictory and it is not yet possible to test Eq. (4.6) adequately. It is also possible in *inclusive* reactions that scaling may set in at lower energies if instead of considering $f(s, p_\perp^2, x)$ one uses $f(s, n^2, x)$; i.e., one looks at the s dependence at fixed n^2 and x rather than fixed $p_\perp^2$ and x. The differential cross sections, $d^2\sigma/dt\, dM^2$ (M is the missing mass), may also scale sooner if plotted at fixed n^2 and M^2 rather than t and M^2.[30]

In summary the empirical evidence suggests that the variable n^2 may have some deep and fundamental dynamical significance. It will be of great interest on the one hand to extend these empirical studies and, on the other hand, to try to understand the role of n^2 from a dynamical point of view.

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