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Empirical Formula for Pion Production in Proton-Proton Collisions up to 1500 GeV*

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A semiempirical formula which well represents the double-differential cross section for pion production in high-energy pp collisions is presented. The formula agrees with all data between 20 and 1500 GeV to within a factor of ~ 2 and is consistent with scaling and limiting fragmentation. Also presented is a set of momentum spectra at some interesting energies.

I. INTRODUCTION

To add to the numerous models[1] of particle production in high-energy collisions, such as the statistical model, the fireball model, the jet model, the multiperipheral model, the isobar model, the thermodynamic model, the diffraction-dissociation model, the emission model, etc., countless papers on inclusive reactions have been initiated by the appearance of the hypotheses of limiting fragmentation[2] and scaling.[3] However, in spite of all these efforts, because of the tremendous complexity of high-energy collisions, it remains impractical if not impossible to calculate the momentum and angular distribution of produced particles for "daily use" by means of these models.

A simple analytical expression which can represent particle production cross section is of great value for many purposes, such as planning experiments, designing high-energy secondary beams (pion, kaon, antiproton, muon, and even neutrino), performing radiation-shield calculations, and performing a Monte Carlo calculation of total absorption nuclear cascade detectors, to mention a few. Semiempirical formulas of Cocconi, Koester, and Perkins (CKP)[4] and of Trilling[5] for pion production proved to be quite useful during the design studies of multihundred-GeV proton synchrotrons, but they are generally considered rather qualitative. Eight-parameter formulas[6] give fairly quantitative representations of pion, kaon, and antiproton production between 10 and 35 GeV and have been widely in use. A five-parameter formula[7] represents the pion production in proton-nuclei collisions between 10 and 70 GeV well and may be extrapolated to 500 GeV.[8]

In this paper we present a four-parameter formula which fits the pion production data in pp collisions between 20 and 1500 GeV. The formula is consistent with scaling and limiting fragmentation and may therefore be extrapolated to much higher energies. Constant improvement of the formula, as new data become available, will of course increase the reliability at all energies. Nevertheless, even with the existing fragmentary data above 100 GeV, it is believed that the formula presented here should be reliable to within a factor of 2 over a large incident-energy region up to 1500 GeV. At 500 GeV, the highest energy of proton synchrotrons for years to come, the formula fits all data points to within a factor of 1.5.

II. DATA ON $p+p \to \pi^{\pm}$ + anything

The data utilized in the present analyses are from Allaby *et al.*[9] at 19.2 GeV/c, Anderson *et al.*[10] at 30 GeV/c, and Ratner *et al.*[11] and Bertin *et al.*[12] at the CERN Intersecting Storage Rings (ISR) corresponding to laboratory momenta of up to 1500 GeV/c.

To obtain the laboratory differential cross section from the ISR data, the center-of-mass (c.m.) differential cross section is obtained from the published invariant cross section I $(=E^* d^3\sigma/dp^{*3})$ by

$$\frac{d^2\sigma}{dp^* d\Omega^*} = \frac{p^{*2}}{E^*} I , \tag{1}$$

where p^* and E^* are c.m. momentum and energy of the pions produced. The corresponding laboratory quantities are then obtained by the Lorentz transformation. Thus

$$\frac{d^2\sigma}{dp\, d\Omega} = \frac{E^* p^2}{E p^{*2}} \frac{d^2\sigma}{dp^* d\Omega^*} , \tag{2}$$

which is simply Ip^2/E.

III. SCALING LAW AND EMPIRICAL FORMULA

That the invariant cross section I is independent of the incident energy but depends only on X $(=p_l/p_m$, where p_l is the longitudinal momentum of the pion produced, and p_m is the maximum kinematically allowed value of $p_l)$ and on p_t, the transverse momentum of the pion $(=p \sin\theta \approx p\theta$ where θ is the production angle), suggests a linear scaling law for the momentum spectrum, namely,

$$\frac{d^2\sigma}{dp\, d\Omega} = \frac{p^2}{E} I(X, p_t) , \tag{3}$$

and for high-energy secondaries, $p \approx E$, thus

$$\frac{d^2\sigma}{dp\,d\Omega} = p_m X I(X, p_t) \text{ (linear scaling law) .} \qquad (4)$$

Therefore, for given X and p_t the differential cross section scales linearly with p_m, or essentially the momentum of the incident proton. Equation (4) holds for any Lorentz frame of reference, and is identical with the scaling law obtained by Vander Velde[13] utilizing the hypothesis of limiting fragmentation.

Now the kinematics requires that the momentum spectrum vanish at $X=1$. Detailed data between 10 and 70 GeV show that the longitudinal-momentum distribution is well approximated by e^{-BX^C} with $C>1$,[6-8] and the transverse-momentum distribution follows closely e^{-Dp_t}.[4,6,7] Thus Eq. (4) can be well represented by

$$Y = \frac{d^2\sigma}{dp\,d\Omega} = A p_m X(1-X) e^{-BX^C - Dp_t} \,, \qquad (5)$$

where A is the normalization constant, B and C dictate the shape of the longitudinal-momentum spectrum, and D is related to the average transverse momentum.

IV. LEAST-SQUARES ANALYSES AND RESULTS

Least-squares analyses were carried out to determine the four parameters A, B, C, and D by minimizing the quantity

$$Q = \frac{1}{F} \sum_i \left(\frac{\log_{10} Y_i^e - \log_{10} Y_i^c}{\Delta Y_i^e / Y_i^e} \right)^2 , \qquad (6)$$

where the superscripts e and c refer to the experimental and calculated values, respectively, ΔY_i^e is the error pertaining to ith experimental datum, and F is the number of degrees of freedom. The result is shown in Table I. The Q's indicate that good fits were obtained.

The goodness of fit can be visualized in Figs. 1–3. Figure 1 shows the invariant cross section of π^- production plotted against X. The broken lines, calculated by Eq. (5) for 500 and 1500 GeV, are independent of the incident energy as expected by scaling and the distribution for limiting fragmentation. In Figs. 2 and 3 are plotted the differential cross sections of π^- and π^+ production at various incident momenta in a way convenient for "daily use."[14] To obtain the number of pions produced per GeV/c per sr, one simply divides the differential cross section by the pp absorption cross section.

TABLE I. Values of parameters.

Pion	A	B	C	D	Data	Q
Negative	51.403	5.732	1.333	4.247	274	0.91
Positive	77.793	3.558	1.333	4.727	210	1.23

V. DISCUSSION

As can be seen from the figures, Eq. (5) fits within a factor ~2 all data points varying six orders of magnitude in differential cross sections and two orders in incident energy. The empirical formula can thus be considered as a reasonably reliable representation of the differential cross sections.

At lower secondary momenta, the momentum spectra for different production angles are believed to cross over,[6] as evidenced by the pion production data in proton-nuclei collision. Such characteristics can be taken care of by replacing the term Dp_t by $D\theta(p - E\cos^F\theta)$ when data in the low-momentum region become available.

For pion production in pp collisions the data do support scaling and limiting fragmentation, thus leading to the linear scaling law [Eq. (4)]. However, there is evidence[12] that for antiprotons this is not the case, indicating that the limiting distribution may not have been reached even at 1500

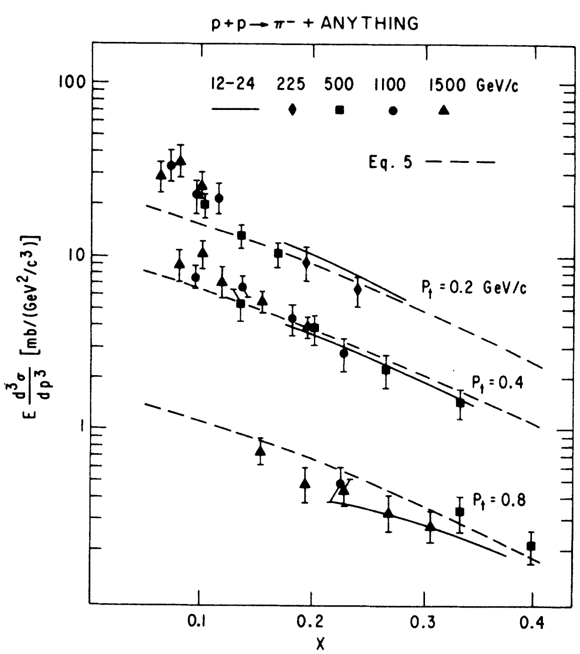

FIG. 1. Invariant cross section for $p+p\to\pi^-+$ anything vs X, showing the scaling behavior. The broken lines are calculated by Eq. (5). (See text.) Data are from Ref. 12.

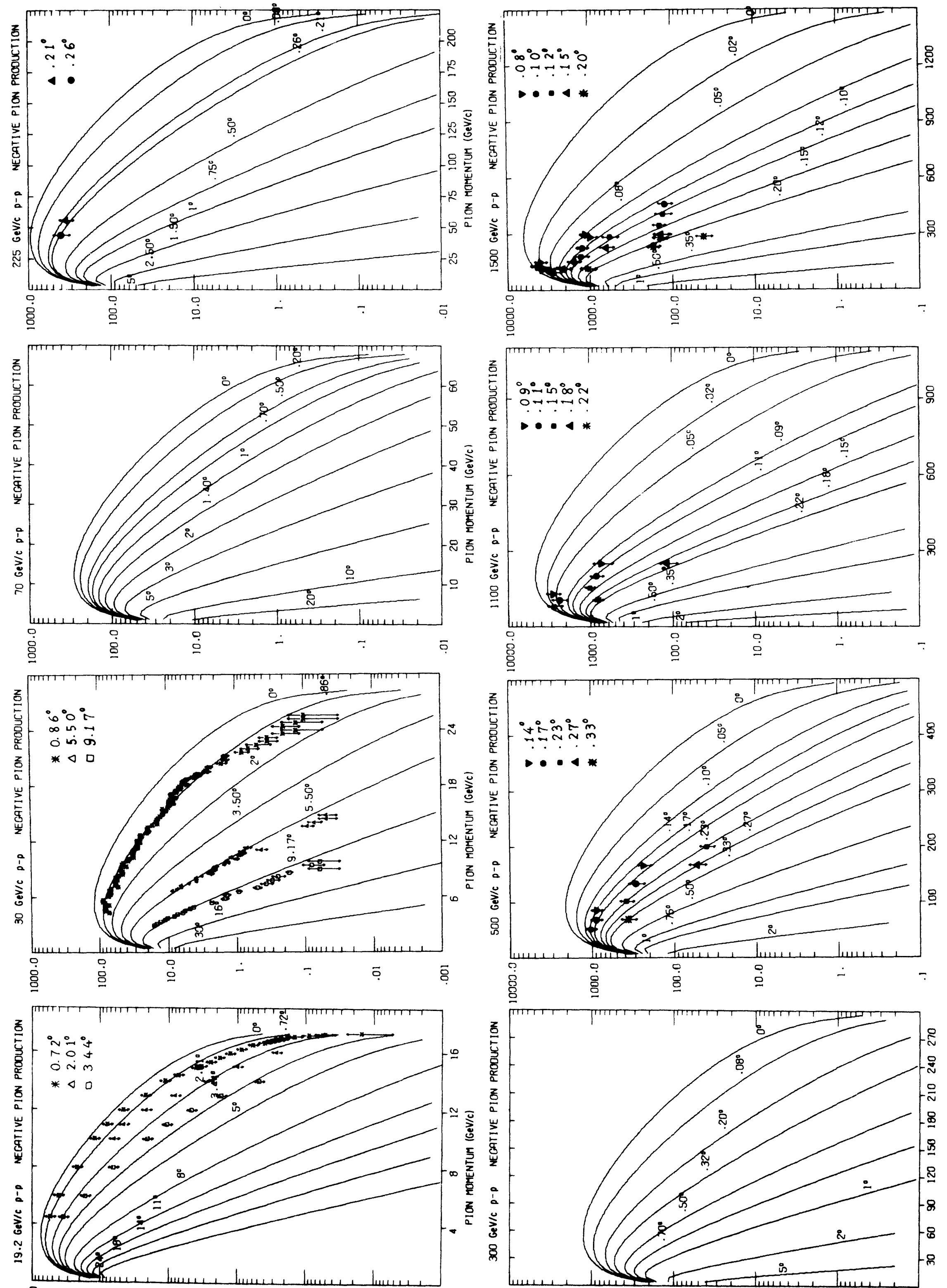

FIG. 2. Double-differential cross section for $p+p\rightarrow\pi^-+$ anything at 19.2, 30, 70, 225, 300, 500, 1100, and 1500 GeV/c calculated by Eq. (5). Data are from Refs. 9, 10, and 12.

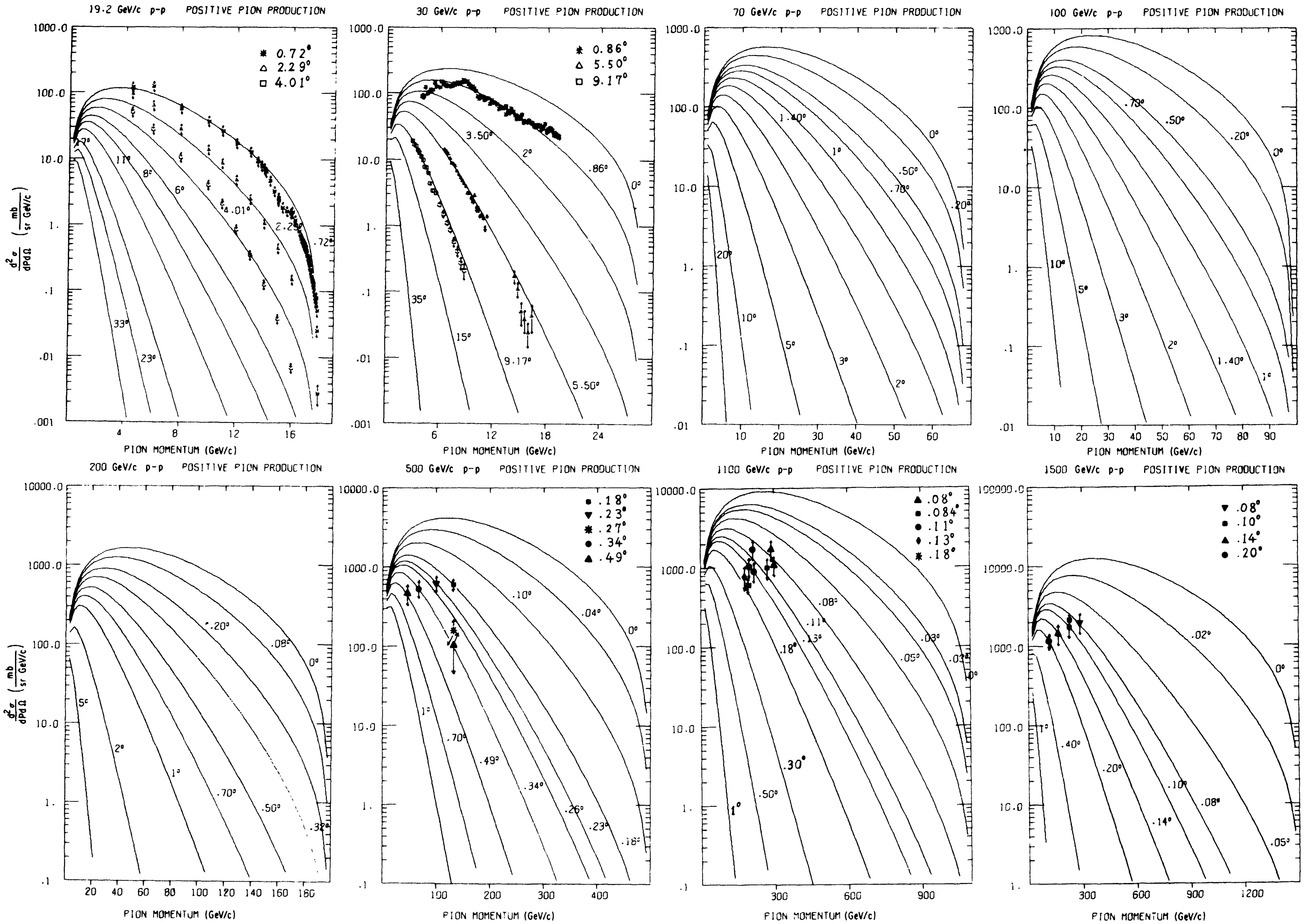

FIG. 3. Double-differential cross section for $p+p\to\pi^+ +$ anything at 19.2, 30, 70, 100, 200, 500, 1100, and 1500 GeV/c calculated by Eq. (5). Data are from Refs. 9, 10, 11, and 12.

GeV. It is worthwhile to point out that for pion production in p-nuclei collisions between 10 and 70 GeV, the data are more consistent with a square root rather than a linear scaling law.[7,8] It is not clear whether p-nuclei collisions behave differently from pp collisions or the normalization of those p-nuclei experiments[15] are really unreliable. A simple measurement of zero-degree π^- production from p-nuclei collisions as a function of incident energy should suffice to clarify the situation.

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