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Two-Variable Expansions and $\bar{p}n \to 3\pi$ Annihilations at Rest

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The experimental Dalitz plot for $\bar{p}n \to \pi^+\pi^-\pi^-$ annihilations at rest is analyzed, using previously suggested two-variable expansions of decay amplitudes, based on the representation theory of the group O(4). A fair fit to the Dalitz plot is obtained by minimizing the χ^2 function, assuming that the annihilation proceeds from a protium state of definite (but unspecified) angular momentum J (or from several such states). Fits with 2, 4, 6, and 9 free parameters are considered, the values of the parameters are essentially stable with respect to the "cutoff" of the expansion, and the solutions are unique. The fit is purely kinematical, in that no assumptions are made about the initial and final state or the annihilation dynamics. An analysis of the contribution of various parts of the Dalitz plot to the over-all χ^2 suggests that the fit would be considerably improved by including, e.g., ρ- and f-meson final-state resonances explicitly. We compare the O(4) expansions with a power-series expansion in Dalitz-Fabri variables and also with various models that have been recently applied, in particular the Veneziano model and its generalizations and various final-state-interaction models.

I. INTRODUCTION

In a previous paper[1] (hereafter referred to as I) a general formalism was presented for analyzing the Dalitz-plot distributions for three-body decays, involving particles with arbitrary spins. Here we apply the formalism to study the $\bar{p}n \to 3\pi$ annihilations at rest, using the experimental data of Anninos *et al.*[2,3] We also compare some features of our treatment with those of other approaches to three-pion annihilations.

The formalism provides two-variable expansions of the decay amplitudes for the process $1 \to 2+3+4$ in terms of the transformation matrices of the group O(4). It is a generalization of a method proposed earlier for treating three-body decays in the case when all particles involved have spin zero. The method has already been applied to analyze data on $K \to 3\pi$ and $\eta \to 3\pi$ decays.[4]

The entire approach, making use of two-variable expansions of decay amplitudes in terms of functions, provided by the O(4) group, is a modification of a similar approach to relativistic scattering amplitudes.[5] There, relativistic invariance is used to derive two-variable expansions in terms of basis functions[6] of the homogeneous Lorentz group O(3, 1) (for spin-zero particles), or in terms of the transformation matrices of O(3, 1) (for particles with arbitrary spins).[7]

The O(4) expansions to be applied below are obtained in the following manner. We consider the decay $1 \to 2+3+4$ in a frame of reference resembling the center-of-mass system for scattering, i.e., we put $\vec{p}_3 = -\vec{p}_4$ and $\vec{p}_1 = \vec{p}_2$. Making use of a formalism suggested by Feldman and Matthews,[8] we can keep track separately of the dependence of the helicity amplitudes for the decay on the spin variables and on various energies and angles. Each amplitude depends dynamically on two variables only, e.g., the Mandelstam variables or any suitable combinations of these. We choose these independent variables to be the spherical coordinates of one of the vectors figuring in the problem, e.g., the momentum p_1. Equivalently we can consider these variables to be the parameters, figuring in the Wigner boost operator[9] that transforms the momentum p_1 from its rest frame to the center-of-mass-like system under consideration. For a scattering process, physically allowed values of the momentum p_1 range over the entire upper sheet of the mass shell $p_1^2 = m_1^2$. For decays, however, the kinematic region is finite and p_1 only varies over a "hyperbolic cup," close to the vertex $(m_1, 0, 0, 0)$ of the hyperboloid. As in our previous publication I we map this cup continuously onto a four-dimensional sphere.[4] The decay amplitudes are then functions, defined over this entire O(4) sphere [or over the parameters of a corresponding O(4) group transformation]. This allows us to expand these amplitudes in terms of the transformation matrices of O(4), or in the special case of spinless particles in terms of basis functions of irreducible representations of O(4).

The O(4) expansions are explicitly two-variable expansions; the amplitudes are presented as infinite double sums of known functions, depending on an energy-type variable α and an angle θ (see Sec. II). Let us note the following features of the O(4) expansions: The total angular momentum of two of the final-state particles is displayed explicitly and the angle θ is contained in O(3) rotation matrices, which ensures the standard behavior of the amplitudes on the boundary of the physical region, where we have $\cos\theta=\pm 1$. Each term in the expansion has the correct behavior at the physical threshold and pseudothreshold. The expansion is in terms of functions that are orthogonal over the Dalitz plot, so that it can be trivially inverted. The orthogonality also decreases the correlation among the experimental errors in the expansion coefficients, which in turn contributes to the stability of the expansions with respect to truncations.

The entire dynamics of the reaction are represented by the expansion coefficients, which we call Lorentz amplitudes. In this article we make no attempt at any dynamical approach at all. Instead, we take the experimental data on $\bar{p}n\to 3\pi$ annihilations[2,3] and use them to determine the Lorentz amplitudes by a best-fit procedure, For comparison we also make a fit to the same data using a power-series expansion of the amplitudes in terms of the Dalitz-Fabri variables.[10]

Generally speaking, our expansions of three-body decay amplitudes are not suitable for describing processes involving five particles, like the annihilation $\bar{p}n\to 3\pi$, where the amplitudes depend on five independent kinematical variables. However, if one considers annihilations for a fixed initial energy, and only detects the energies of the final particles (and not, say, the angles under which they are emitted), then the kinematics of the process will be the same as in the case of a three-body decay of one particle.

The data[2,3] that we are considering refer to the reaction $\bar{p}n\to\pi^+\pi^-\pi^-$ initiated by antiprotons brought to rest in a deuterium bubble chamber and interpreted as annihilations on free neutrons (a slow outgoing spectator proton was observed in each case). This allows us to treat the annihilation as the decay of one "particle," namely protium and to apply the O(4) expansions directly. The total energy and angular momentum of the protium then corresponds to the mass and spin of "particle" 1 in the reaction $1\to 2+3+4$. It is commonly assumed that the annihilation proceeds from an S state.[2,11] Recent investigations,[3,12] however, contradict this assumption and it now seems that P and higher states play an important role. In this paper we make no attempt to distinguish between the different possible initial states and in particular we do not assume that an initial S state dominates.

In Sec. II we present the basic expansion formulas and give expressions for the relevant experimental quantities. The data and the fitting procedure are discussed in Sec. III and the numerical results are presented in tables and summarized in Sec. IV. Section V is devoted to conclusions, comparison with other treatments, and future outlook.

II. EXPANSIONS OF DECAY AMPLITUDES AND EXPRESSIONS FOR EXPERIMENTAL QUANTITIES

Consider the decay

$$1\to 2+3+4 \tag{1}$$

in the center-of-mass frame of particles 3 and 4 ($\vec{p}_1=\vec{p}_2,\ \vec{p}_3=-\vec{p}_4$), and introduce the O(4) variables used in I as

$$\cos\alpha=1-\frac{[(m_1+m_2)^2-s][(m_1-m_2)^2-s]}{2m_1{}^2R^2s}, \tag{2}$$

$$\cos\theta=\frac{2s(t-m_1{}^2-m_3{}^2)+(s+m_1{}^2-m_2{}^2)(s+m_3{}^2-m_4{}^2)}{\{[-s+(m_1+m_2)^2][-s+(m_1-m_2)^2][s-(m_3+m_4)^2][s-(m_3-m_4)^2]\}^{1/2}}, \tag{3}$$

where

$$R=\frac{\{[(m_1+m_2)^2-(m_3+m_4)^2][(m_1-m_2)^2-(m_3+m_4)^2]\}^{1/2}}{2m_1(m_3+m_4)}, \tag{4}$$

and

$$s=(p_3+p_4)^2,\quad t=(p_2+p_4)^2,\quad u=(p_2+p_3)^2 \tag{5}$$

are the Mandelstam variables (see Fig. 1).

The O(4) expansion of the helicity amplitudes, obtained in I, can be written as

$$f_{\lambda_1\lambda_2\lambda_3\lambda_4}(\alpha,\theta)=\sum_{J=\max(|\lambda|,|\mu|)}^{\infty}\ \sum_{L=|J-s_2|}^{J+s_2}\ \sum_{n=\max(s_1,L)}^{\infty}\ \sum_{\nu=-\min(s_1,L)}^{\min(s_1,L)}\ \sum_{\bar\lambda_2\bar\eta_2\bar\eta_3\bar\eta_4}(2J+1)$$

$$\times\prod_{i=2}^{4}f_0(a_i,-\lambda_i,\eta_i\bar\eta_i)(-1)^{-\lambda_1+\bar\lambda_2+\lambda_3-\lambda_4}\begin{pmatrix}L & s_2 & J\\ -\lambda_1 & \lambda_2 & \lambda\end{pmatrix}\begin{pmatrix}L & s_2 & J\\ -\bar\lambda_2-\mu & \lambda_2 & \mu\end{pmatrix}$$

$$\times A_L^{\nu n}(\bar\lambda_2\lambda_3\lambda_4\eta_1\bar\eta_2\bar\eta_3\bar\eta_4)d^{\nu n *}_{L s_1\lambda_1}(-\alpha)d^{J*}_{\lambda\mu}(-\theta)\,, \tag{6}$$

where m_i, s_i, λ_i, $m_i\cosh a_i$, and η_i are the mass, spin, helicity, energy, and intrinsic parity of the ith particle, respectively. The label J is the total angular momentum of the pair 3 and 4, L can be interpreted as the total angular momentum of the final state in a frame in which particle 2 is at rest, and $|\nu|$ and n are the lower and upper limits on L for J fixed [and they label representations of O(4)]. The quantities in brackets are the usual O(3) $3j$ symbols, $d^{\nu n}_{Ls\lambda}(\alpha)$ and $d^J_{\lambda\mu}(\theta)$ are O(4) and O(3) transformation matrices, $\lambda=\lambda_1-\lambda_2$, $\mu=\lambda_3-\lambda_4$, and

$$f_0(a,\lambda,\eta\bar\eta)=\tfrac{1}{2}(e^{a\lambda}+\eta\bar\eta e^{-a\lambda})$$

is an O(3, 1) finite transformation matrix. The expansion coefficients $A_L^{\nu n}$ are the "Lorentz amplitudes." They carry the dynamics and in a phenomenological approach are to be determined from a best fit to experimental data.

In the special case of spinless particles ($s_i=\lambda_i=0$), expansion (6) reduces to the one considered previously,[4] i.e.,

$$f(\alpha,\theta)=\sum_{n=0}^{\infty}\sum_{l=0}^{n}A_{nl}\phi_{nl}(\alpha,\theta)\,, \tag{7}$$

with

$$\phi_{nl}(\alpha,\theta)=e^{-il\pi/2}\,\frac{2^{l+1/2}\Gamma(l+1)}{2\pi}\left((2l+1)\frac{(n+1)\Gamma(n-l+1)}{\Gamma(n+l+2)}\right)^{1/2}(\sin\alpha)^l C^{l+1}_{n-l}(\cos\alpha)P_l(\cos\theta)\,. \tag{8}$$

[$C^{l+1}_{n-l}(\cos\alpha)$ and $P_l(\cos\theta)$ are Gegenbauer and Legendre polynomials.]

It is now a simple matter to express experimental quantities like decay rates, angular distributions, polarizations, etc. in terms of the Lorentz amplitudes $A_L^{\nu n}$. The general formulas are quite cumbersome, so we shall only consider the differential decay rate in the case when at least one of the final particles, which we shall identify with particle 2, has spin $s_2=0$.

For $s_2=0$ expansion (6) simplifies to

$$f_{\lambda_i}(\alpha,\theta)=\sum_{Jn\nu}\sum_{\bar\eta_3\bar\eta_4}\prod_{i=3}^{4}f_0(a_i,-\lambda_i,\eta_i\bar\eta_i)A_J^{\nu n}(\lambda_3\lambda_4\eta_1\bar\eta_3\bar\eta_4)D^{\nu n}_{J\,\lambda_3-\lambda_4\ s_1\lambda_1}(0,\theta,0,\alpha,00)\,, \tag{9}$$

where $D^{\nu n}_{J_1M_1J_2M_2}(\phi,\theta,\psi,\alpha,\eta,\chi)$ is a general O(4) finite transformation matrix (see I).

The decay probability for an unpolarized initial particle of spin s_1 is

$$W(\alpha,\theta)=\frac{1}{2s_1+1}\sum_{\lambda_1\lambda_3\lambda_4}|f_{\lambda_1 0\lambda_3\lambda_4}(\alpha,\theta)|^2\,. \tag{10}$$

We substitute expansion (9) into expression (10), make use of the symmetries of the D functions and use the O(4) Clebsch-Gordan coefficients[4] to expand the product of two O(4) D functions. We have

$$D^{\nu n}_{J\lambda_3-\lambda_4 s_1\lambda_1}(g)D^{\nu' n' *}_{J'\lambda_3-\lambda_4 s_1\lambda_1}(g)=(-1)^{2n'-J'-s_1+\lambda_3-\lambda_4-\lambda_1}\sum_{VNjs}\left(\begin{matrix}\nu n & \nu' n'\\ J\lambda_3-\lambda_4 & J'-\lambda_3+\lambda_4\end{matrix}\middle|\begin{matrix}VN\\ J0\end{matrix}\right)$$

$$\times\left(\begin{matrix}\nu n & \nu' n'\\ s_1\lambda_1 & s_1-\lambda_1\end{matrix}\middle|\begin{matrix}VN\\ s0\end{matrix}\right)D^{VN}_{j0s0}(g)\,. \tag{11}$$

The O(4) Clebsch-Gordan coefficients are

$$\begin{pmatrix} \nu_1 n_1 & \nu_2 n_2 & \nu n \\ J_1 M_1 & J_2 M_2 & JM \end{pmatrix} = (-1)^{J_1-J_2+M}[(2J_1+1)(2J_2+1)(2J+1)(n+\nu+1)(n-\nu+1)]^{1/2}\begin{pmatrix} J_1 & J_2 & J \\ M_1 & M_2 & -M \end{pmatrix}$$

$$\times\begin{Bmatrix} \frac{1}{2}(n_1+\nu_1) & \frac{1}{2}(n_1-\nu_1) & J_1 \\ \frac{1}{2}(n_2+\nu_2) & \frac{1}{2}(n_2-\nu_2) & J_2 \\ \frac{1}{2}(n+\nu) & \frac{1}{2}(n-\nu) & J \end{Bmatrix}$$

[the curly brackets are O(3) $9j$ symbols[13]]. We substitute (9) and (11) into (10) and perform the summation over λ_1 using

$$\sum_{\lambda_1} (-1)^{s_1-\lambda_1}\begin{pmatrix} s_1 & s_1 & s \\ \lambda_1 & -\lambda_1 & 0 \end{pmatrix} = (2s_1+1)^{1/2}\delta_{s0}\,.$$

Thus, only $s=0$ remains in the sum, which in turn forces $V=0$ in (11). The special type of D function that now figures in the expression for the decay probability reduces to a basis function (8), i.e.,

$$D^{0N}_{j000}(\phi, -\theta, -\phi, \alpha, 0, 0) = \frac{\sqrt{2}\pi}{N+1}\,\phi_{Nj}(\alpha, -\theta)\,.$$

The decay probability for $s_2=0$, s_1, s_3, and s_4 arbitrary can finally be written as

$$W(\alpha,\theta) = \sum_{\lambda_3\lambda_4}\ \sum_{\overline{\eta}_3\overline{\eta}_3'\overline{\eta}_4\overline{\eta}_4'}\ \sum_{N=0}^{\infty}\sum_{L=0}^{N} B_{NL}(\lambda_3\lambda_4\overline{\eta}_3\overline{\eta}_3'\overline{\eta}_4\overline{\eta}_4')\,\phi_{NL}(\alpha,\theta)\prod_{i=3}^{4} f_0(a_i, -\lambda_i, \eta_i\overline{\eta}_i)\, f_0^*(a_i, -\lambda_i, \eta_i\overline{\eta}_i')\,. \tag{12}$$

It should be noted that the decay probability in this case ($s_2=0$, initial particle unpolarized) is expressed in terms of the same O(4) functions ϕ_{NL} as it would be, if all particles involved had zero spin. The only difference is the presence of the O(3, 1) transformation matrices $f_0(a, -\lambda, \eta\overline{\eta})$.

In the $\overline{p}n$ annihilation under consideration, particles 3 and 4 also have spin zero and hence we may finally write the decay probability as

$$W(\alpha,\theta) = \sum_{N=0}^{\infty}\sum_{L=0}^{N} B_{NL}\,\phi_{NL}(\alpha,\theta)\,, \tag{13}$$

which coincides with the formula for spinless particles. In our analysis we make use of expansion (13), which is completely insensitive to the angular momentum of the initial state. Thus, if several different momenta were present in the initial protium state (S, P, and higher waves), then we would have to take an incoherent sum of the contributions from all levels, which simply amounts to redefining the coefficients B_{NL}.

The angular momentum s_1 of the initial state does of course figure in expansion (9) of the amplitudes themselves and only vanishes from (13) because of the averaging over initial spins. General D functions would figure in the decay rates for polarized initial states and the expansions would then yield information on the initial angular momentum.

In our case particles 3 and 4 are identical, hence the helicity amplitudes must satisfy

$$f_{\lambda_i}(\alpha,\theta,\phi) = f_{\lambda_i}(\alpha, \pi-\theta, \phi+\pi)\,.$$

This implies that only even values of L will figure in (13). Note also that for even L all coefficients B_{NL} are real.

For comparison we shall also make an analysis of the same $\overline{p}n$ data using a two-variable power-series expansion in terms of the Dalitz-Fabri[10] variables x and y. If T_2, T_3, and T_4 are the kinetic energies of the final particles, then

$$x = \sqrt{3}\,\frac{T_3-T_4}{Q}\,,\qquad y = \frac{3T_2-Q}{Q}\,, \tag{14}$$

where $Q = T_2+T_3+T_4 = m_1-m_2-m_3-m_4$.

The expansion to be used is

$$W(x,y) = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} R_{km}x^k y^m\,. \tag{15}$$

Since particles 3 and 4 are identical, we have k = even. Further aspects of the O(4) and Dalitz-Fabri expansions, in particular the relation between them, were discussed earlier.[4]

III. DISCUSSION OF THE DATA AND THE FITTING PROCEDURE

The Dalitz plot used for the analysis[3] contained 4509 events distributed in 538 bins, each of the size $0.15\times 0.0625\ \mathrm{GeV}^2$. This Dalitz plot (soon to be published[3]) has essentially the same structure as the one presented earlier[2] (on the basis of a smaller sample of events), that has raised a considerable amount of theoretical interest.[14-17] As usual a number of events N_i and a statistical error $N_i^{1/2}$ is associated with each bin.

The decay probability is in this case related to the number of events in each bin by the formula

$$\overline{W}(\alpha_i, \theta_i) = N_i, \tag{16}$$

where $\overline{W}(\alpha_i, \theta_i)$ is the decay probability (10) integrated over the area of the ith bin. For our purposes it was sufficient to replace this integral for $\overline{W}(\alpha_i, \theta_i)$ by the function $W(\alpha, \theta)$, evaluated at the geometrical center of each bin, the coordinates of which are α_i and θ_i, multiplied by the area of the bin. The decay probability is then parametrized using a truncated O(4) expansion (13):

$$\overline{W}_{N_0}(\alpha_i, \theta_i) = \sum_{N=0}^{N_0} \sum_{L=0}^{N} B_{NL}\phi_{NL}(\alpha_i, \theta_i) A_i \tag{17}$$

or a truncated xy expansion (15)

$$\overline{W}_{M_0}(\alpha_i, \theta_i) = \sum_{m=0}^{M_0} \sum_{k=0}^{M_0-m} R_{km} x_i^k y_i^m A_i , \tag{18}$$

where A_i is the area of the ith bin (A_i is constant over the Dalitz plot, except at the boundary of the physical region).

We fit the data by minimizing the χ^2-function

$$\chi^2 = \sum_i \left(\frac{N_i - \overline{W}_{K_0}(\alpha_i, \theta_i)}{N_i^{\,1/2}}\right)^2, \tag{19}$$

where K_0 is either N_0 as in (17) or M_0 as in (18) and where the sum is over all bins. The minimization is performed with respect to the expansion coefficients in (17) or (18), but in order to decrease the number of parameters, varied by the computer, it is convenient to put, e.g.,

$$\overline{W}_{N_0}(\alpha_i, \theta_i) = B_{00}\left[1 + \sum_{NL}{}' b_{NL}\phi_{NL}(\alpha_i, \theta_i)\right] A_i$$

$$= B_{00}\tilde{W}_{N_0}(\alpha_i, \theta_i), \tag{20}$$

where $b_{NL} = B_{NL}/B_{00}$ and the prime on the sum indicates that the term $N = L = 0$ is excluded. One of the conditions for a minimum of χ^2 then expresses B_{00} in terms of the other parameters b_{NL} at the minimum, i.e.,

$$\frac{\partial \chi^2}{\partial B_{00}} = 0 \tag{21}$$

implies that

$$B_{00} = \frac{\sum_i \tilde{W}_{N_0}(\alpha_i, \theta_i)}{\sum_i [\tilde{W}_{N_0}(\alpha_i, \theta_i)]^2/N_i} . \tag{22}$$

The same procedure is used for expansion (18). The best-fit values of the parameters (expansion coefficients) are then found by minimizing χ^2, using the computer program MINUIT[18] modified to include random starting points.[4] In general many solutions are found in the multidimensional space of parameters, corresponding to local minima. In each case we did, however, find a global minimum, distinguished from the others by an obviously lower value of χ^2.

Our approach in this paper is purely nondynamical, i.e., we put in no assumptions about the behavior of the expansion coefficients, about the nature of the final or initial states, etc. In particular the rather obvious presence of ρ^0- and f-meson bands in the Dalitz plot was ignored.

The purpose of our numerical study is to test the suitability of O(4) expansions for representing data with a complex structure. Thus, we are interested in the over-all quality of the fit (the value of χ^2 divided by the number of degrees of freedom), the number of parameters necessary to achieve a reasonable fit, and the stability of the values of the parameters with respect to the cutoff N_0. Finally we wish to establish which dynamical features of the process are well represented and which are distorted. Some information of this type is obtained by studying the relative contribution of various parts of the Dalitz plot to the value of χ^2.

We performed fits to two different Dalitz plots (representing the same data). The first was simply the experimental Dalitz plot, as presented by the experimentalists.[2,3] The second was obtained from the first by partly taking into account final-state Coulomb interactions. This we did in the simplest possible manner, corresponding to a first-order perturbation-theory calculation in a nonrelativistic limit. The effect of such a correction is to divide the number of events N_i in each bin by the factor[19]

$$C = 1 - \pi\alpha\left(\frac{e_2 e_3}{v_{23}} + \frac{e_3 e_4}{v_{34}} + \frac{e_2 e_4}{v_{24}}\right) , \tag{23}$$

where e_i is the charge of the ith particle (in units of the electron charge), α is the finite structure constant, and v_{ij} is the relative velocity between the ith and jth particle. This way of including Coulomb corrections corresponds to taking into account only the longest range part of the electromagnetic interactions, lumping the other parts together with final-state strong interactions. It would of course be possible to make a somewhat more consistent calculation of the contribution of electromagnetic interactions to the $\bar{p}n \to 3\pi$ reaction (along the lines of similar calculations for $K \to 3\pi$ decays[20]). Although the corrections (23) do somewhat improve the fits to the data, it does not seem to make a great difference, and we did not pursue the problem of electromagnetic interactions further.

Finally let us note that expansions (13) and (15) can readily be used to fit the spectrum of the posi-

TABLE I. Coefficients in the O(4) two-variable expansion obtained by a fit to the data on $\bar{p}n \to 3\pi$ annihilations at rest. The errors correspond to an increase of χ^2 by one unit.

NDF	536		534		532		529	
Coulomb corrections	no	yes	no	yes	no	yes	no	yes
χ^2	1726.4	1698.0	1368.2	1322.7	1348.4	1323.5	1287.4	1251.4
χ^2/NDF	3.2	3.2	2.6	2.5	2.5	2.5	2.4	2.4
A_{00}	1301.1	1254.6	1269.8	1281.1	1422.1	1297.9	1281.2	1264.7
A_{10}/A_{00}	-0.343 ± 0.009	-0.342 ± 0.009	-0.061 ± 0.048	-0.087 ± 0.051	-0.189 ± 0.048	-0.131 ± 0.047	0.002 ± 0.062	-0.021 ± 0.062
A_{20}/A_{00}			-0.202 ± 0.030	-0.188 ± 0.031	-0.086 ± 0.046	-0.123 ± 0.047	-0.166 ± 0.049	-0.139 ± 0.052
A_{22}/A_{00}			-0.413 ± 0.028	-0.406 ± 0.028	-0.475 ± 0.039	-0.474 ± 0.041	-0.476 ± 0.046	-0.476 ± 0.046
A_{30}/A_{00}					-0.032 ± 0.020	-0.029 ± 0.016	-0.187 ± 0.046	-0.202 ± 0.047
A_{32}/A_{00}					0.064 ± 0.021	-0.022 ± 0.022	-0.109 ± 0.055	-0.107 ± 0.055
A_{40}/A_{00}							0.109 ± 0.023	0.113 ± 0.024
A_{42}/A_{00}							0.093 ± 0.028	0.089 ± 0.028
A_{44}/A_{00}							0.628 ± 0.092	0.617 ± 0.092

tive pion. For the O(4) expansion, it is simply necessary to integrate both sides of (13) over the angle θ, obtaining

$$W(\alpha) = \int W(\alpha, \theta) d(t-u)$$

$$= \beta(s) \int_0^{\pi} W(\alpha, \theta) \sin\theta \, d\theta$$

$$= \beta(s) \sum_N B_{N_0} \phi_{N_0}(\cos\alpha) , \qquad (24)$$

where

$$\beta(s) = \tfrac{1}{2} \left(\frac{[-s+(M+m)^2][-s+(M-m)^2](s-4m^2)}{s} \right)^{1/2} . \qquad (25)$$

Similarly we can obtain an xy expansion of $W(\alpha)$ by appropriately integrating formula (15) over $(t-u)$.

IV. NUMERICAL RESULTS

The results of the numerical fits to the $\bar{p}n \to 3\pi$ Dalitz plot, using both the original data and the Coulomb corrected data are shown in Tables I and II for the O(4) expansion (13) and the power-series expansion (15), respectively. The errors in the coefficients correspond to an increase of χ^2 by one unit.

A comparison of the two tables shows that the χ^2 values are comparable, but slightly better for the xy expansion. In neither case did we reach a minimum of χ^2/NDF (NDF is the number of degrees of freedom, defined as the number of bins less the number of free parameters), so the fits might be improved by adding further parameters. We did not attempt that simply because the minimization, with such a structured Dalitz plot and so many parameters was taking up too much computer time. As mentioned above, a glance at the tables shows that the inclusion of Coulomb corrections would always tend to slightly improve the fit, without essentially changing the parameters.

It should be noted that the O(4) expansions are considerably more stable than the power series ones with respect to truncation, i.e., with respect to the addition of further terms. Indeed, along each row of Table I the entries essentially coincide within the errors. The same cannot be said of the entries in Table II.

Each entry in the tables corresponds to the results of several minimizations, starting from random initial numbers. In each case it was possible to choose a global minimum, although local ones did show up in the six and nine parameter fits.

It is of considerable interest to look more closely at the structure of the experimental Dalitz plot.[2,3]

TABLE II. Coefficients in the power-series expansion obtained by a fit to the data on $\bar{p}n \to 3\pi$ annihilations at rest. The errors correspond to an increase of χ^2 by one unit.

NDF	536		534		532		529	
Coulomb corrections	no	yes	no	yes	no	yes	no	yes
χ^2	1911.2	1877.3	1395.9	1362.5	1230.4	1201.0	1076.6	1049.7
χ^2/NDF	3.6	3.5	2.6	2.6	2.3	2.3	2.0	2.0
R_{00}	519.1	501.6	533.5	524.8	396.4	390.9	298.8	294.2
R_{01}/R_{00}	0.421 ± 0.058	0.412 ± 0.059	1.445 ± 0.063	1.442 ± 0.062	2.132 ± 0.169	2.131 ± 0.184	3.982 ± 0.408	3.912 ± 0.389
R_{02}/R_{00}			2.599 ± 0.142	2.558 ± 0.047	6.563 ± 0.564	6.458 ± 0.570	15.732 ± 1.710	15.583 ± 1.675
R_{20}/R_{00}			-1.594 ± 0.047	-1.642 ± 0.089	-0.374 ± 0.187	-0.459 ± 0.196	1.323 ± 0.730	1.214 ± 0.699
R_{03}/R_{00}					3.404 ± 0.522	3.337 ± 0.538	-3.139 ± 1.260	-2.880 ± 1.229
R_{21}/R_{00}					-8.007 ± 0.878	-7.856 ± 0.925	-4.495 ± 1.712	-4.130 ± 1.672
R_{04}/R_{00}							-16.033 ± 2.116	-15.692 ± 2.046
R_{22}/R_{00}							-17.267 ± 4.000	-17.904 ± 3.936
R_{40}/R_{00}							-3.091 ± 1.460	-3.045 ± 1.393

The Dalitz plot shows apparent resonance bands in the t and u variables [i.e., in the invariant mass-squared $M^2(\pi^+\pi^-)$] corresponding to the ρ^0 and f mesons. A particular strong enhancement occurs at the $\pi^-\pi^-$ threshold, where $t \simeq u \simeq M_f^2$ (M_f is the mass of the f meson). On the other hand, for $t \simeq u \simeq 1.08$ GeV^2 there is a very prominent "hole" in the center of the Dalitz plot.

If we now look in greater detail at our O(4) fits to the data, in order to determine which parts of the Dalitz plot give the largest contribution to χ^2, we find the following. The over-all fit is reasonably good, in particular it reproduces the hole in the Dalitz plot and the enhancement at the $(\pi^-\pi^-)$ threshold. Bins giving larger and smaller contributions to the χ^2 function are distributed rather randomly, but a certain pattern does emerge. In regions with a small number of events, i.e., the hole in the middle and also regions of small t and large u (and vice versa), the χ^2 contributions vary greatly from bin to bin (say 18 in one bin and 0 in the neighboring one). This seems to be indicating that the χ^2 method itself is not too applicable in these regions (with large statistical errors) and that it might be preferable to consider some other statistical criterion instead, e.g., the "likelihood of observation," as suggested by Gopal *et al.*[16] Where the density of events is higher, the contribution to χ^2 seems to vary smoothly over larger areas. A consistently larger contribution to χ^2 comes from the regions of the apparent ρ-meson bands, somewhat less so from the f-meson ones, and both these

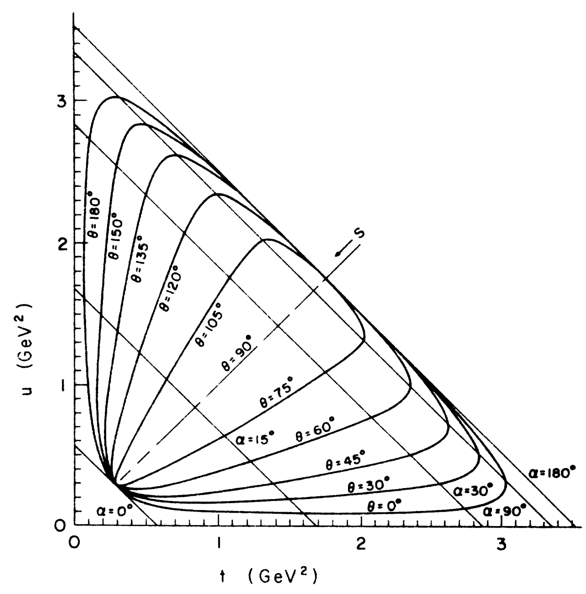

FIG. 1. Dalitz plot for $\bar{p}n \to 3\pi$ decay parametrized in terms of the Mandelstam variables (t, u) and the O(4) variables (α, θ). The boundary of the physical decay region is $\cos\theta = \pm 1$.

regions are underrepresented in our fits.

We have also used the integrated O(4) expansion (24) and the expansion in powers of y (14) to fit the spectrum of the positive pion. Since the results are of no particular interest, we do not reproduce them here. Similarly, we do not give fits to the Dalitz plot using the O(4) expansion (9) of the amplitudes themselves, instead of the decay probability. This essentially amounts to using a different cutoff procedure and for the annihilation Dalitz plot invariably gave higher values of χ^2 (and various ambiguities).

V. CONCLUSIONS

The essence of this paper is that we have taken two-variable expansions of decay amplitudes, based on the representation theory of the group O(4) and applied them to analyze the Dalitz plot for $\bar{p}n \to 3\pi$ annihilation at rest. The formalism is applicable to processes of the type $1 \to 2+3+4$, where the masses and spins of all particles are arbitrary. In principle the method could (and should) be generalized to include reactions with a larger number of particles, in particular the five-point function under consideration. In the case of annihilations at rest we simply assumed that the initial state is a state of definite angular momentum and energy, i.e., could be considered to be a one-particle state. It was shown in Sec. II that as long as only the spin-averaged value of the decay rate is considered, the actual value of the $\bar{p}n$ angular momentum is irrelevant and an O(4) expansion of the decay rate is obtained in terms of the same functions as for spin-zero particles (whereas the helicity amplitudes themselves are expanded in terms of functions that do depend on the spins).

The same O(4) expansions have been applied previously to analyze $K^\pm \to \pi^\pm\pi^\pm\pi^\mp$ and $\eta \to \pi^+\pi^-\pi^0$ Dalitz plots[4] and the obtained fits were much better than in the present case. This is of course not in the least surprising, since much more phase space is available in the $\bar{p}n$ annihilation than in the K and η decays. Also dynamics seem to be much more complicated, as demonstrated by the complex structure of the annihilation Dalitz plot, as opposed to the smooth plots for K and η decays.

Let us try to put our approach into a somewhat broader context. The most standard way of treating similar Dalitz plots is in terms of various final-state-interaction models.[21] The amplitudes can be split into two parts, one corresponding to a sum over intermediate-final-state two-particle resonances, represented, e.g., by Breit-Wigner-type terms, the other corresponding to a nonresonant background (which may or may not be ignored). An example of this approach is the rising-phase-shift model of Gleeson *et al.*,[17] which gives a qualitatively fair description of the $\bar{p}n \to 3\pi$ data (χ^2/NDF ~ 2.36 with 14 parameters). Alternatively, the Veneziano model[14] and its generalizations[15,16,22–24] have been applied to this process and also give a fair or good description of the data. Most of the fits were performed to the spectra only, however, Gopal *et al.*[16] fit the Dalitz plot itself. They do not use the χ^2 method, so it is difficult to compare their fit with ours. Other types of final-state-interactions models exist in the literature, both for annihilations at rest and in flight.[25,26] The final-state-interaction approach to nucleon annihilations in general has been criticized, e.g., from the point of view of duality.[27] The pronounced difference between annihilations at rest and in flight suggests that important dynamical effects are present in the initial-state nucleon-antinucleon state.[28,29] More general treatments based on five-point functions, e.g., B_5 have been proposed.[30,31]

Our O(4) expansion approach is completely kinematical, the only input being the reasonable "kinematic behavior" of the basis functions $\phi_{nl}(\alpha, \theta)$ (at the threshold, pseudothreshold, boundary of physical region, etc.). No assumptions at all about the dynamics are made, yet the fits are stable, unique, and reasonably good with a small number of parameters. The distribution of contributions to χ^2 from various regions of the Dalitz plot (see Sec. IV) suggests that a considerable improvement can be achieved by incorporating the ρ and f bands directly, e.g., by adding the appropriate Breit-Wigner terms.

We plan to return to the problem of $\bar{p}n \to 3\pi$ annihilations in the context of two-variable expansions, using a more dynamical approach. One possibility is to indeed split the amplitudes into resonant and nonresonant parts, specify the resonant parts and use the O(4) expansions for the nonresonant part only. Another possibility is to investigate the behavior of Lorentz amplitudes (i.e., the expansion coefficients), in the presence of resonances, using various models, and then to incorporate this knowledge in the phenomenological fits. Also knowledge concerning initial bound states could, in principle, be expressed in terms of the Lorentz amplitudes. Finally a much more complete discussion, including the dependence on the initial energy can be given, once the expansions are generalized to the case of many-particle reactions.

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