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Canonical Trace Anomalies*

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We discuss the anomalies present in broken-scale-invariance trace identities which result from assuming that products of hadronic currents have a canonical singularity structure at short distances. The analysis is performed qualitatively in configuration space and quantitatively in momentum space. Canonical anomalies are found in trace identities involving two electromagnetic currents, or two axial-vector currents or their divergences. There are related canonical anomalies in trace identities involving three or four currents. They can be represented by the anomalous trace equation $\theta_\lambda^{\lambda\,\text{anomalous}}(x)=\theta_\lambda^\lambda(x)+(R/32\pi^2)\tilde{F}^i_{\mu\nu}\tilde{F}_i^{\mu\nu}$, where $\tilde{F}^i_{\mu\nu}=\partial_\mu F^i_\nu-\partial_\nu F^i_\mu+h_{ijk}F^j_\mu F^k_\nu$, with the F^i_μ external fields coupled to the SU(3)×SU(3) currents, and h_{ijk} the structure constants of SU(3)×SU(3). The electromagnetic current trace anomaly is related to the high-energy cross section for $e^-e^+\to\gamma\to$ hadrons, and via P0T (partially zero trace) to the coupling of a scalar meson to photons. These are connected by

$$(12\pi^2F_\sigma)g_{\sigma\gamma\gamma}=R=\lim_{p^2\to\infty}\frac{\sigma(e^-e^+\to\gamma\to\text{hadrons})}{\sigma(e^-e^+\to\gamma\to\mu^-\mu^+)},$$

where F_σ is defined by $\langle 0|\theta^\mu_\mu|\sigma\rangle=m_\sigma{}^2F_\sigma$. The axial-vector current anomalies are related to the high-energy cross sections for $e^-\bar{\nu}_e\,(\mu^-\bar{\nu}_\mu)\to$ hadrons; they do not affect previous estimates of the $\sigma\pi\pi$ coupling made using broken scale invariance and P0T.

I. INTRODUCTION

The general consequences of exact and approximate symmetries of field theories can be expressed in terms of Ward identities relating different Green's functions. Considerable attention has been paid to the Ward identities associated with the SU(3)×SU(3) current algebra and, more recently, to the trace and conformal identities associated with scale invariance.[1,2] It is however well known that in perturbation theory Ward identities may acquire anomalies, because of singularities which render naive manipulations invalid. Adler succeeded in understanding the axial-vector-current anomaly in the context of perturbation theory,[3] and Callan and Symanzik used perturbation theory to demonstrate the existence of anomalies in the trace identities of scale invariance.[4,5]

However, it is not at all clear that perturbation theory is relevant to the physics of the hadronic currents and the hadronic stress tensor. On the contrary, the logarithmic corrections which are found in perturbation theory to violate scaling in deep-inelastic electron scattering seem either to be absent from the data or to be small.[6] So we have the question: Which of the results of perturbation theory should be believed? In particular, do the anomalies which occur in perturbation theory actually occur in the Ward identities of hadronic physics?

One framework for answering the latter question has been provided by Wilson,[1] who showed that the axial-vector-current anomalies can be regarded as consequences of the short-distance singularity structure of current products. If one has a model for such short-distance behavior, one can deduce what anomalies exist and how they are interrelated. The deep-inelastic scattering experiments suggest that the light-cone singularities of current commutators resemble those of a canonically manipulated field theory with charged fermions and uncharged boson gluons.[7] It is natural to extend this model to other short-distance singularities of current products, including disconnected parts. What Ward-identity anomalies occur in such a canonical model?

As pointed out by Wilson,[1] the axial-vector-current anomaly does occur in such a canonical model, so it might be called a "canonical anomaly" to distinguish it from other anomalies which appear in perturbation theory and reflect deviations from canonical short-distance behavior. An interesting question is whether there are also canonical anomalies among the Ward identities of broken scale invariance. The Callan-Symanzik anomalies[4] are associated with the fact that in perturbation theory canonical singularities are modified by logarithmic factors; accordingly, their presence is excluded by the assumption of canonical singularity structure. However, there are other anomalies in trace identities: For example, there is one in the trace identity involving Green's functions with two electromagnetic currents.[5] It will be shown that this and certain other anomalies are to be expected on the basis of canonical singularity structure,[8,9] i.e., that they are canonical anomalies in trace identities.

In this paper we first discuss, on the basis of Wilson's simple power-counting arguments,[1] what trace identities are vulnerable to anomalies, showing in particular that the trace identity for two electromagnetic currents may be expected to break down. We then calculate the anomalies in this trace identity and that involving two axial-vector currents or axial-vector-current divergences using a model for short-distance singularities based on fundamental fermion and boson fields ("partons").[10]

The calculations are performed in momentum space[9]; since the anomalies are determined by the short-distance singularity structure, all models with the same behavior in this region have the same anomalies. Further, lowest-order perturbation-theory graphs have the canonical singularity structure. Hence the anomalies found on inserting lowest-order graphs into the trace identities will be the same as those which would obtain in the real world if the postulated canonical singularity structure is correct. And it is easier to calculate some simple Feynman graphs than to perform the configuration-space analysis. It is emphasized that the use of perturbation theory is just an algorithm, and in the theoretical context outlined above higher-order calculations are meaningless.

We also study the trace identities involving more than two currents. Canonical anomalies appear in trace identities involving three or four currents, and are related by current algebra to the two-current anomalies. A compact representation for all the canonical trace anomalies is the equation

$$\theta_\lambda^{\lambda\,\text{anomalous}}(x)=\theta_\lambda^\lambda(x)+\frac{R}{32\pi^2}\tilde{F}^{\,i}_{\mu\nu}\tilde{F}^{\mu\nu}_{i}\,,$$

where $\tilde{F}^{\,i}_{\mu\nu}=\partial_\mu F^{\,i}_\nu-\partial_\nu F^{\,i}_\mu+h_{ijk}F^{\,j}_\mu F^{\,k}_\nu$, with the $F^{\,i}_\mu$ external fields coupled to the SU(3)×SU(3) currents, and h_{ijk} the structure constants of SU(3) ×SU(3). R is related to the charges of the fundamental constituent fields,

$$R=\sum_{\substack{i\\ \text{spin}\ \frac{1}{2}}}Q_i^{\,2}+\tfrac{1}{4}\sum_{\substack{i\\ \text{spin}\ 0}}Q_i^{\,2}\,.$$

The phenomenological consequences of these anomalies are then discussed. The θ^μ_μ-J_λ-J_ν anomaly is shown to be proportional to the coefficient of $1/q^2$ in the high-energy total cross section for $e^-e^+\to\gamma\to$ hadrons, and the θ^μ_μ-A^+_λ-A^-_ν anomaly is connected with the high-energy cross sections for $\bar{e}\nu_e$ $(\bar{\mu}\nu_\mu)\to$ hadrons. These applications just require the trace θ^μ_μ of the hadronic energy-momentum tensor to be "soft," as postulated in theories of broken scale invariance. If it is further assumed [PCDC[11] (partially conserved dilation current) or P0T[12] (partially zero trace)] that there is a scalar meson σ that dominates matrix elements of θ^μ_μ, then the anomalous trace identities can be used to study its couplings to photons and to pions.

The coupling to photons is found to be proportional to the anomaly and hence to the total cross section for $e^+e^-\to\gamma\to$ hadrons:

$$g_{\sigma\gamma\gamma}=\frac{R}{12\pi^2F_\sigma}\,, \tag{1.1}$$

where

$$R\equiv\lim_{s\to\infty}\frac{\sigma(e^+e^-\to\gamma\to\text{hadrons})}{\sigma(e^+e^-\to\gamma\to\mu^+\mu^-)}\,; \tag{1.2}$$

the coupling $g_{\sigma\gamma\gamma}$ is defined by the Lagrangian

$$\mathcal{L}_{\sigma\gamma\gamma}=-\tfrac{1}{2}e^2g_{\sigma\gamma\gamma}\sigma F_{\mu\nu}F^{\mu\nu}\,, \tag{1.3}$$

where $F_{\mu\nu}$ is the electromagnetic field, $\alpha=e^2/4\pi$ is the fine-structure constant, and F_σ is the scale analog of the pion decay constant F_π, estimated to be of order 150 MeV. The important point is that $g_{\sigma\gamma\gamma}$ is thus predicted to be rather small, provided that R is of order unity or smaller. In the model for short-distance and light-cone behavior based on three triplets of fractionally charged quarks, which is so far consistent[7] with experiment, R would have the value 2. For $m_\sigma\cong 700$ MeV and $\Gamma(\sigma\to\pi\pi)\cong 400$ MeV, for example, we get $\Gamma(\sigma\to\gamma\gamma)\cong 0.2\,R^2$ keV. This small $\sigma\gamma\gamma$ coupling means that in the two-photon process, $e^\pm e^\pm\to e^\pm e^\pm+$hadrons, such a scalar isoscalar dipion resonance would make a small contribution to the cross section, over an order of magnitude smaller than the Born approximation cross section for $\gamma\gamma\to\pi\pi$. (Of course, by Watson's theorem, the resonance would nevertheless be detectable in the s-wave phase shift.)

Other estimates[13,14] of $g_{\sigma\gamma\gamma}$ have tended to be considerably larger; if these turn out to be experimentally valid, and $e^+e^-\to\gamma\to$ hadrons scales as $1/q^2$ with a coefficient of order unity so that (1.1) is badly violated, this would be good evidence against the utility of the PCDC[11] or POT[12] hypothesis. This is a test of PCDC or POT analogous to the use of Crewther's relation[8] for $\Gamma(\pi^0 \to \gamma\gamma)$ as a test of PCAC.[15] It is a more limited test, since it can only be used to disprove PCDC or POT: If $g_{\sigma\gamma\gamma}$ is indeed given by (1.1) with $R \sim O(1)$, then it will probably be too small to separate from the nonresonant background in $\gamma\gamma \to \pi\pi$.

The $\theta^\mu_\mu - \partial_\lambda A^\lambda - \partial_\nu A^\nu$ trace anomaly was neglected by previous authors[11,12] in scale-invariance calculations of a large value for $\Gamma(\sigma \to \pi\pi)$. We find that their results are unaffected by the anomaly.[16]

II. ANALYSIS IN CONFIGURATION SPACE

Naively we expect a simple Ward identity to relate the vertex function

$$\Delta_{\mu\nu}(p, -p) \equiv \int d^4x\, d^4y\, e^{ip\cdot y} \langle T^*(\theta^\lambda_\lambda(x) J_\mu(y) J_\nu(0))\rangle_\Omega \tag{2.1}$$

to the vacuum-polarization tensor

$$\Pi_{\mu\nu}(p, -p) \equiv i \int d^4x\, e^{ip\cdot x} \langle T^*(J_\mu(x) J_\nu(0))\rangle_\Omega . \tag{2.2}$$

When $\theta^{\mu\nu}$ is the "improved" stress energy tensor of Callan, Coleman, and Jackiw,[17] its trace is

$$\theta^\mu_\mu(x) = \partial_\mu D^\mu(x), \tag{2.3}$$

where $D^\mu(x)$ is the dilation current. Hereafter we will often write $\theta^\mu_\mu \equiv \theta$. The integrated dilation charge, $D(x_0) \equiv \int d^3x\, D^0(\vec{x}, x_0)$, defines the scale dimension d of a field ϕ by the commutation relation

$$[D(x_0), \phi(\vec{x}, x_0)] = -i(x\cdot\partial + d)\phi(x) . \tag{2.4}$$

If we use Eq. (2.3) in Eq. (2.1), integrate by parts, and neglect any possible complications due to the presence of surface terms, then the resulting expression may be evaluated using the equal-time commutation relation (2.4). Assuming asymptotic scale invariance, the scale dimension of both space and time components of J^μ is three, and we obtain the trace identity,

$$\Delta_{\mu\nu}(p, -p) = \left(2 - p\cdot \frac{\partial}{\partial p}\right)\Pi_{\mu\nu}(p, -p) . \tag{2.5}$$

We will show below that relation (2.5) actually fails in any theory possessing asymptotic scale invariance as proposed by Wilson.[1] To understand why the argument leading to (2.5) breaks down, we shall extend Wilson's analysis of the axial-vector anomaly to the case under consideration here. Wilson's analysis offers qualitative insight into how Ward-identity anomalies arise from a configuration-space point of view. It provides a necessary, though not sufficient, criterion for the existence of canonical anomalies. We will see that the integration by parts may give rise to nontrivial surface terms in the derivation of Ward identities involving three or more operators of sufficiently large scale dimension. The anomalies are just these nontrivial surface contributions.

Consider first a Ward identity which relates a two-point function to a one-point function, e.g.,

$$\int d^4x \langle T^*(\theta(x)\phi(0))\rangle_\Omega = id\langle \phi(0)\rangle_\Omega . \tag{2.6}$$

Equation (2.6) is obtained by the same argument which led to (2.5). The equal-time commutator (2.4) arises when we integrate by parts and the derivative ∂_μ acts on the step functions $\theta(t)$ which appear in the definition of the time-ordered product,[18] i.e.,

$$T^*(\theta(x)\phi(0)) = \theta(x_0)\theta(x)\phi(0) + \theta(-x_0)\phi(0)\theta(x) .$$

Since we are concerned with possible complications due to surface terms at the origin, where the operator product is singular, we shall follow a more careful procedure to obtain (2.6). We write the left-hand side of (2.6) as

$$\lim_{\epsilon\to 0+}\left[\int_\epsilon^\infty dx_0 \int d^3x\, \theta(x)\phi(0) + \int_{-\infty}^{-\epsilon} dx_0 \int d^3x\, \phi(0)\theta(x)\right]. \tag{2.7}$$

Following Wilson,[1] we simplify the analysis by adopting a Euclidean space-time metric, O(4), so that the light-cone singularity collapses to the origin. Then the integrands in (2.7) are finite and unambiguous as long as we keep $\epsilon > 0$, and we may study in a well-defined way their behavior as $\epsilon \to 0+$. Now when we use Eq. (2.3) and integrate by parts, we find just the contribution of the surface at $\pm\epsilon$ and $\pm\infty$. In the absence of massless scalar particles, the surfaces at $\pm\infty$ cannot contribute. From the surfaces at the origin we find

$$\lim_{\epsilon\to 0+} \{-D(\epsilon)\phi(0) + \phi(0)D(-\epsilon)\}, \tag{2.8}$$

which is just the equal-time commutator (2.4), and we obtain precisely the relation (2.6). The more careful analysis has in this case only confirmed the usual result. Thus Wilson's analysis shows that canonical anomalies cannot occur in Ward identities involving products of only two operators.

But let us now consider the three-point function (2.1). Before we can proceed with the configuration-space analysis, we must make a simple kinematical observation. Equations (2.1) and (2.2) must have the gauge-invariant forms

$$\Delta_{\mu\nu}(p,-p)=(p_\mu p_\nu-g_{\mu\nu}p^2)\Delta(p^2), \tag{2.9}$$

$$\Pi_{\mu\nu}(p,-p)=(p_\mu p_\nu-g_{\mu\nu}p^2)\Pi(p^2). \tag{2.10}$$

The naive Ward identity (2.5) is then

$$\Delta(p^2)=-2p^2\frac{\partial}{\partial p^2}\Pi(p^2). \tag{2.11}$$

Now concentrate on the case when the currents are on the photon mass shell, $p^2=0$. Then we can write

$$\Delta(0)=-\tfrac{1}{12}\int d^4x\,d^4y\,y^\mu y^\nu e^{-ip\cdot y}\times\langle T^*(\theta(x)J_\mu(y)J_\nu(0))\rangle_\Omega. \tag{2.12}$$

Notice that because of gauge invariance, the short-distance singularity has been softened by two powers of y. This is just the configuration-space analog of the fact, well known in perturbation theory, that one power of convergence is gained for each gauge-invariant vertex.

We now define the integrand in (2.12) by again adopting the O(4) metric and excluding the singularities with the restrictions

$$|x_0|\geq\epsilon,\quad |y_0|\geq\delta,\quad |x_0-y_0|\geq\eta. \tag{2.13}$$

We use Eq. (2.3) and integrate by parts, with the result

$$\begin{aligned}\Delta(0)=\tfrac{1}{12}\Big[&\int_\delta^\infty dy_0\int d^3y\,e^{ip\cdot y}y^\mu y^\nu\langle D(y_0+\eta)J_\mu(y)J_\nu(0)-J_\mu(y)D(y_0-\eta)J_\nu(0)\\&\qquad+J_\mu(y)D(\epsilon)J_\nu(0)-J_\mu(y)J_\nu(0)D(-\epsilon)\rangle_\Omega\\&+\int_{-\infty}^{-\delta}dy_0\int d^3y\,e^{ip\cdot y}y^\mu y^\nu\langle D(\epsilon)J_\nu(0)J_\mu(y)-J_\nu(0)D(-\epsilon)J_\mu(y)\\&\qquad+J_\nu(0)D(y_0+\eta)J_\mu(y)-J_\nu(0)J_\mu(y)D(y_0-\eta)\rangle_\Omega\Big].\end{aligned} \tag{2.14}$$

First consider the integrand when $|y_0|$ is large, i.e., $|y_0|\gg\epsilon,\delta,\eta$. Then the terms combine in pairs to form equal-time commutators, as in the example of the two-point function considered above. This is what we expect from the naive manipulations which yield (2.5) and (2.12).

Next consider the region for which $|y_0|\cong\delta$. If the first two terms are to associate unambiguously into an equal-time commutator, the products must be evaluated in the order

$$(D(y_0+\eta)J_\mu(y))J_\nu(0)-(J_\mu(y)D(y_0-\eta))J_\nu(0), \tag{2.15}$$

so the operator must be defined by taking the limit $\eta\to0$ before $\delta\to0$, i.e., we must have $\delta\gg\eta$. In the region of integration defined by

$$y_0\cong\delta\gg\eta, \tag{2.16}$$

the two terms in (2.15) become an equal-time commutator. Similarly the third and fourth terms combine to form an equal-time commutator in the region

$$y_0\cong\delta\gg\epsilon. \tag{2.17}$$

So with the choice $\delta\gg\eta,\epsilon$ all terms become equal-time commutators. Taking the limits[19] $\eta\to0$ and $\epsilon\to0$ and using Eq. (2.4), we find

$$\begin{aligned}\Delta(0)=-\tfrac{1}{12}i\Big[&\int_\delta^\infty dy_0\int d^3y\,e^{ip\cdot y}y^\mu y^\nu\Big(6+y\cdot\frac{\partial}{\partial y}\Big)\langle J_\mu(y)J_\nu(0)\rangle_\Omega\\&+\int_{-\infty}^{-\delta}dy_0\int d^3y\,e^{ip\cdot y}y^\mu y^\nu\Big(6+y\cdot\frac{\partial}{\partial y}\Big)\langle J_\nu(0)J_\mu(y)\rangle_\Omega\Big].\end{aligned} \tag{2.18}$$

Integrating by parts once more, we find

$$\Delta(0)=\Big[-2p^2\frac{\partial}{\partial p^2}\Pi(p^2)\Big]_{p^2=0}+\Delta_s(0), \tag{2.19}$$

where $\Delta_s(0)$ is the surface term,

$$\Delta_s(0) \equiv -\tfrac{1}{12} i \left\{ \int_\delta^\infty dy_0 \int d^3y \, \frac{\partial}{\partial y_\alpha} [e^{ip\cdot y} y^\mu y^\nu y_\alpha \langle J_\mu(y) J_\nu(0) \rangle_\Omega] + \int_{-\infty}^{-\delta} dy_0 \int d^3y \, \frac{\partial}{\partial y_\alpha} [e^{ip\cdot y} y^\mu y^\nu y_\alpha \langle J_\nu(0) J_\mu(y) \rangle_\Omega] \right\}. \tag{2.20}$$

Comparing (2.19) with the naive trace identity (2.11), we see that $\Delta_s(0)$ is the anomalous contribution. To see whether $\Delta_s(0)$ may be nonzero, we use the assumption of scale invariance at short distances, according to which the product of two currents has a c-number singularity proportional to δ^{-6}. Since the numerator of the integrand is proportional to δ^6, we see it is indeed possible that $\Delta_s(0)$ may make a finite, nonvanishing contribution to the trace identity.

The reader who studies Wilson's configuration-space analysis[1] of the V-V-A anomaly[3] will find that the V-V-A anomaly arises in a different way from the trace anomaly. The V-V-A anomaly comes from the region of configuration space where the three currents are all "pinching" against each other, to give a δ^{-9} singularity. We have just seen that the trace anomaly is due to the δ^{-6} singularity of the two electromagnetic currents alone.

The configuration-space analysis gives a simple necessary condition for deciding whether other scale-invariance Ward identities have canonical anomalies. Consider a vertex consisting of θ, and n currents $J^i_{\mu_i}$:

$$\int \prod_{i=1}^n d^4y_i \exp\Big(i \sum_{i=1}^n p_i y_i \Big) \Big\langle T^*\Big(\theta(0) \prod_{i=1}^n J^i_{\mu_i}(y_i) \Big) \Big\rangle_\Omega .$$

As in (2.12), the quantities which may have anomalous surface-term contributions are of the form

$$\Delta_{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_r} = \int \prod_{i=1}^n d^4y_i \prod_{j=1}^r y_{i\nu_j} \Big\langle T^*\Big(\theta(0) \prod_{i=1}^n J^i_{\mu_i}(y_i) \Big) \Big\rangle_\Omega , \tag{2.21}$$

where the factors y_{ν_j} correspond to momenta in the anomalies. Carrying through the analysis as before, we exclude the regions $|y_i^0| \ge \delta_i$ and $|y_i^0 - y_j^0| \ge \eta_{ij}$ from the integrations in Eq. (2.21). As before, we may take the limits $\eta_{ij} \to 0$ and pick up equal-time commutator terms. The anomalies are then proportional to sums of quantities of forms analogous to Eq. (2.20):

$$(\Delta_s)_{\mu_1 \cdots \mu_n \nu_1 \cdots \nu_r} \propto \int_{|y_i^0| \ge \delta_i} \prod_{i=1}^{n-1} d^4y_i \frac{\partial}{\partial y_1^\alpha} \Big\{ y_1^\alpha \prod_{j=1}^r y_{i\nu_j} \Big\langle T^*\Big(\prod_{i=1}^{n-1} J^i_{\mu_i}(y_i) J^n_{\mu_n}(0) \Big) \Big\rangle_\Omega \Big\} .$$

These quantities may have nonzero terms coming from the surfaces $|y_i^0| = \delta_i$ if (counting powers of the y_i)

$$4n - 4 + r - 3n \le 0 .$$

Since $r \ge 0$, this condition reduces to $n \le 4$. In other words, there may be canonical anomalies in trace identities involving not more than four currents. The two-current anomalies will be discussed in Secs. III and IV, and three- and four-current anomalies in Sec. V.

III. MOMENTUM-SPACE ANALYSIS OF TRACE IDENTITY FOR TWO ELECTROMAGNETIC CURRENTS

The configuration-space analysis suggests that the naive Ward identity may fail provided there is scale invariance at short distances and the currents have dimension three. Any model satisfying those conditions must be examined in detail to determine whether an anomalous contribution is indeed present and to find its value. This is particularly simple in the class of models which assume that the leading short-distance and light-cone singularities of products of hadronic currents are given by simple canonical theories. These models are popular because they seem to provide a correct zeroth-order picture of scaling in deep-inelastic electron scattering.[7]

Consider a model in which the fundamental constituents are a collection of spin-$\frac{1}{2}$ fields $\psi_i(x)$ corresponding to particles of charge eQ_i and mass m_i. The electromagnetic current is

$$J^\mu(x) = \sum_i Q_i \overline{\psi}_i(x) \gamma^\mu \psi_i(x) \tag{3.1}$$

and the trace of the stress tensor is

$$\theta(x) = \sum_i m_i \overline{\psi}_i(x) \psi(x) . \tag{3.2}$$

Our configuration-space analysis has shown that the possible "anomalous" contribution to the naive Ward identity, (2.5), arises from the leading singularity when the space-time interval between the two electromagnetic currents approaches zero. Then all we have to do is calculate both sides of

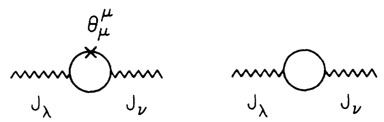

FIG. 1. Lowest-order contributions to $\Delta_{\lambda\nu}$ and $\Pi_{\lambda\nu}$ in models with fundamental constituent fields. These graphs are denoted $\Delta^c_{\lambda\nu}$ and $\Pi^c_{\lambda\nu}$ in the text.

(2.5) at the canonical level, which means that we calculate the lowest-order triangle and vacuum-polarization diagrams (Fig. 1). Any difference between the left- and right-hand sides is then due to canonical singularities, which according to our hypothesis are the same as those in nature.

Let us make one thing perfectly clear: This procedure does not imply any commitment to the view that perturbation theory is a reliable guide to the physics of hadronic currents. We are certainly not asserting that the hadronic Green's functions $\Delta^{\mu\nu}$ and $\Pi^{\mu\nu}$ are equal to their canonical counterparts $\Delta_c^{\mu\nu}$ and $\Pi_c^{\mu\nu}$ which are given by lowest-order perturbation theory. Rather we are asserting, on the basis of the hypothesis of canonical singularities and the configuration-space analysis of Sec. II, that the hadronic anomaly is equal to the canonical anomaly, i.e., that

$$\Delta^{\mu\nu}(p,-p)-\left(2-p\cdot\frac{\partial}{\partial p}\right)\Pi^{\mu\nu}(p,-p) = \Delta_c^{\mu\nu}(p,-p)-\left(2-p\cdot\frac{\partial}{\partial p}\right)\Pi_c^{\mu\nu}(p,-p). \tag{3.3}$$

So our assumptions simply require that we evaluate $\Delta_{\mu\nu}$ and $\Pi_{\mu\nu}$ in lowest-order perturbation theory, with J_μ and θ given by (3.1) and (3.2). The lowest-order diagrams are shown in Fig. 1. The triangle diagrams have a superficial linear divergence but are in fact finite because of the two gauge-invariant vertices. A straightforward calculation gives

$$\Delta_c^{\mu\nu}(p,-p) = -\frac{1}{\pi^2}(p^\mu p^\nu - g^{\mu\nu}p^2) \times \sum_i Q_i^2\,\frac{m_i^2}{p^2}(m_i^2 A_i - 1) - \frac{1}{4\pi^2}\sum_i m_i^2 g^{\mu\nu}, \tag{3.4}$$

where the subscript c stands for "canonical" and A_i is given by

$$A_i(p^2) \equiv \frac{2}{(p^4-4m_i^2p^2)^{1/2}}\ln\frac{p^2-(p^4-4m_i^2p^2)^{1/2}}{p^2+(p^4-4m_i^2p^2)^{1/2}}. \tag{3.5}$$

The term $-(\alpha/\pi)\sum_i m_i^2 g^{\mu\nu}$ is discarded because it violates gauge invariance. It cannot be compensated by adding a term proportional to $\sum_i m_i^2 p^\mu p^\nu / p^2$, since this would introduce a spurious photon pole. The vacuum-polarization diagram has a superficial quadratic divergence, which is reduced to an actual logarithmic divergence by gauge invariance. We have just the standard result[20]

$$\Pi_c^{\mu\nu}(p,-p) = \frac{1}{12\pi^2}(p^\mu p^\nu - g^{\mu\nu}p^2)\sum_i Q_i^2\left\{\ln\frac{\Lambda^2}{m_i^2} - 6\int_0^1 dz\, z(1-z)\ln\left[1-\frac{p^2}{m_i^2}z(1-z)\right]\right\}, \tag{3.6}$$

where Λ is the cutoff energy. The quantity which appears on the right-hand side of the Ward identity (2.5) is then found to be

$$\left(2-p\cdot\frac{\partial}{\partial p}\right)\Pi_c^{\mu\nu}(p,-p) = -\frac{1}{\pi^2}(p^\mu p^\nu - g^{\mu\nu}p^2)\sum_i\left[-\tfrac{1}{6}Q_i^2 + \frac{m_i^2}{p^2}(m_i^2A_i - 1)\right]. \tag{3.7}$$

The difference between (3.4) and (3.7) is the anomaly. The Ward identity with the anomalous contribution included is then

$$\Delta^{\mu\nu}(p,-p) = \left(2-p\cdot\frac{\partial}{\partial p}\right)\Pi^{\mu\nu}(p,-p) - \frac{1}{6\pi^2}R(p^\mu p^\nu - g^{\mu\nu}p^2), \tag{3.8}$$

where we have defined

$$R \equiv \sum_i Q_i^2. \tag{3.9}$$

If we allow for the possibility that the currents have both spin-0 and spin-$\frac{1}{2}$ constituents, then (3.8) is still correct with

$$R \equiv \sum_{\text{spin }\frac{1}{2}} Q_i^2 + \tfrac{1}{4}\sum_{\text{spin }0} Q_i^2. \tag{3.10}$$

However, we will not discuss the case of spin-0 fields in the rest of the paper, as experimental evidence from deep-inelastic scattering favors a model with fermions predominating. The phenomenological consequences of the anomaly (3.8) are discussed in Secs. VI and VII.

IV. TRACE IDENTITIES INVOLVING TWO AXIAL-VECTOR CURRENTS

We now discuss the canonical trace anomalies involving the two-point functions $\langle T^*(A_\lambda^+(x)A_\mu^-(0))\rangle_\Omega$, $\langle T^*(A_\lambda^+(x)\partial^\mu A_\mu^-(0))\rangle_\Omega$, and $\langle T^*(\partial^\lambda A_\lambda^+(x)\partial^\mu A_\mu^-(0))\rangle_\Omega$, where $A_\mu^\pm$ indicate positively and negatively charged $\Delta S=0$ axial-vector currents. We will use the following notation for Green's functions:

$$\Delta_{\mu_1\cdots\mu_{m+r}}^{J\cdots JA\cdots AD\cdots D}(p_1\cdots p_n)\equiv\int\prod_{i=1}^n d^4x_i\exp\left(i\sum_{i=1}^n p_i\cdot x_i\right)\left\langle T^*\left(\theta(0)\prod_{i=1}^m J_{\mu_i}(x_i)\prod_{i=m+1}^{m+r}A_{\mu_i}(x_i)\prod_{i=m+r+1}^{n}\partial^{\mu_i}A_{\mu_i}(x_i)\right)\right\rangle_\Omega$$

and

$$(2\pi)^4\delta^4\left(\sum_{i=1}^n p_i\right)\Pi_{\mu_1\cdots\mu_{m+r}}^{J\cdots JA\cdots AD\cdots D}(p_1\cdots p_n)\equiv i\int\prod_{i=1}^n d^4x_i\exp\left(i\sum_{i=1}^n p_i\cdot x_i\right)\times\left\langle T^*\left(\prod_{i=1}^m J_{\mu_i}(x_i)\prod_{i=m+1}^{m+r}A_{\mu_i}(x_i)\prod_{i=m+r+1}^{n}\partial^{\mu_i}A_{\mu_i}(x_i)\right)\right\rangle_\Omega .$$

It has been pointed out by several authors[21] that in lowest-order perturbation theory there are no anomalies in current-algebra Ward identities involving triple and double products of quark bilinears, except for the $A_\mu V_\lambda V_\nu$ and $A_\mu A_\lambda A_\nu$ anomalies. This is because the Green's functions are ambiguous, and polynomials can be added to make the Ward identities be satisfied. For the same reason, there are no canonical anomalies in these current-algebra Ward identities, and in particular those relating $\Delta_{\lambda\mu}^{A^+A^-}$, $\Delta_\lambda^{A^+D^-}$, and $\Delta^{D^+D^-}$:

$$-ip^\lambda\Delta_{\lambda\mu}^{A^+A^-}(p,q)=\Delta_\mu^{D^+A^-}(p,q)+\Pi_\mu^{D^+A^-}(-q,q) \tag{4.1}$$

and

$$-iq^\mu\Delta_\mu^{D^+A^-}(p,q)=\Delta^{D^+D^-}(p,q)+\Pi^{D^+D^-}(p,-p)+\Pi^{\theta\sigma}(-q-p,p+q), \tag{4.2}$$

where we have introduced the field $\sigma(x)$ defined by

$$[Q_5^\pm(t),\partial^\lambda A_\lambda^\mp(\vec{x},t)]=-i\sigma(\vec{x},t).$$

As noted in Sec. II, Wilson's arguments[1] indicate that Ward identities relating two-point Green's functions and vacuum expectation values of fields are free of anomalies:

$$-ip^\lambda\Pi_{\lambda\mu}^{A^+A^-}(p,-p)=\Pi_\mu^{D^+A^-}(p,-p), \tag{4.3}$$

$$iq^\nu\Pi_\nu^{D^+A^-}(q,-q)=\Pi^{D^+D^-}(q,-q)+\langle\sigma(0)\rangle_\Omega. \tag{4.4}$$

We shall use these Ward identities (4.1)–(4.4) to connect the canonical trace anomalies involving the Green's functions appearing in them.

Consider first the trace identity relating $\Delta_{\lambda\mu}^{A^+A^-}$ and $\Pi_{\lambda\mu}^{A^+A^-}$. The form of the anomaly as a function of momentum is constrained by Wilson's short-distance power-counting arguments. Since $\langle T^*(A_\lambda^+(x)A_\mu^-(0))\rangle_\Omega$ diverges as δ^{-6} when $x\sim\delta\sim 0$, only its zeroth, first, and second moments with respect to space-time coordinates can give anomalous surface terms when integrated over d^4x as in (2.20). As in the V_λ-V_ν case discussed earlier, the second moments determine terms in the anomaly of second-order in the field momenta; constant terms in the anomaly are determined by the zeroth moment. Hence the anomalous trace identity must take the form

$$\Delta_{\lambda\mu}^{A^+A^-}(p,-p)=\left(2-p\cdot\frac{\partial}{\partial p}\right)\Pi_{\lambda\mu}^{A^+A^-}(p,-p)+(Ap^2+B)g_{\lambda\mu}+Cp_\lambda p_\mu, \tag{4.5}$$

where A, B, and C are parameters to be determined.

Similarly, the trace identity involving $\Delta_\mu^{D^+A^-}(p,-p)$ and $\Pi_\mu^{D^+A^-}(p,-p)$ has only an anomaly of first order in the field momentum, which is proportional to the integral of a first moment of $\langle T^*(\partial^\lambda A_\lambda^+(x)A_\mu^-(0))\rangle_\Omega$:

$$\Delta_\mu^{D^+A^-}(p,-p)=\left(2-p\cdot\frac{\partial}{\partial p}\right)\Pi_\mu^{D^+A^-}(p,-p)+Dp_\mu, \tag{4.6}$$

where D is another parameter to be determined.[22]

Analogously, the anomaly in the trace identity relating the $\Delta^{D^+D^-}(p,-p)$ and $\Pi^{D^+D^-}(p,-p)$ may contain constant and quadratic terms:

$$\Delta^{D^+D^-}(p,-p)=\left(2-p\cdot\frac{\partial}{\partial p}\right)\Pi^{D^+D^-}(p,-p)+Ep^2+F. \tag{4.7}$$

First we note that broken scale invariance implies that at short distances (large momenta) the axial-vector current is asymptotically conserved. Hence we deduce that in Eq. (4.5) $A + C = 0$. Then, multiplying (4.5) by ip^λ and using the Ward identities (4.1) and (4.3), we get

$$\Delta_\mu^{D^+A^-}(p,-p) = \left(2 - p\cdot\frac{\partial}{\partial p}\right)\Pi_\mu^{D^+A^-}(p,-p) - ip_\mu B\,. \tag{4.8}$$

Comparing Eqs. (4.6) and (4.8) we see that $D = -iB$; note also that $A + C \neq 0$ would have been inconsistent with the linear form of the anomaly in Eq. (4.6). Similarly, using Eqs. (4.2), (4.4), and (4.6) we deduce that

$$\Delta^{D^+D^-}(p,-p) = \left(2 - p\cdot\frac{\partial}{\partial p}\right)\Pi^{D^+D^-}(p,-p) + iDp^2\,. \tag{4.9}$$

Comparing Eqs. (4.7) and (4.9) we deduce that $E = iD$, $F = 0$.[23]

Thus the anomalies in Eqs. (4.5)–(4.7) contain just two independent parameters, A and B, which are to be determined from the canonical model of short-distance behavior. By chiral and SU(3) symmetry at short distances the leading anomaly A is related to the anomaly in the trace identity:

$$A = \frac{R}{8\pi^2}\,. \tag{4.10}$$

By explicit calculation of the asymptotic behavior of the $\partial^\lambda A_\lambda^+ - \partial^\nu A_\nu^-$ Green's function we deduce

$$E = -\frac{1}{\pi^2}\sum_i m_i^2\,, \tag{4.11}$$

where the m_i are the masses of the fundamental fermion fields contributing to the axial-vector current. Inserting expressions (4.10) and (4.11) into the anomalous trace identities (4.5)–(4.7) using the relations between parameters obtained above, we obtain the final expressions:

$$\Delta_{\lambda\nu}^{A^+A^-}(p,-p) = \left(2 - p\cdot\frac{\partial}{\partial p}\right)\Pi_{\lambda\nu}^{A^+A^-}(p,-p) + \frac{R}{8\pi^2}(p^2 g_{\lambda\nu} - p_\lambda p_\nu) - \frac{g_{\lambda\nu}}{\pi^2}\sum_i m_i^2\,, \tag{4.12}$$

$$\Delta_\nu^{D^+A^-}(p,-p) = \left(2 - p\cdot\frac{\partial}{\partial p}\right)\Pi_\nu^{D^+A^-}(p,-p) + i\frac{p_\nu}{\pi^2}\sum_i m_i^2\,, \tag{4.13}$$

$$\Delta^{D^+D^-}(p,-p) = \left(2 - p\cdot\frac{\partial}{\partial p}\right)\Pi^{D^+D^-}(p,-p) - \frac{p^2}{\pi^2}\sum_i m_i^2\,. \tag{4.14}$$

The phenomenological implications of these anomalies for high-energy cross sections and (via POT) for couplings of a scalar isoscalar meson are discussed in Secs. VI and VII.

V. CANONICAL TRACE ANOMALIES WITH MORE THAN TWO CURRENTS

In this section we discuss canonical anomalies for Green's functions involving more than two currents. The naive trace identity for a Green's function consisting of m currents and $n-m$ divergences is[22]

$$\Delta_{\mu_1\cdots\mu_m}^{J\cdots JD\cdots D}(p_1\cdots p_n) = \left(4 - n - \sum_{i=1}^{n-1} p_i\cdot\frac{\partial}{\partial p_i}\right)\Pi_{\mu_1\cdots\mu_m}^{J\cdots JD\cdots D}(p_1\cdots p_n)\,, \tag{5.1}$$

where the notation is defined as in Sec. IV.[24] From the discussion in Sec. II we know that canonical anomalies are possible for $n=3$ and $n=4$.

In the three-current case, power counting indicates that a canonical anomaly may be of zeroth or first order in the current momenta. Since we must form a tensor with three Lorentz indices, it must in fact be linear in the momenta:

$$\Delta_{\mu\nu\tau}^{J_iJ_jJ_k}(p_1,p_2,p_3) = \left(1 - \sum_{a=1}^{2} p_a\cdot\frac{\partial}{\partial p_a}\right)\Pi_{\mu\nu\tau}^{J_iJ_jJ_k}(p_1,p_2,p_3) + \sum_{a=1}^{2}\alpha^a_{\mu\nu\tau\omega}p_a^\omega\,, \tag{5.2}$$

where

$$\alpha^a_{\mu\nu\tau\omega} \equiv A^a g_{\mu\nu}g_{\tau\omega} + B^a g_{\mu\tau}g_{\nu\omega} + C^a g_{\mu\omega}g_{\nu\tau} + D^a\epsilon_{\mu\nu\tau\omega}\,, \tag{5.3}$$

with A^a, B^a, C^a, D^a constants and $a = 1, 2$. (The dependence of A^a, etc. on i, j, and k has been suppressed in the notation.)

We now consider the constraints imposed by chiral symmetry. Contract (5.3) with p_1^μ:

$$-ip_1^\mu \Delta_{\mu\nu\tau}^{J_iJ_jJ_k}(p_1,p_2,p_3) = -i\left(2-\sum_{a=1}^{2} p_a\cdot\frac{\partial}{\partial p_a}\right)p_1^\mu \Pi_{\mu\nu\tau}^{J_iJ_jJ_k}(p_1,p_2,p_3) - i\sum_{a=1}^{2}\alpha^a_{\mu\nu\tau\omega}p_a^\omega p_1^\mu . \tag{5.4}$$

We now insert the chiral Ward identities

$$-ip_1^\mu \Delta_{\mu\nu\tau}^{J_iJ_jJ_k}(p_1,p_2,p_3) = \Delta_{\nu\tau}^{D_iJ_jJ_k}(p_1,p_2,p_3) + \Pi_{\nu\tau}^{D_iJ_jJ_k}(-p_2-p_3,p_2,p_3) + ih_{ijm}\Delta_{\nu\tau}^{J_mJ_k}(p_1+p_2,p_3) + ih_{ikm}\Delta_{\nu\tau}^{J_jJ_m}(p_2,p_1+p_3), \tag{5.5}$$

$$-ip_1^\mu \Pi_{\mu\nu\tau}^{J_iJ_jJ_k}(p_1,p_2,p_3) = \Pi_{\nu\tau}^{D_iJ_jJ_k}(p_1,p_2,p_3) + ih_{ijm}\Pi_{\nu\tau}^{J_mJ_k}(-p_3,p_3) + ih_{ikm}\Pi_{\nu\tau}^{J_jJ_m}(p_2,-p_2) \tag{5.6}$$

into (5.4). It has been shown[21] that in lowest-order perturbation theory Eq. (5.5) does not have an anomaly; therefore, by the reasoning of Secs. II and III, Eq. (5.5) has no canonical anomaly. Equation (5.6) may also have a canonical anomaly[3] quadratic in momentum, but it would not contribute to (5.4) because of the factor

$$\left(2-\sum_{a=1}^{2} p_a\cdot\frac{\partial}{\partial p_a}\right).$$

We have therefore suppressed this anomaly in writing (5.6). Substituting now into (5.4) we find

$$\begin{aligned}\Delta_{\nu\tau}^{D_iJ_jJ_k}(p_1,p_2,p_3) &+ ih_{ijm}\Delta_{\nu\tau}^{J_mJ_k}(p_1+p_2,p_3) + ih_{ikm}\Delta_{\nu\tau}^{J_jJ_m}(p_2,p_1+p_3)\\ &= \left(1-\sum_{a=1}^{2} p_a\cdot\frac{\partial}{\partial p_a}\right)\Pi_{\nu\tau}^{D_iJ_jJ_k}(p_1,p_2,p_3)\\ &\quad+\left(2-\sum_{a=1}^{2} p_a\cdot\frac{\partial}{\partial p_a}\right)\left[ih_{ijm}\Pi_{\nu\tau}^{J_mJ_k}(-p_3,p_3) + ih_{ikm}\Pi_{\nu\tau}^{J_jJ_m}(p_2,-p_2)\right] - i\sum_{a=1}^{2}\alpha^a_{\mu\nu\tau\omega}p_a^\omega p_1^\mu .\end{aligned} \tag{5.7}$$

To get relations between the anomalies in (5.7) we must now use the trace identities

$$\Delta_{\nu\tau}^{D_iJ_jJ_k}(p_1,p_2,p_3) = \left(1-\sum_{a=1}^{2} p_a\cdot\frac{\partial}{\partial p_a}\right)\Pi_{\nu\tau}^{D_iJ_jJ_k}(p_1,p_2,p_3) + \gamma g_{\nu\tau}, \tag{5.8}$$

$$\Delta_{\nu\tau}^{J_mJ_k}(p_1+p_2,p_3) = \left(2-p_3\cdot\frac{\partial}{\partial p_3}\right)\Pi_{\nu\tau}^{J_mJ_k}(-p_3,p_3) + \frac{R}{8\pi^2}\delta_{mk}(g_{\nu\tau}p_3^2 - p_{3\nu}p_{3\tau}) + \epsilon_{mk}g_{\nu\tau}, \tag{5.9}$$

$$\Delta_{\nu\tau}^{J_jJ_m}(p_2,p_1+p_3) = \left(2-p_2\cdot\frac{\partial}{\partial p_2}\right)\Pi_{\nu\tau}^{J_jJ_m}(p_2,-p_2) + \frac{R}{8\pi^2}\delta_{jm}(g_{\nu\tau}p_2^2 - p_{2\nu}p_{2\tau}) + \epsilon_{jm}g_{\nu\tau}. \tag{5.10}$$

By power counting and Lorentz invariance, (5.8) may have a canonical anomaly $\gamma g_{\nu\tau}$, where γ is a constant. As shown in the previous sections, (5.9) and (5.10) may also have canonical anomalies, quadratic and constant in the momenta; the former are related to (3.8), and the latter are denoted by $\epsilon_{mk}, \epsilon_{jm}$.

Substituting (5.8), (5.9), and (5.10) into (5.7), we find

$$-i\sum_{a=1}^{2}\alpha^a_{\mu\nu\tau\omega}p_a^\omega p_1^\mu = ih_{ijk}\frac{R}{8\pi^2}\left[g_{\nu\tau}(p_3^2-p_2^2) - p_{3\nu}p_{3\tau} + p_{2\nu}p_{2\tau}\right] + (\gamma+\phi)g_{\nu\tau}, \tag{5.11}$$

where

$$\phi \equiv i(h_{ijm}\epsilon_{mk} + h_{ikm}\epsilon_{jm}).$$

From (5.11) we see that

$$\gamma+\phi=0$$

and that

$$A^1+B^1=-C^1=A^2=B^2=-\tfrac{1}{2}C^2=h_{ijk}R/8\pi^2,$$

$$D^2=0.$$

Analogous arguments using the Ward identities obtained by contracting on p_2^ν allow us to conclude that

$$A^1 = -\tfrac{1}{2}B^1 = h_{ijk}R/8\pi^2\,,$$

$$D^1 = 0\,.$$

Thus we have the following form for the canonical anomalies in the three-current trace identity, (5.2):

$$h_{ijk}\frac{R}{8\pi^2}(-g_{\mu\nu}p_{1\tau} + 2g_{\mu\tau}p_{1\nu} - g_{\nu\tau}p_{1\mu} + g_{\mu\nu}p_{2\tau} + g_{\mu\tau}p_{2\nu} - 2g_{\nu\tau}p_{1\mu})\,. \tag{5.12}$$

Also, the anomalies in trace identities for $\Delta^{DJJ}_{\mu\nu}$ are determined by those for Δ^{DJ}_{μ}. However, we pursue these anomalies no further here.

We now discuss trace identities involving four currents, which by power counting may have canonical anomalies of zeroth order in momentum:

$$\Delta^{J_iJ_jJ_kJ_m}_{\mu\nu\tau\omega}(p_1,p_2,p_3,p_4) = \left(-\sum_{a=1}^{3} p_a\cdot\frac{\partial}{\partial p_a}\right)\Pi^{J_iJ_jJ_kJ_m}_{\mu\nu\tau\omega}(p_1,p_2,p_3,p_4) + \alpha_{\mu\nu\tau\omega}\,, \tag{5.13}$$

where $\alpha_{\mu\nu\tau\omega}$ is of the form of Eq. (5.3). Contracting with $-ip_1^\mu$ we have

$$-ip_1^\mu\Delta^{J_iJ_jJ_kJ_m}_{\mu\nu\tau\omega}(p_1,p_2,p_3,p_4) = -\left(1-\sum_{a=1}^{3} p_a\cdot\frac{\partial}{\partial p_a}\right)ip_1^\mu\Pi^{J_iJ_jJ_kJ_m}_{\mu\nu\tau\omega}(p_1,p_2,p_3,p_4) - i\alpha_{\mu\nu\tau\omega}p_1^\mu\,. \tag{5.14}$$

Because of the results obtained in lowest-order perturbation theory[21] and the discussion of Secs. II and III, the chiral Ward identity for $p_1^\mu\Delta^{J_iJ_jJ_kJ_m}_{\mu\nu\tau\omega}$ does not have a canonical anomaly. There could[3] be an anomaly of first order in momenta in the chiral Ward identity for $p_1^\mu\Pi^{J_iJ_jJ_kJ_m}_{\mu\nu\tau\omega}$, but this would vanish in (5.14) because of the factor

$$\left(1-\sum_{a=1}^{3} p_a\cdot\frac{\partial}{\partial p_a}\right).$$

Therefore we may use chiral Ward identities free of anomalies to evaluate (5.14). The result is

$$\begin{aligned}
&\Delta^{D_iJ_jJ_kJ_m}_{\nu\tau\omega}(p_1,p_2,p_3,p_4) + \Pi^{D_iJ_jJ_kJ_m}_{\nu\tau\omega}(-p_2-p_3-p_4,p_2,p_3,p_4)\\
&\quad + ih_{ijn}\Delta^{J_nJ_kJ_m}_{\nu\tau\omega}(p_1+p_2,p_3,p_4) + ih_{ikn}\Delta^{J_jJ_nJ_m}_{\nu\tau\omega}(p_2,p_1+p_3,p_4) + ih_{imn}\Delta^{J_jJ_kJ_n}_{\nu\tau\omega}(p_2,p_3,p_1+p_4)\\
&\qquad = \left(1-\sum_{a=1}^{3} p_a\cdot\frac{\partial}{\partial p_a}\right)\big[\Pi^{D_iJ_jJ_kJ_m}_{\nu\tau\omega}(p_1,p_2,p_3,p_4) + ih_{ijn}\Pi^{J_nJ_kJ_m}_{\nu\tau\omega}(p_1+p_2,p_3,p_4)\\
&\qquad\qquad + ih_{ikn}\Pi^{J_jJ_nJ_m}_{\nu\tau\omega}(p_2,p_1+p_3,p_4) + ih_{imn}\Pi^{J_jJ_kJ_n}_{\nu\tau\omega}(p_2,p_3,p_1+p_4)\big] - i\alpha_{\mu\nu\tau\omega}p_1^\mu\,.
\end{aligned} \tag{5.15}$$

By power counting, the trace identity for $\Delta^{D_iJ_jJ_kJ_m}_{\nu\tau\omega}$ may have a canonical anomaly of zeroth order in momentum, but this is forbidden by the Lorentz tensor structure. The left-hand side of (5.15) may be evaluated using the canonical anomalies (5.12). The result is that

$$\alpha_{\mu\nu\tau\omega} = \frac{R}{8\pi^2}[g_{\mu\nu}g_{\tau\omega}(h_{ikn}h_{jnm} - h_{imn}h_{jkn}) + g_{\nu\omega}g_{\mu\tau}(h_{imn}h_{jkn} - h_{ijn}h_{kmn}) + g_{\nu\tau}g_{\mu\omega}(h_{ijn}h_{kmn} - h_{ikn}h_{jnm})]\,. \tag{5.16}$$

Thus the canonical trace anomalies for three and four currents are directly related by current algebra, independently of chiral anomalies, to the two-current trace anomalies.

In fact it is possible to use the generating functional formalism[3,25] to obtain a compact representation of all the canonical trace anomalies which are due to the leading singularities in the current products. We introduce the functional

$$Z \equiv \left\langle T^*\left(\exp i\int d^4x[J^i_\mu(x)F^\mu_i(x) + \theta(x)S(x)]\right)\right\rangle_\Omega\,, \tag{5.17}$$

where the $J^i_\mu(x)$ are SU(3)×SU(3) currents, $\theta(x)$ the trace of the hadronic energy-momentum tensor, $F^i_\mu(x)$ external vector fields, and $S(x)$ an external scalar field. The functional derivatives of Z with respect to the external fields, evaluated with the external fields equal to zero, are the current Green's functions of the theory. The connected Green's functions are generated by a functional W: $e^{iW} = Z$. The canonical trace identities (5.1) may be generated by the following procedure:

(1) Apply to W the operator

$$\mathcal{G} = \left[-i\frac{\delta}{\delta S(x)} - iF^\mu_b(x)\left(3 + x_\alpha\cdot\frac{\partial}{\partial x_\alpha}\right)\frac{\delta}{\delta F^\mu_b(x)}\right];$$

(2) apply n operators

$$-i\frac{\delta}{\delta F^{\mu_i}_{a_i}(x_i)} \quad (i=1,\ldots,n)$$

to $\mathcal{g}W$;

(3) set the external fields $S(x)$, $F^{\mu}_{a}(x)=0$;

(4) multiply the result by $\exp(i\sum_{i=1}^{n-1}x_i\cdot p_i)$ and integrate with respect to x and x_i, $i=1,\ldots,n-1$, setting $x_n=0$.

The resulting expression is the usual trace identity for n SU(3)×SU(3) currents $J^{a_i}_{\mu_i}$ with momenta $p_{i\mu}$. To obtain the canonical anomalies we replace the hadronic trace in expression (5.17) for Z by the anomalous trace equation

$$\theta^{\text{anomalous}}(x)=\theta(x)+\frac{R}{32\pi^2}\tilde{F}^{i}_{\mu\nu}(x)\tilde{F}^{\mu\nu}_{i}(x), \qquad (5.18)$$

where $\tilde{F}^{i}_{\mu\nu}(x)=\partial_\mu F^{i}_{\nu}(x)-\partial_\nu F^{i}_{\mu}(x)+h_{ijk}F^{j}_{\mu}(x)F^{k}_{\nu}(x)$, the h_{ijk} being SU(3)×SU(3) structure constants.[24] The expression (5.18) is analogous to the representations written down by previous authors[3,25] for the axial-vector current anomalies. Note that it has a chiral-invariant form, as expected from the fact that the higher anomalies are short-distance effects and are related by current algebra to the two-current anomalies, and are not affected by axial-vector anomalies.

VI. PHENOMENOLOGICAL APPLICATIONS OF TRACE ANOMALIES AT HIGH ENERGIES

In this section are discussed the phenomenological implications of the anomalous trace identities (3.8) and (4.12)–(4.14) at high energies. One of the basic assumptions of broken scale invariance is that the energy-momentum tensor trace is "soft." This means that it is composed of operators with scale dimension less than 4 – generalized mass terms.[1] Masses are generally supposed to be negligible in certain kinematic regions of certain processes, notably in high-energy processes involving very virtual currents, and deep-inelastic scattering experiments support this supposition. Thus, in the present case it is expected that as $|p^2|\to\infty$,

$$\Delta_{\mu\nu}(p,-p)\ll\Pi_{\mu\nu}(p,-p).$$

This expectation is borne out to all orders in perturbation theory, where it follows from Zimmermann's extension of Weinberg's theorem to the Minkowski region of momentum space.[26] Thus for large $|p^2|$

$$\left(2-p\cdot\frac{\partial}{\partial p}\right)\Pi_{\mu\nu}(p,-p)\approx\frac{R}{6\pi^2}(p_\mu p_\nu-g_{\mu\nu}p^2). \qquad (6.1)$$

Using Eq. (6.1) we find[9] that as $p^2\to\infty$,

$$\Pi_{\mu\nu}(p,-p)\sim(g_{\mu\nu}p^2-p_\mu p_\nu)\left(\frac{R}{12\pi^2}\ln p^2+c\right), \qquad (6.2)$$

where c is an unknown constant. From (6.2) we find that $\Pi_{\mu\nu}$ has asymptotically the absorptive part

$$\int e^{+ip\cdot x}\langle 0|[J_\mu(x),J_\nu(0)]|0\rangle d^4x \simeq -(g_{\mu\nu}p^2-p_\mu p_\nu)\left(\frac{R}{6\pi}\right). \qquad (6.3)$$

This is the same asymptotic behavior of the absorptive part as has previously been deduced from the parton model[27] and from assuming canonical short-distance behavior for the disconnected part of the current commutator[7]:

$$[J_\mu(x),J_\nu(0)]\underset{x_\mu\to 0}{\sim}\frac{-iR}{3\pi^3}(g_{\mu\nu}x^2-2x_\mu x_\nu)\times\epsilon(x_0)\delta'''(x^2)+\cdots. \qquad (6.4)$$

The absorptive part of $\Pi_{\mu\nu}$ is related to the total cross section for $e^-e^+\to\gamma\to$ hadrons as follows:

$$\sigma(e^-e^+\to\gamma\to\text{hadrons}) = -\frac{8\pi^2\alpha^2}{3(p^2)^2}\int e^{ip\cdot x}\langle 0|[J_\mu(x),J^\mu(0)]|0\rangle d^4x \approx\frac{4\pi\alpha^2}{3p^2}R,$$

using Eq. (6.3). This means that as $p^2\to\infty$

$$\frac{\sigma(e^-e^+\to\gamma\to\text{hadrons})}{\sigma(e^-e^+\to\gamma\to\mu^-\mu^+)}\simeq R.$$

Preliminary indications from Frascati and CEA (Cambridge Electron Accelerator) for $q^2\geq 4$ GeV^2 are consistent with $R\sim O(1)$, and are consistent with a model of three triplets of fractionally charged quarks, which gives $R=2$.[9,28]

The fact that using the softness of θ^μ_μ and the anomaly (3.8) we recover the results of canonical manipulations of current commutators emphasizes the canonical nature of the anomaly.[29] If the anomaly were absent, one would have

$$\Pi_{\mu\nu}(p,-p)\sim(g_{\mu\nu}p^2-p_\mu p_\nu)c$$

as $p^2\to\infty$, so that the absorptive part would be $o(p^2)$, and $\sigma(e^-e^+\to\gamma\to\text{hadrons})$ would fall faster than $1/p^2$, in contrast with canonical expectations.[27]

The close relationship between the trace anomaly and the asymptotic cross section for $e^+e^-\to\gamma\to$ hadrons is also very clear in configuration space. The details of the configuration-space calculation are given by Crewther,[8] but the major qualitative features are evident in the analysis of Sec. II. In Sec. II we saw that the anomaly is determined by the coefficient of the sixth-order singularity of $\langle J_\mu(x)J_\nu(0)\rangle_\Omega$ as $x\to 0$. This leading singularity determines the leading asymptotic behavior in p^2 of the Fourier transform

$$\int d^4x\, e^{ip\cdot x}\langle J^\mu(x)J_\mu(0)\rangle_\Omega,$$

which is in turn proportional to $\sigma(e^+e^-\to\gamma\to$ hadrons). The connection is familiar to students of equal-time commutators: The ϵ^{-6} singularity is just the quadratically divergent c-number Schwinger term, which is well known to determine the asymptotic behavior of $\sigma(e^+e^-\to\gamma\to$ hadrons).

The situation in field-algebra[30] models with regard to the trace anomaly deserves comment. In a simple field-algebra model, the space components of the electromagnetic currents have dimension 1, and the simple Wilson[1] arguments yield no anomaly. The lowest-order graphs, corresponding in momentum space to the canonical calculation, are indicated in Fig. 2, and they transparently yield no anomaly. This is consistent with canonical manipulations of the current commutator. In a simple field-algebra model this is less singular than x^{-6}, and $\sigma(e^-e^+\to\gamma\to$ hadrons) falls faster than $1/s$. In fact at the canonical level one has

$$i\int dx e^{ip\cdot x}\langle T^*(J_\mu(x)J_\nu(0))\rangle_\Omega \propto (g_{\mu\nu}p^2-p_\mu p_\nu)\frac{p^2}{p^2-m_V{}^2+i\epsilon}, \tag{6.5}$$

where m_V is the mass of the vector-meson field. Equation (6.5) has an absorptive part $\propto\delta(p^2-m_V{}^2)$. In a field algebra with strong interactions there will also be higher-order graphs like those of Fig. 3, with internal loops. It is clear that these graphs will yield anomalies, which will however be of the Callan-Symanzik type, being proportional to $\partial\Pi_{\mu\nu}/\partial g$, where g is a hadronic coupling constant, e.g., $\mathcal{L}_{\rm int}=g\rho_\mu\bar\psi\gamma^\mu\psi$. Such graphs correspond to the breakdown of canonical manipulations.

It is clear that by an argument analogous to that of the previous section the quadratic anomaly in Eq. (4.12) is related to the experimentally inaccessible cross sections for $e^-\bar\nu_e\to$ hadrons and $\mu^-\bar\nu_\mu\to$ hadrons at high energies. The interesting anomaly (4.14) is related to the structure functions corresponding to the nonconserved parts of the currents. These should be relatively suppressed by a power of p^2, and furthermore appear experimentally multiplied by lepton masses. Hence their observability is regrettably minimal.

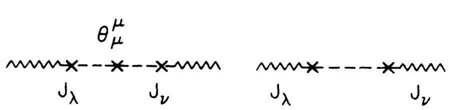

FIG. 2. Lowest-order canonical contributions to $\Delta_{\lambda\nu}$ and $\Pi_{\lambda\nu}$ in a field-algebra model. Vector-meson propagators are denoted by dashes.

VII. PHENOMENOLOGICAL APPLICATIONS OF TRACE ANOMALIES AT LOW ENERGIES

In this section we consider low-energy implications of the trace anomalies (3.8) and (4.12)–(4.14), when used in conjunction with POT, starting with the electromagnetic-current case. We introduce two form factors W_1 and W_2 for $\Delta_{\lambda\mu}(q,p)$:

$$\Delta_{\lambda\mu}(q,p)=(-p_\lambda q_\mu+g_{\lambda\mu}q\cdot p)W_1(q,p) + (q_\lambda q_\rho-g_{\lambda\rho}q^2)(p^\rho p_\mu-g^\rho_\mu p^2)W_2(q,p),$$

so that

$$\Delta_{\lambda\mu}(q,-q)=(q_\lambda q_\mu-g_{\lambda\mu}q^2)[W_1(q,-q)-3q^2W_2(q,-q)]\,. \tag{7.1}$$

Then

$$\Delta_{\lambda\mu}(q,-q)=\left(2-q\cdot\frac{\partial}{\partial q}\right)\Pi_{\lambda\mu}(q)-\frac{R}{6\pi^2}(q_\lambda q_\mu-g_{\lambda\mu}q^2), \tag{7.2}$$

where

$$\Pi_{\lambda\mu}(q,-q)\equiv(q_\lambda q_\mu-g_{\lambda\mu}q^2)\Pi(q^2),$$

so that

$$\left(2-q\cdot\frac{\partial}{\partial q}\right)\Pi_{\lambda\mu}(q,-q)=-2q^2(q_\lambda q_\mu-g_{\lambda\mu}q^2)\frac{\partial\Pi(q^2)}{\partial q^2}\,. \tag{7.3}$$

Since $\Pi(q^2)$ is nonsingular at $q^2=0$, we deduce, comparing (7.1)–(7.3), that $W_1(0,0)=-R/6\pi^2$, $W_1(q,-q)$ $(q\neq 0)$ and $W_2(q,-q)$ being unknown and arbitrary. In the spirit of POT it is assumed that W_1 and W_2 are maximally smooth apart from poles in $r^2\equiv(q+p)^2$ due to a scalar meson σ (dilaton). In fact the maximally smooth coupling for $\sigma\gamma\gamma$ implies

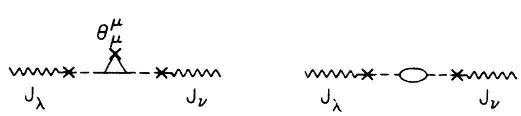

FIG. 3. Next-to-lowest-order contributions to $\Delta_{\lambda\nu}$ and $\Pi_{\lambda\nu}$ in a field-algebra model with interactions.

$$W_1(q,p) = -\frac{c}{m_\sigma^2 - (q+p)^2}, \qquad (7.4)$$
$$W_2(q,p) = 0$$

for some constant c. From Eq. (7.4) we see that $c = m_\sigma^2 R/6\pi^2$. c is proportional to the $\sigma\gamma\gamma$ coupling: $c = 2g_{\sigma\gamma\gamma} m_\sigma^2 F_\sigma$, where $g_{\sigma\gamma\gamma}$ is defined by the interaction Lagrangian ($F_{\mu\nu}$ is the electromagnetic field)

$$\mathcal{L}_{\sigma\gamma\gamma} = -\tfrac{1}{2} e^2 g_{\sigma\gamma\gamma} \sigma F_{\mu\nu} F^{\mu\nu},$$

and the decay constant F_σ is defined by

$$\langle 0 | \theta^\mu_\mu(0) | \sigma \rangle = m_\sigma^2 F_\sigma .$$

Thus we estimate that

$$g_{\sigma\gamma\gamma} = \frac{R}{12\pi^2} \frac{1}{F_\sigma} \qquad (7.5)$$

(see also Refs. 8 and 9). This can be compared with the broken-scale-invariance estimate[11,12]

$$g_{\sigma\pi\pi} \simeq \frac{m_\sigma^2}{2F_\sigma}, \qquad (7.6)$$

where $g_{\sigma\pi\pi}$ is defined by

$$\mathcal{L}_{\sigma\pi\pi} = g_{\sigma\pi\pi} \sigma \vec{\pi} \cdot \vec{\pi} .$$

The ratio of coupling constants is independent of the parameter F_σ, and so is less theoretically uncertain:

$$\frac{g_{\sigma\gamma\gamma}}{g_{\sigma\pi\pi}} \simeq \frac{R}{6\pi^2 m_\sigma^2} . \qquad (7.7)$$

The prediction (7.5) for the $\sigma\gamma\gamma$ coupling constant deserves a certain number of comments:

(a) Essentially the same coupling was obtained by Schwinger,[31] who performed a lowest-order perturbation theory calculation analogous to that of Steinberger,[32] to evaluate the rate for a scalar meson to decay into two photons via a fermion loop. Since the trace identities had not at that time been formulated, the result did not strike him as anomalous.

(b) If we set $R = 0$ then we recover the prediction $g_{\sigma\gamma\gamma} = 0$ of Kleinert, Staunton, and Weisz,[33] who explicitly ignored anomalies. Kleinert, Staunton, and Weisz also indicated ways of obtaining $g_{\sigma\gamma\gamma} = 0$ in the absence of the anomaly, but as they pointed out, such a result requires nonmaximal smoothness for the $\Delta_{\mu\nu}$ vertex. Equation (7.5) seems to be the legitimate prediction of broken scale invariance and POT for $g_{\sigma\gamma\gamma}$.

(c) Our prediction for the $\sigma\gamma\gamma$ coupling seems considerably smaller than most other estimates in the literature.[13] Two *a priori* unknown constants appear in (7.5). As discussed in Sec. II, R is expected to be of order unity, and we have a preference (not inconsistent with experiment) for the three-fractionally-charged-triplet value $R = 2$. The decay constant F_σ can be estimated if we identify σ with the apparent scalar isoscalar dipion resonance ϵ (700).[34] Using the broken-scale-invariance estimate[11,12] (7.6) for $g_{\sigma\pi\pi}$ and estimating $\Gamma(\epsilon \to \pi\pi) \simeq 400$ MeV we find $F_\sigma \approx 150$ MeV. This value is not inconsistent with the POT estimate

$$g_{\sigma N\bar{N}} \approx M_N / F_\sigma ,$$

where $\mathcal{L}_{\sigma N\bar{N}} = g_{\sigma N\bar{N}} \sigma\bar{\psi}\psi$, or with the sketchy experimental information on $g_{\sigma N\bar{N}}$; however, any value for F_σ between 100 and 200 MeV is certainly respectable. Taking $F_\sigma \approx 150$ MeV we find[35]

$$\Gamma(\epsilon \to 2\gamma) \approx 0.2 R^2 \text{ keV}, \qquad (7.8)$$

which compares with the following other theoretical estimates.[13,14]

Sarker: 6 keV

Bramon and Greco: 6 keV

Schrempp-Otto, Schrempp, and Walsh: 22 keV

Lyth[36]: ≤ 1 keV.

The first three of these estimates use finite-energy sum rules or pole dominance of dispersion relations. Neither our estimate nor the others should probably be regarded as better than order-of-magnitude values; everybody treats the scalar meson in the narrow-resonance approximation, which is likely to be bad for the ϵ(700) meson ($\Gamma_\epsilon \sim 400$ MeV), quite apart from the conjectural quality of the particle's existence.

However, if R is of the size suggested by theoretical prejudice, our estimate (7.8) does seem significantly smaller than most others.[13] Whether there is a noticeable signal from the ϵ in the $\gamma\gamma \to \pi\pi$ cross section will depend on the magnitude of the background at $s \sim m_\epsilon^2$. For instance, consider a a Breit-Wigner propagator for the ϵ,

$$\frac{1}{s - m_\epsilon^2 + i m_\epsilon \Gamma_\epsilon},$$

which is unrealistic but may serve as a qualitative guide. The amplitude for $\gamma\gamma \to \epsilon \to \pi\pi$ at $s = m_\epsilon^2$ is then

$$\mathfrak{M}_\epsilon(s = m_\epsilon^2) = i \frac{8e^2}{9\pi} (\epsilon_1 \cdot \epsilon_2) \frac{R}{(1 - 4m_\pi^2/m_\epsilon^2)^{1/2}}, \qquad (7.9)$$

where we have used (7.7) and where the ϵ_i are photon polarizations. For comparison, the s-wave contribution of the Born term to $\gamma\gamma \to \pi\pi$ (which come from the seagull diagram and represents the amplitude exactly at threshold according to the low-energy theorem) is

$$\mathfrak{M}_{\rm Born}=2e^2\epsilon_1\cdot\epsilon_2\,.$$

If $\mathfrak{M}_{\rm Born}$ approximately represents the background at $s=m_\epsilon{}^2$, then we have

$$\sigma_{\gamma\gamma\to\pi\pi}(s=m_\epsilon{}^2)\sim\sigma_{\rm Born}\,|1+i(0.15)R|^2$$
$$\sim\sigma_{\rm Born}[1+(0.02)R^2]$$

and the ϵ gives only a 10% enhancement to the background. Since no one can claim to understand the background to within 10%, this means that there would be no significant ϵ signal in the cross section. On the other hand, if it turns out that the background at $s=m_\epsilon{}^2$ is much smaller than $\mathfrak{M}_{\rm Born}$, then there might be an observable ϵ signal in the cross section. However, in either case, according to Watson's theorem (inelastic effects which set in at $\sqrt{s}=4m_\pi$ are still small at $\sqrt{s}=m_\epsilon\sim700$ MeV) the amplitude for $\gamma\gamma\to\pi\pi$ will still have the standard $\pi\pi$ phase shift, so that the meson is still observable, though the strength of the coupling may not be.

There seems to be no reason why the $\epsilon\gamma\gamma$ coupling should not be small; the implications of unitarity for the process $\gamma\gamma\to\pi\pi$ have been studied by Carlson and Tung[37] and by Lyth[14] with a view to getting information on the $\epsilon\gamma\gamma$ coupling. Both papers write down Omnes-type solutions: Lyth allows terms in the left-hand cut in addition to the pion Born term, and concludes that the $\epsilon\gamma\gamma$ coupling is not completely constrained. His calculations in fact assume elasticity, and that any scalar isoscalar resonance is narrow; however, his order-of-magnitude estimate of an upper bound for $g_{\epsilon\gamma\gamma}$, based on the likely magnitude of the left-hand cut, is encouragingly close to our estimate.

We now turn to the anomalies (4.12)–(4.14) involving axial-vector currents. Several previous authors[11,12,38] have used these Ward identities, neglecting anomalies, in conjunction with POT, to make predictions on scalar-meson couplings to pions in particular. We study whether these results are affected by taking anomalies into account – they seem not to be changed, but our arguments are not watertight. A common discussion[12,38] of the $\sigma\pi\pi$ coupling proceeds somewhat as follows. The θ^μ_μ-$\partial^\lambda A^+_\lambda$-$\partial^\nu A^-_\nu$ Green's function is given the following low-energy parametrization:

$$\Delta^{D^+D^-}(p,q)\simeq\frac{A+B(p^2+q^2)+Cr^2}{(m_\pi{}^2-p^2)(m_\pi{}^2-q^2)(m_\sigma{}^2-r^2)}\ ,\qquad(7.10)$$

where $r=p+q$ is the momentum associated with θ^μ_μ. Using the chiral low-energy theorem (4.2) and single-particle dominance we obtain

$$\Delta^{D^+D^-}(p,0)=-\Pi^{D^+D^-}(p,-p)+\Pi^{\theta\sigma}(-p,p)$$
$$=-\frac{F_\pi{}^2m_\pi{}^4}{m_\pi{}^2-p^2}+\frac{3F_\pi{}^2m_\pi{}^2m_\sigma{}^2}{m_\sigma{}^2-p^2}\ .\qquad(7.11)$$

Using a naive trace identity and single-particle dominance we obtain

$$\Delta^{D^+D^-}(p,-p)=\left(2-p\cdot\frac{\partial}{\partial p}\right)\Pi^{D^+D^-}(p,-p)$$
$$=\frac{2F_\pi{}^2m_\pi{}^4(m_\pi{}^2-2p^2)}{(m_\pi{}^2-p^2)^2}\ .\qquad(7.12)$$

Comparing these expressions with the parametrization (7.10) we see that

$$A=2F_\pi{}^2m_\pi{}^6m_\sigma{}^2\,,$$
$$B=-2F_\pi{}^2m_\pi{}^4m_\sigma{}^2\,,$$
$$C=F_\pi{}^2m_\pi{}^4(m_\pi{}^2-m_\sigma{}^2)\,,$$

which yields an on-mass-shell $\sigma\pi\pi$ coupling (7.6).

As shown in Sec. IV, the trace identity (7.12) has an anomaly in models with a fundamental fermion structure; however, we can still find two ways of deriving (7.6). Neither of these is completely convincing, which is why we sketch them both.

(a) The parametrization (7.7) is inconsistent with the anomalous Ward identity (4.14): The simplest consistent parametrization for the θ^μ_μ-$\partial^\lambda A^+_\lambda$-$\partial^\nu A^-_\nu$ vertex is

$$\frac{A'+B'(p^2+q^2)+C'r^2}{(m_\pi{}^2-p^2)(m_\pi{}^2-q^2)(m_\sigma{}^2-r^2)}+D'+E'(p^2+q^2)+F'r^2\,,\qquad(7.13)$$

where we have allowed a contact term quadratic in the field momenta. In the single-particle-dominance approximation the anomalous trace identity becomes

$$\Delta^{D^+D^-}(p,-p)=\frac{2F_\pi{}^2m_\pi{}^4(m_\pi{}^2-2p^2)}{(m_\pi{}^2-p^2)}-\frac{\sum_i m_i{}^2p^2}{\pi^2}\ .\qquad(7.14)$$

Using (7.14) and the nonanomalous[21] chiral Ward identity (7.11) we find

$$A'=2F_\pi{}^2m_\pi{}^6m_\sigma{}^2\,,\quad B'=-2F_\pi{}^2m_\pi{}^4m_\sigma{}^2\,,$$
$$C'=F_\pi{}^2m_\pi{}^4(m_\pi{}^2-m_\sigma{}^2)\,,\quad D'=0\,,$$
$$E'=-\frac{\sum_i m_i{}^2}{2\pi^2}\,,\quad F'=\frac{\sum_i m_i{}^2}{2\pi^2}\ .$$

When we go to the mass shell, the contact terms do not contribute to the coupling constant, and as $A=A'$, etc., the same on-mass-shell coupling (7.6) is found as before.

The parametrization (7.13) is not the only one that could be chosen. However, other choices either seem to have a larger number of parameters, which are hence not all determined by the Ward identities so that no prediction can be obtained, or else are not consistent with all the low-energy theorems.

(b) Alternatively, Crewther's method[11] of obtaining the $\sigma\pi\pi$ coupling could be used. In this derivation, only θ and one of the axial divergences are taken off the hadronic mass shell, and (7.6) is obtained from a resulting trace identity. According to Wilson's analysis[1] (see also Sec. II) anomalies can only arise if three or more fields are taken off the mass shell in deriving low-energy theorems. Hence Crewther's method is not subject to anomalies, and his derivation of (7.6) is not affected.

VIII. DISCUSSION

We have discussed which anomalies arise in the trace identities of broken scale invariance if canonical behavior of strong interactions is assumed. We have also discussed the phenomenological relevance of these anomalies to high-energy processes, and via POT to the couplings of a scalar isoscalar meson σ. In particular we obtain a connection

$$g_{\sigma\gamma\gamma}=\frac{1}{12\pi^2 F_\sigma}\lim_{p^2\to\infty}\frac{\sigma(e^-e^+\to\gamma\to\text{hadrons})}{\sigma(e^-e^+\to\gamma\to\mu^-\mu^+)} \quad (8.1)$$

between the two-photon coupling of such a meson and asymptotic e^-e^+ annihilation cross sections.

Apart from the phenomenological testing of broken scale invariance, canonical singularity structure, and POT via Eq. (8.1), there remain several interesting open theoretical questions. There is the question of what canonical anomalies are present in conformal Ward identities, and whether they are simply related to the canonical trace anomalies.[10] The answer to this question may provide clues to the significant problem of what are the conformal analogs of the Callan-Symanzik anomalies.[4] As concerns the applications of the anomalies, there might be other ways of measuring the axial trace anomalies (4.12)–(4.14) which would shed light on the question of the quark "mass." Finally, in our low-energy applications of the trace anomaly (3.8) using POT we treated the σ particle in a simple pole approximation. It may be possible to take into account finite-width effects and unitarity, as has been done by other authors studying the isoscalar $\pi\pi$ s wave.[14,37,39]

ACKNOWLEDGMENTS

We wish to thank S. Coleman, R. Crewther, and D. Lyth for useful discussions.

APPENDIX: TRACE ANOMALY IN HIGHER ORDERS OF PERTURBATION THEORY

From the point of view adopted in this paper – asymptotic scale invariance realized by canonical behavior at short distances – the trace anomaly and the V-V-A anomaly are due to similar physical phenomena and are equally likely to be realized in nature. However, if we choose to study products of currents from the perspective of perturbation theory, then there is an important distinction between the two anomalies. The axial-vector anomaly has the remarkable property that it is not modified by higher-order corrections,[3] i.e., the value of the anomaly computed from the simple triangle graph is actually exact to all orders. This property is not shared by the trace anomaly.

This difference between the two anomalies is easily understood in a qualitative way. It is possible to define a regulator which leaves chiral symmetry invariant, e.g., Pauli-Villars regulators chosen in chiral singlets. Then the usual chiral Ward identities are still valid in the regulated theory. This condition makes it possible to prove the nonrenormalizability theorem for the axial-vector Ward identity. But in the case of the trace anomaly, the underlying scale symmetry is violently broken by the large mass which is introduced in the regulator. The usual scale-invariance Ward identities are modified by new terms depending on the regulator mass. As we take the regulator mass to infinity, these extra terms give rise to higher-order corrections to the trace anomaly.[40]

We now calculate the leading radiative correction to the trace anomaly in fermion electrodynamics. The anomalous trace identity for the fourth-order quantities has the form

$$\Delta_{\mu\nu}^{(4)}(p)=\left(2-p\cdot\frac{\partial}{\partial p}\right)\Pi_{\mu\nu}^{(4)}(p)+C^{(4)}(p_\mu p_\nu-g_{\mu\nu}p^2)\,. \quad (A1)$$

This form is dictated by the following requirements:

(1) The anomaly must be a polynomial in p because the absorptive parts of $\Delta_{\mu\nu}$ and $\Pi_{\mu\nu}$ are regular and therefore satisfy the nonanomalous trace identity.

(2) Dylan's version of Weinberg's theorem[26] fixes the degree of the polynomial as quadratic.

(3) There is gauge invariance.

We can easily calculate $C^{(4)}$ by using Weinberg's theorem. According to the theorem, in the deep Euclidean region $\Delta_{\mu\nu}^{(4)}(p)$ diverges at most like p, while $\Pi_{\mu\nu}^{(4)}(p)$ diverges like p^2. Therefore the leading divergence of $\Pi^{(4)}_{\mu\nu}$ must be canceled by the anomaly. Where

$$\Pi^{(4)}_{\mu\nu}(p)=(p_\mu p_\nu-g_{\mu\nu}p^2)\Pi^{(4)}(p^2) \tag{A2}$$

we have

$$-2p^2\frac{\partial}{\partial p^2}\Pi^{(4)}(p^2)\xrightarrow[p^2\to\infty]{}C^{(4)}. \tag{A3}$$

For larger momenta $\Pi^{(4)}$ is given by[41]

$$\Pi^{(4)}(p^2)\underset{p^2\to\infty}{\sim}+\left(\frac{\alpha}{3\pi}+\frac{\alpha^2}{4\pi^2}\right)\ln\left(\frac{p^2}{m^2}\right) \tag{A4}$$

so that

$$C^{(4)}=-\frac{2\alpha}{3\pi}-\frac{\alpha^2}{2\pi^2}. \tag{A5}$$

The second-order term obtained here is seen to agree with the result of the calculation of Sec. III.

We emphasize once again that from our point of view the presence of high-order-perturbation-theory corrections is irrelevant when considering the trace anomaly in hadronic physics. This is because we invoke the hypothesis that the leading singularities of products of hadronic currents are given by canonical models, so that only the canonical singularities (which may be calculated from lowest-order diagrams) are relevant.

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*Work supported by the U. S. Atomic Energy Commission.

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[3]For a review of the axial-vector current anomalies see S. L. Adler, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser *et al*. (MIT Press, Cambridge, Mass., 1971), Vol. I.

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[6]Even if logarithmic corrections did appear eventually at high energies, there would be a region in which neglecting them was valid. In any case it would be important to know what anomalies occurred in the skeleton theory of Ref. 1.

[7]See Fritzsch and Gell-Mann, Ref. 2, and references quoted therein. Important references on the light cone are R. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971), and Y. Frishman, Phys. Rev. Letters 25, 966 (1970). For a recent review see C. H. Llewellyn Smith, in Proceedings of the Fourth International Conference on High Energy Collisions, Oxford, 1972 (Oxford Univ. Press, to be published).

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[10]We are at present studying in a similar model the analogous anomalies in Ward identities involving the conformal current.

[11]See Gell-Mann and Carruthers, Ref. 2. Also see P. Carruthers, Phys. Rev. D 2, 2265 (1970); R. J. Crewther, Phys. Letters 33B, 305 (1970); Phys. Rev. D 3, 3152 (1971).

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[15]Of course, another view of Crewther's relation is possible: to accept PCAC and to regard the relation as a test of the assumed singularity structure.

[16]This conclusion has also been reached by Crewther (private communication).

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[18]We neglect complications due to possible noncovariant seagull terms. The possible effect of such terms must be examined for each particular case. For the trace anomaly, as for the axial-vector anomaly, they cannot be invoked to make the anomaly vanish.

[19]The integral is absolutely convergent (cf. Ref. 1); therefore, it is independent of which order we take ϵ, η, $\delta\to 0$, and we are free to choose the order of taking limits which transparently results in equal-time commutators.

[20]See for example J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).

[21]W. A. Bardeen, Phys. Rev. 184, 1848 (1969); R. W. Brown, C.-C. Shih, and B.-L. Young, *ibid*. 186, 1491 (1969). For a discussion of Green's functions with three fermion bilinear densities, see I. S. Gerstein and R. Jackiw, *ibid*. 181, 1955 (1969).

[22]In Secs. IV and V we assume that θ and the current divergences have scale dimensions three as in the quark model.

[23]Note that if the scale dimensions of θ and $\partial_\mu A^\mu$ were less than three, then there would be no anomaly in the trace identity for Π^{DD}. This is because Wilson's power counting arguments would not allow an anomaly quadratic in the momenta and we have just seen that chiral symmetry forbids an anomaly independent of the momenta.

[24]In this section, we will not distinguish between vector and axial-vector currents, which will be denoted by J^i_μ,

where the Latin suffix runs over both axial-vector and vector SU(3) indices. The corresponding group structure functions are denoted by h_{ijk}, where $h_{ijk}=0$ if an odd number of its indices are axial-vector, and $h_{ijk}=f_{ijk}$ otherwise.

[25]See, for example, the elegant discussion by J. Wess and B. Zumino, Phys. Letters 37B, 95 (1971). See also Bardeen, Ref. 21.

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[28]H. Fritzsch and M. Gell-Mann, in *Proceedings of the International Conference on Duality and Symmetry in Hadron Physics*, edited by E. Gotsman (Weizmann Science Press, Jerusalem, 1971).

[29]In a recent paper R. Delbourgo and P. Phocas-Cosmetatos [Lett. Nuovo Cimento 5, 420 (1972)] argue that there is no anomaly in the trace identity involving two electromagnetic currents. They reach this conclusion by using a different energy-momentum tensor whose trace is not soft – a soft trace would not have photon-photon or electron-positron-photon vertices. As we have argued above, the anomaly is necessarily present if the trace is soft.

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[31]J. Schwinger, Phys. Rev. 82, 664 (1951).

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[33]H. Kleinert, L. P. Staunton, and P. H. Weisz, Nucl. Phys. B38, 104 (1972).

[34]Particle Data Group, Rev. Mod. Phys. 43, S1 (1971).

[35]If we took $M_\epsilon \sim 600$ MeV, $\Gamma(\epsilon\rightarrow 2\pi)\sim 500$ MeV, we would obtain $F_\epsilon \sim 135$ MeV, $\Gamma(\epsilon\rightarrow 2\gamma)\approx 0.1(3)R^2$ keV.

[36]In taking this estimate from Lyth we inserted our assumed values for m_ϵ and Γ_ϵ into his upper bound on the product $g_{\epsilon\gamma\gamma}g_{\epsilon\pi\pi}$. For $m_\epsilon \sim 600$ MeV, $\Gamma(\epsilon\rightarrow 2\pi)\sim 500$ MeV his bound becomes 0.6 keV.

[37]C. E. Carlson and W.-K. Tung, Phys. Rev. D 6, 147 (1972).

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Deep-Inelastic Scattering, the Subtraction of Divergent Sum Rules, and Chiral-Symmetry Breaking in the Gluon Model*

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The formal light-cone properties of commutators involving current divergences are studied in the gluon model. Relations are derived which make it possible (in principle) to distinguish the vector- from the (pseudo) scalar-gluon model. In the vector-gluon model these relations provide an experimental determination of the bare quark masses. The additional assumption that the residues of any $\alpha=0$ fixed poles in current scattering amplitudes are polynomials in q^2 makes it possible to relate the σ term in pion-nucleon scattering to convergent integrals over neutrino-scattering structure functions; the polynomial assumption dictates a prescription for subtracting a (linearly) divergent sum rule derived previously. The same technique generates subtracted sum rules for the (neutrino- and spin-dependent) structure functions W_3^- and G_2. With the parton-model assumption that the leading scaling behavior of current-divergence and divergence-divergence scattering is given by free-field theory, it is possible to relate fixed-pole residues in *ep, en, νp,* and *νn* scattering, deep-inelastic data, the σ term, baryon mass differences, and the bare quark masses; approximate values for the bare quark masses and the parameter μ_0 can be obtained.

I. INTRODUCTION

In this paper we study properties of light-cone commutators involving current divergences in the vector-gluon model. All our considerations are formal; that is to say they are untrue in perturbation theory. It is frequently argued that the scaling observed in the SLAC-MIT inelastic electron scattering experiments implies that formal field theory might be relevant to the real world. This argument is not totally compelling (especially since the data do not exclude $\log Q^2$ terms) but at