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There Is No Efimov Effect for Four or More Particles*

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(Received 5 December 1972)

We show that the existence of a zero-energy $(n-1)$-body bound state will not produce an infinite number of n-body bound states for $n \geq 4$. We show this in particular for $n=4$, and sketch the proof for $n>4$.

In some recent papers[1,2] we have investigated a remarkable property of the three-body system first suggested by Efimov[3] – namely that there are an infinite number of bound states of the three-body system when the two-body system has its first zero-energy bound state. This property, which we call the Efimov effect, is easily investigated by studying the homogeneous Faddeev equation,

$$\Psi = K\Psi \ , \tag{1}$$

where K is the kernel of that equation. This kernel depends on the total three-body energy E and on some coupling strength g. Bound states come at values of E and g for which (1) is satisfied, that is, at values of E and g for which K has a unit eigenvalue. All such bound states must come before the first scattering threshold in the problem. If $g < g_0$, where g_0 is the coupling that produces the first zero-energy two-body bound state, that threshold is $E=0$, but for $g > g_0$ it is $E=-b_2$, where b_2 is the binding energy of the most tightly bound two-body state. For $E<0$, the eigenvalues of K are, in general, monotonically increasing with decreasing $|E|$ (strictly we are discussing the S-wave projection of K but in fact much of the argument is true for the unprojected K). Hence an infinite number of bound states for $E \leq 0$ corresponds to an infinite number of eigenvalues of K being larger than unity. A necessary, but not sufficient, condition for this is that tr$K$ diverges as $E \to 0$, $g \to g_0$, since the trace is just the sum of these eigenvalues. It was just by investigating such traces that we proved the existence of the three-body Efimov effect.[1,2]

In this note we extend the analysis to the four-body problem (in detail) and then to a sketch of the argument for the n-body system. We show that the trace does not diverge in the corresponding situation there and hence that there are not an infinite number of bound states. What is the corresponding situation in the four-body case? It is not a zero-energy two-body bound state since if the force is strong enough to just produce a two-body bound state, it will produce a finite-energy three-body state with binding energy b_3. Four-body bound states must then have binding energy $b_4 > b_3$, and states near $E=0$, the four-body threshold, are not four-body bound states. No doubt the existence of a zero-energy two-body state and the corresponding existence of an infinite number of three-body states will produce a dreadful singularity in the four-body system at $E=0$, but it will not correspond to an infinite number of four-body bound states. It is that infinite number of states that we would consider to be a four-body Efimov effect. Hence we look to see what the consequences of a zero-energy three-body bound state are for the four-body system. In general we look to see what the consequences of the first $(n-1)$-body bound state are for the n-body system. In general it is difficult to write down n-body connected equations, but the contribution from the $n-1$ connected kernel is always easy to isolate and we shall assume that the pieces of the kernel coming from $n-2, n-3, \ldots$ amplitudes will not contribute to our n-body Efimov effect since these kernels are all bounded at zero energy if the force is only strong enough to produce a $(n-1)$-body zero-energy bound state. In general we shall find that the singularity of the $n-1$ connected amplitude at a zero-energy bound state is not strong enough to conspire with the n-body continuum and make the traces diverge. We show this first in some detail for the four-body system and then sketch the argument for the n-body case.

The following homogeneous equations for the four-body T matrix have a connected kernel (after iterating once)[4]:

$$T^{ij}=R^k_{ij}G_0(T^{il}+T^{jl}+T^{kl})+R^l_{ij}G_0(T^{ik}+T^{jk}+T^{kl})+(1-t_{ij}G_0t_{kl}G_0)^{-1}t_{ij}G_0t_{kl}(1+G_0t_{ij})G_0(T^{il}+T^{ik}+T^{jk}) . \quad (2)$$

T^{ij} is the contribution to T in which i-j is the "first" pair to interact, and t_{ij} is the two-body t matrix between particles i and j. R^k_{ij} is the connected part of the three-body T matrix, F^k_{ij}, in which i-j is the first pair to interact and k is the third particle involved.

$$F^k_{ij}=t_{ij}+t_{ij}G_0(F^j_{ik}+F^i_{jk})\equiv t_{ij}+R^k_{ij} . \quad (3)$$

There will be a four-body Efimov effect only if the trace of the four-body kernel of (2) diverges as E, the center-of-mass energy, goes to zero and there is a zero-energy three-body bound state. A two-body bound state at zero energy does not concern us, for the reasons discussed above. Clearly then we need only consider the parts of the kernel of (2) that contains R.

The relation of the matrix elements of $R^k_{ij}(E)$ in the four-body center-of-mass system to its three-body matrix element is given by

$$_4\langle\vec{p}\,\vec{q}\,\vec{r}|R^k_{ij}(E)|\vec{p}\,',\vec{q}\,',\vec{r}\,'\rangle_4$$
$$={}_3\langle\tfrac{1}{2}(\vec{p}-\vec{q}),\tfrac{1}{3}(2\vec{r}-\vec{p}-\vec{q})|R(E-\tfrac{2}{3}(\vec{p}+\vec{q}+\vec{r})^2)|\tfrac{1}{2}(\vec{p}\,'-\vec{q}\,'),\tfrac{1}{3}(2\vec{r}\,'-\vec{p}\,'-\vec{q}\,')\rangle_3\delta(\vec{p}+\vec{q}+\vec{r}-\vec{p}\,'-\vec{q}\,'-\vec{r}\,') . \quad (4)$$

($\hbar=m=1$; for simplicity we take all masses equal.) $|\vec{p}\vec{q}\vec{r}\rangle$ is the state where particles $ijkl$ have momenta $\vec{p}$, $\vec{q}$, $\vec{r}$, and $-(\vec{p}+\vec{q}+\vec{r})$, respectively. $|\vec{A},\vec{B}\rangle_3$ is the three-body state where the relative momentum of the pair that interacts first is $\vec{A}$, and $\vec{B}$ is the momentum of the other interacting particle relative to the three-particle center of mass.

The part of the kernel of (2) involving the R's has terms like

$$\frac{_3\langle\frac{1}{2}(\vec{p}-\vec{q}),\frac{1}{3}(2\vec{r}-\vec{p}-\vec{q})|R(E-\frac{2}{3}(\vec{p}+\vec{q}+\vec{r})^2)|\frac{1}{2}(p''-r''),-\frac{2}{3}\vec{q}\,''-\vec{p}\,''-\vec{r}\,''\rangle_3}{E-p''^2-q''^2-r''^2-\vec{p}\,''\cdot\vec{r}\,''-\vec{p}\,''\cdot\vec{q}\,''-\vec{q}\,''\cdot\vec{r}\,''}\,\delta(-\vec{q}-\vec{p}-\vec{r}+\vec{q}\,'') . \quad (5)$$

Other terms differ only in the exact combinations of $\vec{p}$, $\vec{q}$, and $\vec{r}$ that appear in the bras and kets. The trace of the above term is

$$\int d^3p\,d^3q\,\frac{_3\langle\frac{1}{2}(\vec{p}-\vec{q}),-\vec{p}-\frac{1}{3}\vec{q}\,|R(E-\frac{2}{3}q^2)|\,\vec{p},-\frac{2}{3}\vec{q}\rangle_3}{E-p^2-q^2} . \quad (6)$$

We wish to see if this trace diverges as $E\to 0$ for an R corresponding to a three-body bound state at zero energy. We are looking for a divergence associated with small p and q (it is an infrared divergence or long-range effect). We expect that for a finite-range force, $\langle\vec{A}\vec{B}\,|R(\epsilon)|\vec{A}'\vec{B}'\rangle$ is bounded as $\vec{A},\vec{B},\vec{A}',\vec{B}'\to 0$. Hence any divergence will come from the singularity of R at $\epsilon=0$, when there is a zero-energy bound state. In fact, we would expect any Efimov effect to arise from a coincidence of this singularity with the singularity of the four-body propagator in the trace, represented here by the term $(E-p^2-q^2)^{-1}$. This coincidence of singularities corresponds to the coincidence of the four-body threshold, represented by the propagator, and the threshold for the scattering of one from a bound state of the three others, represented by the singularity of R. These coincide at $E=0$ when the three-body bound state has zero energy. Assuming that, for small ϵ, $R(\epsilon)$ can be written

$$R(\epsilon)\sim c/\epsilon^y , \quad (7)$$

the trace, for small E,p,q, is proportional to

$$\int\frac{p^2q^2dp\,dq}{(E-p^2-q^2)(E-\frac{2}{3}q^2)^y} ; \quad (8)$$

for finite p, the q integral will be logarithmically divergent as $E\to 0$ if $y=\frac{3}{2}$. In order that the propagator and the ϵ^{-y} singularity coincide as $E\to 0$ we need $y=2$. We shall now show that in fact $y=1$ or $\frac{1}{2}$.

Consider the three-body D function, or Fredholm denominator. The singularity of R in ϵ comes from its zero at the three-body bound state, since $R(\epsilon) \sim D^{-1}(\epsilon)$. D can be written

$$D(\epsilon) = F(\epsilon) + I(\epsilon) , \quad (9)$$

where $F(\epsilon)$ is real and $I(\epsilon)$ is the imaginary part of D for $\epsilon > 0$. $I(E)$ is real for $\epsilon < 0$. Since D is a real analytic function, $F(\epsilon)$ is analytic at $\epsilon = 0$ and hence near $\epsilon = 0$

$$F(\epsilon) = A + B\epsilon + C\epsilon^2 + \cdots . \quad (10)$$

A zero-energy bound state corresponds to $A = 0$. The leading small-ϵ behavior of I will be $\sqrt{-\epsilon}$ from the two-particle subenergy cut as we have shown in a previous paper.[5] This singularity cancels in the on-shell bound-state three-body amplitude, that is, in the amplitude for the scattering of one from a bound state of the other two, but the $\sqrt{-\epsilon}$ remains in the three-body D. It may not be there if, for example, there is no two-body force, only a three-body force; then I can go like $\epsilon^2 \ln(-E)$, but then the $B\epsilon$ term in $F(\epsilon)$ will dominate for small ϵ. Hence $y = \frac{1}{2}$ or 1 for the three-body case, and, as is easily seen for the n-body case as well, $y = \frac{1}{2}$ is enough to make the trace diverge for the three-body Efimov effect but with four particles it cannot. This is essentially a phase-space argument. We are assuming that if A vanishes in (10) and the $\sqrt{-\epsilon}$ is absent, B does not vanish. Cases in which B vanishes when A does are very special "accident" cases that would have to be treated separately.

We have seen that the singularity of the connected three-body amplitude at a zero-energy three-body bound state is not enough to make the connected four-body kernel have divergent trace, and hence to produce an Efimov effect. It is easy to generalize this argument to the n-body system. Let us again call R_{n-1} the $n-1$ connected amplitude. Again we only study the part of the homogeneous equation involving R_{n-1} since we are only interested in the singularity arising from an $(n-1)$-body bound state at zero energy. The trace of a typical term involving R_{n-1} in the Faddeev-like connected integral equation for the n-body amplitude can be written

$$\int \frac{R_{n-1}(E - [n/2(n-1)]q^2)\, d^3q\, d^3p_1 \cdots d^3p_{n-3}}{E - P_2(\vec{q}, \vec{p}_i)} , \quad (11)$$

where P_2 is a second-order polynomial in the p's and q's and $E - P_2$ is the propagator. In writing (11) we have already assumed that in

$$\langle \vec{A}, \vec{B}, \ldots | R_{n-1}(\epsilon) | \vec{A}', \vec{B}' \ldots \rangle$$

only the small-ϵ behavior can be unbounded and hence have suppressed the dependence on $\vec{A}, \vec{B}, \ldots, \vec{A}', \vec{B}'$. We see that if $R \sim \epsilon^{-y}$, the integral of (11) will diverge for $y = \frac{3}{2}$ for fixed p but requires $y = \frac{3}{2}n - 4$ in order to have the ϵ^{-y} singularity and that of the propagator coincidence. However we showed above that $y = \frac{1}{2}$ or 1 for $n \geq 3$, and hence there is no divergence. The essential point here is that in the part of the equation containing the $n-1$ connected kernel there are $n-2$ free momenta and one propagator. The dimension of the propagator is always the same, but each additional particle gives three more powers of momentum in the numerator. Since the R's do not diverge more and more strongly at an $(n-1)$-body zero-energy bound state the integrals are finite for small ϵ and small momenta for $n > 3$.

One might be concerned that our argument is based only on tr$K$. What if trK^2 diverges? It is easy to show that in fact it does not. It is also clear that if an infinite number of eigenvalues of K exceed 1 we expect tr$K$ and trK^2 to diverge. It is of course possible that K is not a positive definite operator and the positive and negative eigenvalues cancel. But looking at trK^2 or making a partial-wave projection removes this difficulty, and it is easy to see that the partial-wave projection will not change what are essentially dimensional arguments.

In conclusion we have shown that for the four-body system, the singularity of the connected three-body amplitude corresponding to a three-body bound state at zero energy is not strong enough to make the trace of the four-body scattering equation diverge at zero energy and hence to produce an infinite number of four-body bound states. For the n-body case, $n > 4$, the situation is even less singular. Hence the remarkable Efimov effect seems even more remarkably to be a property of the three-body system only.

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*Supported in part by the National Science Foundation.

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[2]S. K. Adhikari and R. D. Amado, Phys. Letters 40B, 11 (1972).

[3]V. Efimov, Phys. Letters 33B, 563 (1970); Yad. Fiz. 12, 1080 (1970) [Sov. J. Nucl. Phys. 12, 598 (1971)].

[4]Cf. R. D. Amado, in *Elementary Particle Physics and Scattering Theory*, edited by M. Chrétien and S. S. Schweber (Gordon and Breach, New York, 1970), Vol. 2, pp. 120–123.

[5]S. K. Adhikari and R. D. Amado, Phys. Rev. C 6, 1484 (1972).