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Inclusive Photon Distributions: Contributions from π^0's and Bremsstrahlung*

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Photons produced in high-energy hadronic collisions come primarily from π^0 decays. If the π^0 inclusive cross section exhibits Feynman scaling, then so does the derived photon spectrum. The scaled photon-spectrum behavior in the central region ($x = 0$) is examined in detail. The bremsstrahlung contribution can be estimated for not too energetic photons, and provides a means for measuring the mean charged multiplicity at very high energies.

I. INTRODUCTION

The study of high-energy hadronic collisions has been greatly facilitated by advances in the understanding of inclusive processes – processes in which not all the final-state particles are specified.[1] The most extensively studied inclusive processes are the single-particle inclusive reactions of the form $(a:c|b)$.[2] If particle c is not a hadron, but rather the decay product of a hadron, d, the observed spectrum is an indirect image of the original inclusive process $(a:d|b)$. In particular, the observation of $(a:\gamma|b)$ yields information primarily about $(a:\pi^0|b)$. Of course the information is not as precise as would be a direct measurement of $(a:\pi^0|b)$ in a coincidence experiment.

There are further complications in interpreting the photon spectrum. Some photons arise from the decays of hadrons other than π^0's, most notably from the decays of η's. The $\eta\to 3\pi^0$ mode is an especially copious source of photons. Each η yields an average of 3.2 photons. Because of the much greater production of π^0's and because the photons from the $\eta\to 3\pi^0$ mode are essentially indistinguishable from "true" π^0 photons we shall generally ignore the complication introduced by η decays.

A second complication is the production of photons by charged-particle bremsstrahlung. For soft photons bremsstrahlung must be taken into account. We shall do this in Sec. VI, in the context of a model.

The basic assumption of the paper is that at high energies hadronic processes exhibit Feynman scaling.[3] Beyond this, we shall need little more than kinematics. Since these kinematics are essential and somewhat unfamiliar, we present them in some detail in Sec. II. These results are combined with Feynman scaling in Sec. III. A numerical example is presented in Sec. IV to clarify the preceding sections. The prescription of Sternheimer[4] for extracting the π^0 spectrum from the observed photon spectrum is examined in Sec. V. The bremsstrahlung contribution to the photon distribution is analyzed in Sec. VI. Finally, the principles derived are applied to an analysis of the CERN ISR (Intersecting Storage Rings) data on $(p:\gamma|p)$.

The principal results are the following:

(a) If $(a:\pi^0|b)$ scales so that the inclusive π^0 differential cross section gives

$$\lim_{s\to\infty}\frac{E}{\sigma_{\rm inel}}\frac{d\sigma}{d^3p}(x,p_\perp^2,s)=\frac{1}{\sigma_{\rm inel}}f_{\pi^0}(x,p_\perp^2)=\bar f_{\pi^0}(x,p_\perp^2),$$

where $p_\perp$ is the component of the π^0 momentum perpendicular to the beam direction and $x=p_\parallel/[(s)^{1/2}/2]$, then the photon spectrum resulting from the π^0 decays also scales,

$$\lim_{s\to\infty}\frac{k}{\sigma_{\rm inel}}\frac{d\sigma}{d^3k}(x,k_\perp^2,s)=\frac{1}{\sigma_{\rm inel}}f_\gamma(x,k_\perp^2)=\bar f_\gamma(x,k_\perp^2),$$

where $k_\perp$ is the component of photon momentum perpendicular to the beam direction and $x=k_\parallel/[(s)^{1/2}/2]$.

(b) $f_\gamma(x,k_\perp^2)$ is continuous as $x\to 0$ except for $k_\perp^2=0$, where we have

$$\lim_{x\to 0}\lim_{k_\perp\to 0}f_\gamma(x,k_\perp^2)=\tfrac{1}{2}\lim_{k_\perp\to 0}\lim_{x\to 0}f_\gamma(x,k_\perp^2).$$

(c) For the photon spectrum arising from π^0 decays, we have

$$\lim_{k_\perp\to 0}f_\gamma(x=0,k_\perp^2)=\frac{2}{m^2}\int_0^\infty dk_\perp^2 f_\gamma(x=0,k_\perp^2)$$

on the assumption that the π^0 distribution has scaled.

(d) For the photon spectrum arising from bremsstrahlung, we derive the result for large s, small $k_\perp$, and $x=0$:

$$\frac{kd\sigma}{\sigma d^3k}(k_\perp,s)\approx\frac{\alpha}{4\pi^2k_\perp^2}\langle n_c\rangle,$$

where α is the fine-structure constant and $\langle n_c\rangle$ is the mean charged multiplicity.

II. DECAY KINEMATICS

The calculation of the photon distribution from a known π^0 distribution is straightforward. If a single π^0 with four-momentum p and mass m decays into two photons, the distribution of photons is given, in invariant form, by

$$k\frac{dN}{d^3k}(p,k)=\frac{1}{\pi}\delta(p\cdot k-\tfrac{1}{2}m^2). \tag{1}$$

Consequently, the Lorentz-invariant cross section for photon production is

$$k\frac{d\sigma}{d^3k}=\int\frac{d^3p}{E}\left(E\frac{d\sigma}{d^3p}(p,s)\right)\frac{1}{\pi}\delta(p\cdot k-\tfrac{1}{2}m^2), \tag{2}$$

where

$$E\frac{d\sigma}{d^3p}(p,s)$$

is the invariant differential cross section for the production of π^0's at a center-of-mass energy squared equal to s. For definiteness we shall assume that the π^0's result from p-p collisions, and the center of mass is that of the p-p system. It will be apparent that all the results apply equally to $(a:\pi^0|b)$ with only trivial modifications, if any.

The π^0's which contribute to the photon spectrum at a given momentum, k, are constrained by Eq. (1) to satisfy

$$E-p'_\parallel=\frac{m^2}{2k}, \tag{3}$$

where $p'_\parallel$ is the component of π^0 momentum parallel to the photon momentum. From (3) we find

$$p'_\parallel=p_0+\frac{k}{m^2}p'^2_\perp, \tag{4}$$

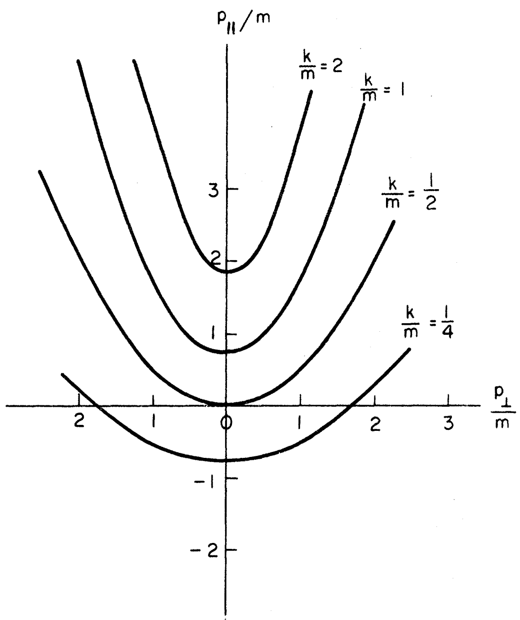

FIG. 1. Cross sections of the paraboloids on which π^0's must lie to contribute at given values of photon momentum.

where $p'_\perp$ is component of π^0 momentum perpendicular to the photon momentum and where

$$p_0 = k - \frac{m^2}{4k}. \tag{5}$$

See Fig. 1. In momentum space, the π^0's contributing at a given photon momentum are confined to a paraboloid whose axis is along the direction of the photon, and whose apex is at $p'_\parallel = p_0$. For large k/m the paraboloid becomes very narrow. The limit $k \to 0$ is degenerate and must be handled with care.

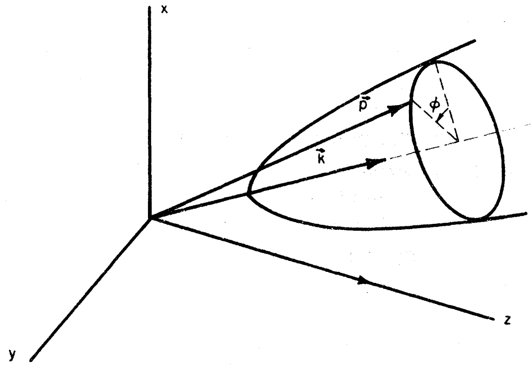

FIG. 2. The geometry for determining the photon spectrum from a given pion spectrum.

The δ function in (2) can be eliminated by integrating over θ, the angle between the π^0 and the photon, with the result

$$k\frac{d\sigma}{d^3k} = \frac{1}{\pi k}\int_{E_0}^{\infty} dE \int_0^{2\pi} d\phi E \frac{d\sigma}{d^3p}(p, s), \tag{6}$$

where ϕ is the azimuthal angle in the plane perpendicular to the photon, and where

$$E_0 = (p_0^2 + m^2)^{1/2}$$
$$= k + \frac{m^2}{4k}. \tag{7}$$

In (6) we have taken the upper limit in the E integration to be infinite and assumed the kinematical limits are incorporated into the π^0 differential cross section. The π^0 three-momentum in (6) is given, in coordinates relative to the photon, by

$$\vec{p} = \left(p'_\parallel, p'_x = \cos\phi\left(\frac{m^2}{k}(p'_\parallel - p_0)\right)^{1/2}, p'_y = \sin\phi\left(\frac{m^2}{k}(p'_\parallel - p_0)\right)^{1/2}\right). \tag{8}$$

The π^0 inclusive cross section is a function of s, p_z the component of momentum parallel to the beam direction, and $p_\perp$, the component perpendicular to it. If the azimuthal angle ϕ is measured away from the plane containing the photon and the beam direction, and if the photon has components of momentum parallel and perpendicular to the beam direction $k_\parallel$ and $k_\perp$, respectively, then

$$p_\perp^2 = \left[\left(\frac{m^2}{k}(E - E_0)\right)^{1/2}\frac{k_\parallel}{k}\cos\phi + \left(E - \frac{m^2}{2k}\right)\frac{k_\perp}{k}\right]^2 + \frac{m^2}{k}(E - E_0)\sin^2\phi, \tag{9}$$

where E is the π^0 energy. See Fig. 2. Thus we have explicitly

$$k\frac{d\sigma}{d^3k}(k, s) = \frac{1}{\pi k}\int_{E_0}^{\infty} dE \int_0^{2\pi} d\phi E \frac{d\sigma}{d^3p}\left(E, p_\perp^2 = \left[\left(\frac{m^2}{k}(E - E_0)\right)^{1/2}\frac{k_\parallel}{k}\cos\phi + \left(E - \frac{m^2}{2k}\right)\frac{k_\perp}{k}\right]^2 + \frac{m^2}{k}(E - E_0)\sin^2\phi, s\right). \tag{10}$$

For $k/m \gg 1$, the paraboloid over which the integration takes place becomes narrow. If it is approximated by one of vanishing width, Eq. (10) becomes

$$k\,\frac{d\sigma}{d^3k}(k,s)\approx\frac{2}{k}\int_{E_0}^{\infty}dE\,E\,\frac{d\sigma}{d^3p}(p',s)\,, \tag{11}$$

where $\vec{p}'$ lies in the same direction as $\vec{k}$. In this limit we can also approximate E_0 by k. Then we have the approximation of Sternheimer[4]:

$$\frac{\partial}{\partial k}\left(k^2\,\frac{d\sigma}{d^3k}(\vec{k},s)\right)=-2E\,\frac{d\sigma}{d^3p}(\vec{p}=\vec{k},s)\,. \tag{12}$$

We postpone until Sec. VI an evaluation of the reliability of Sternheimer's approximation.

III. SCALING

The scaling hypothesis is that as $s\to\infty$, the invariant differential production cross section becomes a function of $p_\perp^2$ and $x=p_\parallel/[(s)^{1/2}/2]$ only. (See Sec. I.) To see how this scaling manifests itself in the photon spectrum arising from π^0 decay, we begin with Eq. (10) and introduce the following new coordinates:

$$\begin{aligned} Q^2&=\frac{m^2}{k}(E-E_0)\,,\\ Q_x&=Q\cos\phi\,,\\ Q_y&=Q\sin\phi\,. \end{aligned} \tag{13}$$

Then we have

$$k\,\frac{d\sigma}{d^3k}(\vec{k},s)=\frac{2}{\pi m^2}\int d^2Q\,E\,\frac{d\sigma}{d^3p}\left(E=\frac{kQ^2}{m^2}+E_0,\,p_\perp^2=\left[\frac{k_\parallel}{k}\,Q_x+\left(\frac{kQ^2}{m^2}+p_0\right)\frac{k_\perp}{k}\right]^2+Q_y^{\,2},\,s\right). \tag{14}$$

If we take $s\to\infty$ with $x_\gamma\neq 0$, in the notation of Sec. I, we have

$$f_\gamma(x_\gamma,k_\perp^2)=\frac{2}{\pi m^2}\int d^2Q\,f_\pi\left(x_\pi=x_\gamma\left(1+\frac{Q^2}{m^2}\right),\,p_\perp^2=\left[Q_x+k_\perp\left(1+\frac{Q^2}{m^2}\right)\right]^2+Q_y^{\,2}\right). \tag{15}$$

Equation (15) makes manifest the scaling of the photon distribution for $x\neq 0$.

The π^0 spectrum is expected to become a function of $p_\perp$ alone as $s\to\infty$ with $p_\perp$ and $p_\parallel$ fixed. This carries over to the photon spectrum as we show below. From (14), we have in the $x_\gamma=0$ limit (with $k_\perp$ and $k_\parallel$ fixed)

$$f_\gamma(0,k_\perp^2)=\frac{2}{\pi m^2}\int d^2Q\,f_\pi\left(0,\,p_\perp^2=\left[\frac{k_\parallel}{k}\,Q_x+\left(\frac{kQ^2}{m^2}+p_0\right)\frac{k_\perp}{k}\right]^2+Q_y^{\,2}\right). \tag{16}$$

Only π^0's with $x_\pi=0$ contribute, since from (14), π^0's with $x_\pi\neq 0$ would have $p_\perp^2\propto s$, and we assume the cross section falls off in $p_\perp^2$. We can change variables in (16) to

$$\begin{aligned} Q_x'&=Q_x+\frac{m^2}{2k_\perp}\,\frac{k_\parallel}{k}\,,\\ Q_y'&=Q_y \end{aligned} \tag{17}$$

to get

$$f_\gamma(0,k_\perp^2)=\frac{2}{\pi m^2}\int d^2Q'\,f_\pi\left(0,\,p_\perp^2=\left(p_0(k_\perp)+\frac{k_\perp}{m^2}\,Q'^2\right)^2+Q_y'^2\right), \tag{18}$$

where

$$p_0(k_\perp)=k_\perp-\frac{m^2}{4k_\perp}\,.$$

From (18), we see that the photon spectrum in the central region is indeed independent of $k_\parallel$.

If $k_\perp\neq 0$, then $f_\gamma(x_\gamma,k_\perp^2)$ [Eq. (15)] joins smoothly to $f_\gamma(0,k_\perp^2)$ [Eq. (18)]. To prove this, let

$$Q_x' = Q_x + \frac{m^2}{2k_\perp}, \qquad Q_y' = Q_y \tag{19}$$

in Eq. (15) and let $x_\gamma \to 0$. Then we find

$$\lim_{x_\gamma \to 0} f_\gamma(x_\gamma, k_\perp^2) = \frac{2}{\pi m^2} \int d^2Q' f_\pi \left(0, p_\perp^2 = \left(p_0(k_\perp) + \frac{k_\perp}{m^2} Q'^2\right)^2 + Q_y'^2\right) = f_\gamma(0, k_\perp^2). \tag{20}$$

On the other hand, if $k_\perp = 0$, the transformation given by (19) cannot be used. Instead, we have from (15),

$$f_\gamma(x_\gamma, 0) = \frac{2}{\pi m^2} \int d^2Q f_\pi(x_\pi = x_\gamma(1 + Q^2/m^2), p_\perp^2 = Q^2). \tag{21}$$

Evaluation of the $k_\perp^2 \to 0$ limit of $f_\gamma(0, k_\perp^2)$ requires some care. Because $p_0(k_\perp) \approx -m^2/4k_\perp$, $p_\perp^2$ is large unless [see Eq. (18)]

$$\frac{k_\perp Q_x'^2}{m^2} - \frac{m^2}{4k_\perp} \approx 0. \tag{22}$$

With

$$Q_x' = Q_x'' \pm \frac{m^2}{2k_\perp}, \qquad Q_y' = Q_y'', \tag{23}$$

we have from (18)

$$p_\perp^2 = Q''^2 \pm 2Q_x'' k_\perp \left(1 + \frac{Q''^2}{m^2}\right) + k_\perp^2 \left(1 + \frac{Q''^2}{m^2}\right)^2. \tag{24}$$

Hence there are two identical contributions to the photon spectrum. We can write (18) as

$$f_\gamma(0, k_\perp^2) = \frac{4}{\pi m^2} \int_{-\infty}^{\infty} dQ_y'' \int_{-m^2/2k_\perp}^{\infty} dQ_x'' f_\pi(0, p_\perp^2 = Q''^2 + 2Q_x'' k_\perp(1 + Q''^2/m^2) + k_\perp^2(1 + Q''^2/m^2)^2). \tag{25}$$

For $k_\perp/m \ll 1$, we separate out an integral over the entire Q'' plane,

$$f_\gamma(0, k_\perp^2) = \frac{4}{\pi m^2} \int d^2Q'' f_\pi(0, p_\perp^2 = Q''^2 + 2Q_x'' k_\perp(1 + Q''^2/m^2) + k_\perp^2(1 + Q''^2/m^2)^2) + E, \tag{26}$$

where we anticipate that E will die exponentially in $m/k_\perp$. In particular,

$$\lim_{k_\perp^2 \to 0} f_\gamma(0, k_\perp^2) = \frac{4}{\pi m^2} \int d^2Q'' f_\pi(0, p_\perp^2 = Q''^2). \tag{27}$$

But from (21)

$$\lim_{x \to 0} f_\gamma(x, 0) = \frac{2}{\pi m^2} \int d^2Q f_\pi(0, p_\perp^2 = Q^2). \tag{28}$$

Thus $f_\gamma(x, k_\perp^2)$ is not continuous at the point $x = 0$, $k_\perp = 0$. To understand how this comes about, examine Eq. (15). Contributions to the integral come from small values of $p_\perp^2$. For $Q_y = 0$,

$$p_\perp^2 = \left[Q_x + k_\perp \left(1 + \frac{Q_x^2}{m^2}\right)\right]^2 \tag{29}$$

while the integral in Eq. (15) extends to $x_\pi = 1$, or

$$\frac{Q_x^2}{m^2} = \frac{1}{x_\gamma} - 1. \tag{30}$$

The condition $p_\perp^2 = 0$ yields, for $(k_\perp/m) \ll 1$,

$$Q_x \approx k_\perp \text{ or } -\frac{m^2}{k_\perp}.$$

Thus the small $p_\perp^2$ regions are near ($Q_x = k_\perp$, $Q_y = 0$) and ($Q_x = -m^2/k_\perp$, $Q_y = 0$). If $x > (k_\perp/m)^2$, then by (30) the second region falls outside the integration domain determined by the kinematical limits. Thus as x decreases to values less than about $(k_\perp/m)^2$, the second region is introduced into the integrations, giving rise to the factor of 2 between (27) and (28).

We can reformulate (27) in an interesting fashion. To do this we first note that

$$\int_0^\infty dp_\perp^2 f_{\pi^0}(0, p_\perp^2) = \tfrac{1}{2}\int_0^\infty dk_\perp^2 f_\gamma(0, k_\perp^2). \tag{31}$$

This relation can be proved directly from (1). It reflects the fact that the central rapidity region for the photons must be twice as heavily populated as the central region for pions.

As a consequence, we have a theorem for the photon spectrum arising from the decays of a scaled π^0 spectrum:

$$\lim_{k_\perp\to 0} f_\gamma(0, k_\perp^2) = \frac{2}{m^2}\int_0^\infty dk_\perp^2 f_\gamma(0, k_\perp^2). \tag{32}$$

Equation (32) is a striking consequence of scaling in hadronic collisions, relating the photon spectrum in the central region at zero transverse momentum, to the spectrum in the central region integrated over transverse momentum. Since the equation is linear in the photon cross section, it is unaffected by uncertainties in over-all normalization.

The integral in Eq. (26) can be expanded in powers of $k_\perp/m$:

$$f_\gamma(0, k_\perp^2) = \frac{4}{\pi m^2}\int_0^\infty \frac{dQ^2}{2}\int_0^{2\pi} d\phi \left\{ f_\pi(0, Q^2) + \left[2Q\cos\phi\, k_\perp\left(1+\frac{Q^2}{m^2}\right) + k_\perp^2\left(1+\frac{Q^2}{m^2}\right)^2\right] f_\pi'(0, Q^2) \right.$$

$$\left. + \tfrac{1}{2}\left[2Q\cos\phi\, k_\perp\left(1+\frac{Q^2}{m^2}\right)\right]^2 f_\pi''(0, Q^2) + O(k_\perp^4/m^4) \right\} \tag{33}$$

$$= \frac{4}{m^2}\int_0^\infty dQ^2 \left[f_\pi(0, Q^2) + k_\perp^2\left(1+\frac{Q^2}{m^2}\right)^2 f_\pi'(0, Q^2) + Q^2 k_\perp^2\left(1+\frac{Q^2}{m^2}\right)^2 f_\pi''(0, Q^2) + O(k_\perp^4/m^4)\right], \tag{34}$$

where primes denote differentiation with respect to Q^2. Assuming $Q^2 f_\pi(0, Q^2)$ and $Q^2 f_\pi'(0, Q^2)$ vanish for $Q^2 = 0$, we integrate by parts to get

$$f_\gamma(0, k_\perp^2) = \frac{4}{m^2}\int_0^\infty dQ^2 f_\pi(0, Q^2)\left[1 + \frac{k_\perp^2}{m^2}\left(2 + \frac{4Q^2}{m^2}\right) + O\left(\frac{k_\perp^4}{m^4}\right)\right]. \tag{35}$$

As $k_\perp$ increases, the E term in Eq. (26) must be considered as well. From Eq. (35), though, we expect a rise in the photon spectrum for transverse momenta increasing from zero to small values. For larger values of $k_\perp$, the E term reduces the value of the right-hand side of (26). We expect E to become significant when $(m^2/2k_\perp) \approx \langle p_\perp\rangle$, where $\langle p_\perp\rangle$ is the average transverse pion momentum.

The general principles outlined here are displayed explicitly in the next section.

IV. A SIMPLE EXAMPLE

An appreciation for the results of the preceding sections can be gained by considering an especially simple example. Suppose that the scaled π^0 distribution is independent of x and given by

$$f_{\pi^0}(x, p_\perp^2) = e^{-ap_\perp^2}, \quad 0 < |x| < 1. \tag{36}$$

From (15) we find

$$f_\gamma(x, k_\perp^2) = \frac{2}{\pi m^2}\int_0^{m^2(1/x-1)} \frac{dQ^2}{2}\int_0^{2\pi} d\theta \exp\left\{-a\left[Q^2 + 2k_\perp Q\cos\theta\left(1+\frac{Q^2}{m^2}\right) + k_\perp^2\left(1+\frac{Q^2}{m^2}\right)^2\right]\right\} \tag{37}$$

$$= \frac{2}{m^2}\int_0^{m^2(1/x-1)} dQ^2 \exp\left\{-a\left[Q^2 + k_\perp^2\left(1+\frac{Q^2}{m^2}\right)^2\right]\right\} I_0\left(2ak_\perp Q\left(1+\frac{Q^2}{m^2}\right)\right), \tag{38}$$

where I_0 is the usual modified Bessel function. Similarly, from (18) we find

$$f_\gamma(0,k_\perp^2) = \frac{4}{am^2}\int_0^\infty dz \exp\left[-a\left(\frac{2k_\perp z}{am^2}+p_0(k_\perp)\right)^2\right]e^{-z}I_0(z)\,. \qquad (39)$$

We can find the $k_\perp \to 0$ limit of the two expressions. From (38),

$$f_\gamma(x,k_\perp^2=0) = \frac{2}{am^2}\left\{1-\exp\left[-am^2\left(\frac{1}{x}-1\right)\right]\right\}, \qquad (40)$$

while from (39) (see the Appendix),

$$\lim_{k_\perp\to 0} f_\gamma(0,k_\perp^2) = \frac{4}{am^2}. \qquad (41)$$

This shows explicitly the factor of 2 associated with interchanging the order of the limits $x \to 0$ and $k_\perp \to 0$, which we proved generally in Eqs. (27) and (28).

The numerical evaluation of Eq. (39) with $am^2 = 0.3$ is shown in Figs. 3 and 4. In Fig. 3 we see that the falloff in $k_\perp$ is much steeper than that of the generating π^0 spectrum. For $k_\perp/m \gg 1$, Eq. (39) becomes

$$f_\gamma(0,k_\perp^2 \gg m^2) \approx \frac{e^{-ak_\perp^2}e^{am^2/2}}{ak_\perp^2}. \qquad (42)$$

In Fig. 4, the quadratic rise away from $k_\perp = 0$ is clearly visible in confirmation of Eq. (35). Also shown is the $x=0$ spectrum arising from a π^0 distribution

$$f_{\pi^0}(0,p_\perp^2) = e^{-bp_\perp} \qquad (43)$$

with the value of b chosen to give the same $\langle p_\perp^2\rangle$ as (36) ($b^2m^2 = 1.8$ corresponds to $am^2 = 0.3$).

Figure 4 shows that two rather different π^0 spectra can give rise to quite similar photon spectra, provided the $\langle p_\perp^2\rangle$ values are roughly the same. It also shows the turnover as the E term in Eq. (26) becomes significant, around $(m^2/2k_\perp)^2 = \langle p_\perp^2\rangle$. Numerical calculations reveal that $f_\gamma(0,k_\perp^2)$ is not terribly sensitive to the parameter a in Eq. (36) if the results are normalized to the same value at $k_\perp = 0$. For example, with $am^2 = 0.2$ the curve differs from that with $am^2 = 0.3$ by no more than 15% in the range $0 \le k_\perp/m \le 1$.

The evaluation of $f_\gamma(x,k_\perp^2)$ [Eq. (38)] is shown in Fig. 5. For small $k_\perp/m$ and x not too small, Eq. (40) is a good representation of the photon spectrum. For very small x, the distribution rises towards the value dictated by Eq. (39). The transition takes place in the region $x \approx m^2/k_\perp^2$.

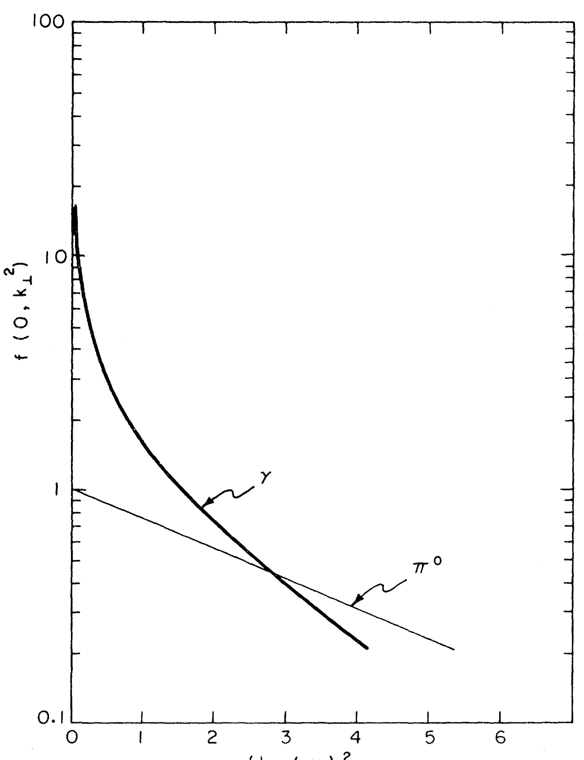

FIG. 3. The photon spectrum at $x = 0$ resulting from a pion spectrum given by $\exp(-ap_\perp^2)$ with $am^2 = 0.3$. The behavior at extremely small values of $k_\perp$ is shown in more detail in Fig. 4.

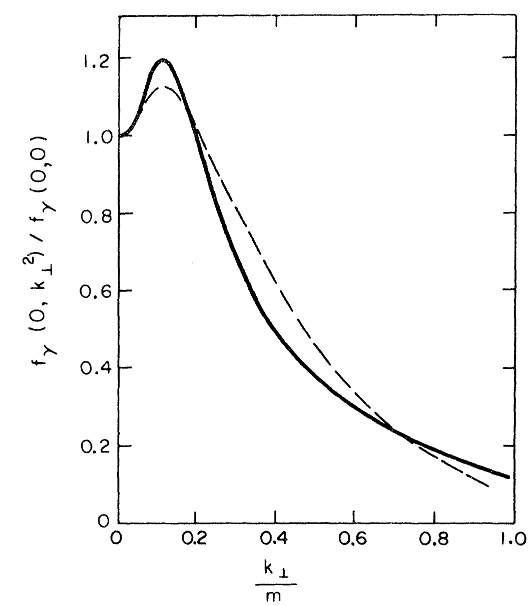

FIG. 4. Photon spectra at $x = 0$ resulting from two pion spectra having the same $\langle p_\perp^2\rangle$: solid line, $\exp(-ap_\perp^2)$ with $am^2 = 0.3$; dashed line, $\exp(-bp_\perp)$ with $b^2m^2 = 1.8$. Both spectra are normalized to unity at $k_\perp = 0$.

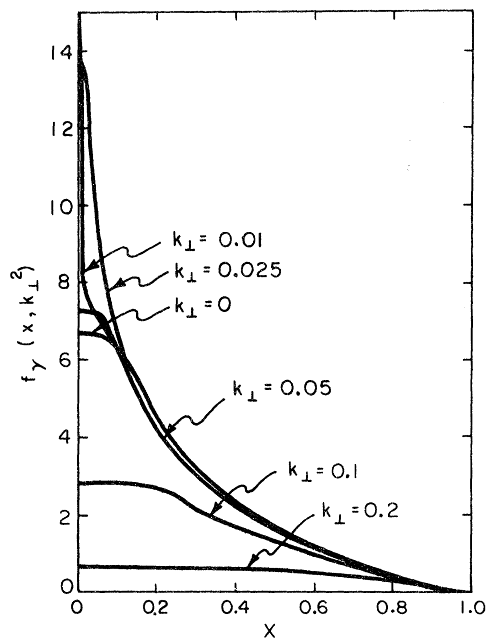

FIG. 5. The photon spectrum $f_\gamma(x,k_\perp^2)$ resulting from a pion spectrum given by $\exp(-ap_\perp^2)$ with $am^2 = 0.3$ ($k_\perp$ in GeV/c).

V. STERNHEIMER'S PRESCRIPTION

At any finite energy, Sternheimer's prescription, Eq. (12), may be used to derive the π^0 spectrum from a known photon spectrum if $k/m \gg 1$. A scaled form of Sternheimer's prescription is also easily obtained. From Eq. (12),

$$2E\frac{d\sigma}{d^3p}(\vec{p}=\vec{k},s) = -\left[\frac{\partial}{\partial k}\left(k^2\frac{d\sigma}{d^3k}(\vec{k},s)\right)\right]_{\vec{k}=\vec{p}_\pi}. \tag{44}$$

In terms of $k_\perp$ and $k_\parallel$ we have

$$2E\frac{d\sigma}{d^3p}(\vec{p}=\vec{k},s) \cong -\left(1+k_\parallel\frac{\partial}{\partial k_\parallel}+k_\perp\frac{\partial}{\partial k_\perp}\right)\left(k\frac{d\sigma}{d^3k}(k,s)\right). \tag{45}$$

Passing to the scaling limit,

$$2f_{\pi^0}(\vec{p}=\vec{k},s) = -\left(1+x\frac{\partial}{\partial x}+k_\perp\frac{\partial}{\partial k_\perp}\right)f_\gamma(x,k_\perp). \tag{46}$$

If $k_\perp = \beta x$ and if

$$F_\gamma(x,\beta) = f_\gamma(x,k_\perp), \tag{47}$$

then

$$\frac{\partial}{\partial x}[xF_\gamma(x,\beta)] = \frac{\partial}{\partial x}[xf_\gamma(x,k_\perp=\beta x)] = \left(1+x\frac{\partial}{\partial x}+x\beta\frac{\partial}{\partial k_\perp}\right)f_\gamma(x,k_\perp=\beta x). \tag{48}$$

Thus we can write (46) as

$$2f_{\pi^0}(x,k_\perp) = -\frac{\partial}{\partial x}[xF_\gamma(x,\beta)]. \tag{49}$$

In effect, k has been replaced by x, and $\tan\theta = k_\perp/k_\parallel$ by $\beta = k_\perp/x$. From (46) the $x=0$ analog is

$$2f_{\pi^0}(0,k_\perp) = -\frac{\partial}{\partial k_\perp}[k_\perp f_\gamma(0,k_\perp)]. \tag{50}$$

We can test this prescription with the model of the previous section. From (42),

$$\frac{\partial}{\partial k}[k_\perp f_\gamma(0,k_\perp)] \approx -2e^{am^2/2}e^{-ak_\perp^2}. \tag{51}$$

Thus the value of the pion distribution we would infer from Sternheimer's prescription differs from the true expression by a constant. It is easy to see how this has happened. The approximation $p_0 \approx k$ is inadequate since $\exp(-ap_0^2)$ differs by a constant from $\exp(-ak^2)$. For less rapidly varying π^0 distributions this problem would not arise.

VI. BREMSSTRAHLUNG

While π^0 decays are the primary source of photons in hadronic collisions, in a restricted kinematic region bremsstrahlung is an important source also. The bremsstrahlung arises primarily from the sudden creation of charged particles, i.e., inner bremsstrahlung analogous to radiative β decay. The general features of the bremsstrahlung contribution can be anticipated by considering the classical formula for the intensity of inner bremsstrahlung from a particle created suddenly with a velocity $\vec{\beta}$ (Ref. 5),

$$k\frac{dN}{d\Omega dk} = \frac{\alpha}{4\pi^2}\beta^2\frac{\sin^2\theta}{(1-\beta\cos\theta)^2}, \tag{52}$$

where k is the photon energy and θ is the angle between the photon direction and the direction of $\vec{\beta}$. For the Lorentz-invariant form we have

$$k\frac{dN}{d^3k} = \frac{\alpha}{4\pi^2k^2}\beta^2\frac{\sin^2\theta}{(1-\beta\cos\theta)^2}. \tag{53}$$

Consider the special case of photons emitted at 90° to the beam direction. As the value of s (the center-of-mass energy squared) increases, more and more charged particles are produced in the forward and backward directions. While the bremsstrahlung from these particles peaks also in the forward and backward directions, a contribution at 90° persists. Indeed, it can be seen from (53) that, with the assumption of incoherence, each particle gives a contribution to the soft-photon spectrum at 90° of $(\alpha/4\pi^2k^2)$. Thus at 90° the bremsstrahlung contribution is

$$\frac{kd\sigma}{\sigma d^3k} \approx \frac{\alpha}{4\pi^2k^2}\langle n_c\rangle\,, \tag{54}$$

where $\langle n_c\rangle$ is the mean multiplicity of charged particles. Since $\langle n_c\rangle$ is believed to grow like ln$s$, this contribution, unlike that from π^0 decay, does not scale, but increases with increasing s.

We shall now treat the bremsstrahlung in a more complete fashion. Our model will be based on a number of assumptions. First, we shall assume that the photons are emitted incoherently from the charged particles. Second, we shall assume that all the created charged particles are pions. Third,

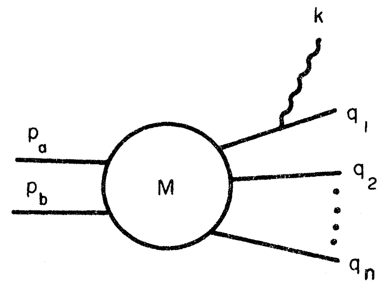

FIG. 6. A typical bremsstrahlung diagram in charged-particle production.

we shall neglect bremsstrahlung from the incident particles. Finally, we shall assume that the relevant diagrams are like that in Fig. 6. We shall restrict ourselves to low-photon momenta and assume that the extrapolation of the hadronic matrix element is negligible. Thus a typical matrix element squared is

$$|M(p_a, p_b, q_1-k, q_2, \ldots, q_n)|^2 e^2 \frac{1}{|(q_1-k)^2-m^2|^2}\sum_i |(2q_1-k)\cdot\epsilon_i|^2\,, \tag{55}$$

where $M(p_a, p_b, q_1, \ldots, q_n)$ is the matrix element for the nonradiative process. The sum is over the photon polarizations. Because we have assumed incoherent production of the photons, and negligible extrapolation of the matrix element, after summing over exclusive processes, we get a form which factorizes between the hadronic production and the emission of bremsstrahlung:

$$[2(2\pi)^3]^2 k q_0 \frac{d\sigma}{d^3k\,d^3q} = 2(2\pi)^3 q_0 \frac{d\sigma}{d^3q} 4\pi\alpha \frac{|\vec{q}\times\hat{k}|^2}{(q\cdot k)}. \tag{56}$$

Making manifest the correspondence with the classical result, we have

$$kq_0 \frac{d\sigma}{d^3k\,d^3q} = \frac{\alpha}{4\pi^2k^2}\beta^2 \frac{\sin^2\theta'}{(1-\beta\cos\theta')^2} q_0 \frac{d\sigma}{d^3q}, \tag{57}$$

where $\beta = |\vec{q}|/q_0$ and θ' is the angle between $\hat{q}$ and $\hat{k}$. The bremsstrahlung spectrum then is

$$k\frac{d\sigma}{d^3k} = \frac{\alpha}{4\pi^2k^2}\int \frac{d^3q}{q_0}\left(q_0\frac{d\sigma}{d^3q}\right)\left(\frac{q}{q_0}\right)^2 \frac{\sin^2\theta'}{[1-(q/q_0)\cos\theta']^2}. \tag{58}$$

Specializing to the case of photons at 90° to the beam direction, we can reexpress the result in terms of angles relative to the beam direction rather than the photon direction. Thus we have

$$\frac{k}{\sigma}\frac{d\sigma}{d^3k} = \frac{\alpha}{4\pi^2k^2}\int \frac{dq_\parallel dq_\perp^2}{2q_0}\left(\frac{q_0}{\sigma}\frac{d\sigma}{d^3q}\right)\int_0^{2\pi} d\phi \frac{1-(q_\perp^2/q^2)\cos^2\phi}{[q_0/q-(q_\perp/q)\cos\phi]^2},$$

$$= \frac{\alpha}{4\pi^2k^2}\int \frac{dq_\parallel \pi dq_\perp^2}{q_0}\left(\frac{q_0}{\sigma}\frac{d\sigma}{d^3q}\right)\left\{1+2\left[\left(1-\frac{q_\perp^2}{q_0^2}\right)^{-3/2}\left(1-\frac{q_\perp^2}{q_0^2}-\frac{m^2}{2q_0^2}\right)-1\right]\right\} \tag{59}$$

$$= \frac{\alpha}{4\pi^2k^2}\langle n_c\rangle\{1+R\}. \tag{60}$$

As we shall see, R is a small correction. The basic result is simply that at 90° each charged particle contributes to the bremsstrahlung according to the classical 90° result for relativistic particles:

$$k\frac{dN}{d^3k} = \frac{\alpha}{4\pi^2k^2}. \tag{61}$$

We can derive an estimate for R by considering a model in which $(q_0/\sigma)(d\sigma/d^3q)$ is a function of $q_\perp$ only, except that it vanishes for $|q_\parallel| > P$, with $s = 4P^2$. With $(q_0/\sigma)(d\sigma/d^3q) = g(q_\perp^2)$, we have

$$R = \frac{2\int dq_\perp^2 \int_{-P}^{P} dq_\parallel [(q_0^2 - q_\perp^2)^{-1/2} - q_0^{-1} - \frac{1}{2}m^2(q_0^2 - q_\perp^2)^{-3/2}]g(q_\perp^2)}{\int dq_\perp^2 \int_{-P}^{P} dq_\parallel g(q_\perp^2) q_0^{-1}}, \tag{62}$$

$$R \cong \frac{2\int dq_\perp^2 g(q_\perp^2)[\ln(m_\perp^2/m^2) - 1]}{\int dq_\perp^2 g(q_\perp^2)[\ln(s/m^2) - \ln(m_\perp^2/m^2)]}$$

$$\cong \frac{2[\langle \ln(m_\perp^2/m^2)\rangle - 1]}{\ln(s/m^2)}, \tag{63}$$

where $m_\perp^2 = q_\perp^2 + m^2$ and where we have assumed $s \gg m^2$. In addition to the $\ln(s/m^2)$ suppression, R is reduced by cancellation between the two terms in the numerator. For example, if $g(q_\perp^2) = \exp(-aq_\perp^2)$,

$$\langle \ln(m_\perp^2/m^2)\rangle = e^{am^2} E_1(am^2), \tag{64}$$

where E_1 is the usual exponential integral function. For a reasonable value of am^2, say, $am^2 = 0.3$, $\langle \ln(m_\perp^2/m^2)\rangle = 1.2$ so that $R \approx 0.4/\ln(s/m^2)$.

For photons with fixed center-of-mass momentum, but not necessarily perpendicular to the beam direction, we have similarly,

$$\frac{k}{\sigma}\frac{d\sigma}{d^3k} = \frac{\alpha}{4\pi^2 k_\perp^2}\langle n_c\rangle[1 + R'], \tag{65}$$

where R' is given by the previous expression for R except that the argument of the pion inclusive differential cross section is shifted by the amount necessary to bring the photon to a 90° orientation.

VII. ANALYSIS OF ISR DATA

Data are available for photon distributions at $s = 900$, 2000, and 2800 GeV^2, at 10°, 16°, 24°, and 90°.[6] Neuhofer *et al.* provide a parametrization of the data for photon energies between 100 MeV and 5 GeV as

$$\frac{k}{\sigma}\frac{d\sigma}{d^3k} = \frac{A}{k_\perp}\exp\left(-\frac{k_\perp}{k_0} - \frac{x}{x_0}\right) \tag{66}$$

with $A = 1.48$ GeV^{-1}, $k_0 = 0.162$ GeV, and $x_0 = 0.083$.

The parametrization in Eq. (66) clearly does not satisfy the requirements of Eq. (32). Still it does permit a reasonable evaluation of the right-hand side of the equation. Thus we have the prediction

$$\lim_{k_\perp \to 0} \bar{f}_\gamma(x = 0, k_\perp) \approx 53 \text{ GeV}^{-2}. \tag{67}$$

Verification of this prediction is obscured by the bremsstrahlung. If at $s = 2800$ GeV^2, $\langle n_c\rangle \approx 10$, then

$$\left(\frac{k}{\sigma}\frac{d\sigma}{d^3k}\right)_{\text{brems}} \approx 2\times 10^{-3} k_\perp^{-2}. \tag{68}$$

The data cited above have been binned by total photon momentum into bins of 100 MeV. The bremsstrahlung contribution is primarily in the first bin. Formally the integrated bremsstrahlung contribution diverges; the actual bremsstrahlung contribution to the measured cross section depends critically on the detection efficiency at low-photon momenta. Comparing the bremsstrahlung prediction with the parametrization of the data, we see that the bremsstrahlung becomes significant in the region $k_\perp \simeq 1$–10 MeV. In principle, careful measurements of the photon spectrum at low transverse momentum could identify the bremsstrahlung by its characteristic $1/k_\perp^2$ behavior. After subtracting the bremsstrahlung, the remaining cross section should conform to the condition imposed by Eq. (32). Figure 7 shows that the data of Neuhofer *et al.* are consistent with this interpretation. The cross section for the lowest transverse momenta lies above the curves anticipated on the basis of a π^0 spectrum like that of Eq. (36). Presumably this is a reflection of the bremsstrahlung contribution.

The data shown in Fig. 7 are not exactly at $x = 0$. At fixed production angle, as $k_\perp$ increases, so does x. Since we expect a falloff in x (cf. Fig. 5), the data might be expected to be below the anticipated curve for $x = 0$ for larger values of $k_\perp$. Such a trend may exist in the 10° data. On the other hand, the choice of a hypothetical π^0 spectrum at $x = 0$ is quite arbitrary and different spectra would give somewhat different photon distributions. (See Fig. 4.)

Using the Sternheimer prescription and the parametrization of the photon data of Neuhofer *et al.*, we have

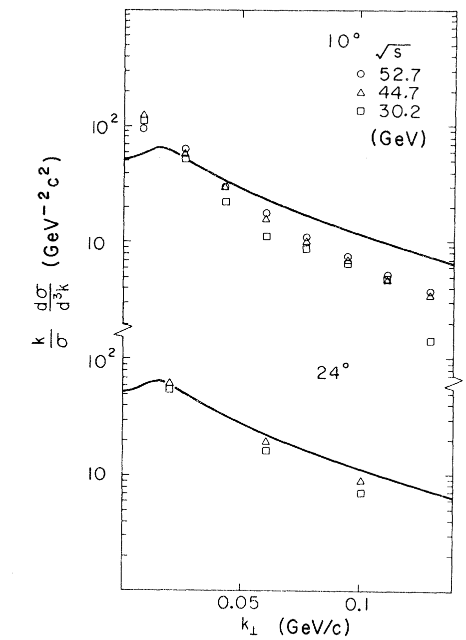

FIG. 7. Some of the data of Ref. 7 for $k d\sigma/\sigma d^3k$ at low transverse photon momenta and fixed angles of 10° and 24° away from the beam direction. The curves are predictions for $x = 0$ photons based on an assumed pion spectrum proportional to $\exp(-ap_\perp^2)$ with $am^2 = 0.3$. The normalization for the curve is determined by Eq. (32). See Eq. (69). The data points have values of x between 0 and 0.05.

$$\frac{E}{\sigma}\frac{d\sigma}{d^3p}(\vec{k}, s) \approx -\frac{1}{2}\left(1 + k_\perp\frac{\partial}{\partial k_\perp} + x\frac{\partial}{\partial x}\right)\frac{A}{k_\perp}\exp\left(-\frac{x}{x_0} - \frac{k_\perp}{k_0}\right)$$
$$\simeq \frac{1}{2}\left(\frac{x}{x_0} + \frac{k_\perp}{k_0}\right)\frac{A}{k_\perp}\exp\left(-\frac{x}{x_0} - \frac{k_\perp}{k_0}\right). \quad (69)$$

In particular, at $x = 0$, we have

$$\frac{E}{\sigma}\frac{d\sigma}{d^3p}(x=0) \simeq \frac{A}{2k_0}e^{-p_\perp/k_0}. \quad (70)$$

Using $\sigma_{\text{inel}} = 33$ mb, we have for the π^0 distribution

$$E\frac{d\sigma}{d^3p}(x=0) \simeq 150\, e^{-6.2p_\perp}\ \text{mb}/(\text{GeV}^2/c^3). \quad (71)$$

Data taken by the Saclay-Strasbourg Collaboration at the CERN Intersecting Storage Rings in the same range ($s = 900$–2800 GeV^2) gave essentially identical π^+ and π^- distributions at $x = 0$.[7] These distributions could be parametrized by

$$E\frac{d\sigma}{d^3p} \simeq 140\, e^{-6.25p_\perp}\ \text{mb}/(\text{GeV}^2/c^3) \quad (72)$$

in confirmation of the prediction that at $x = 0$ the π^+, π, and π^0 distributions must coincide at very high energies.

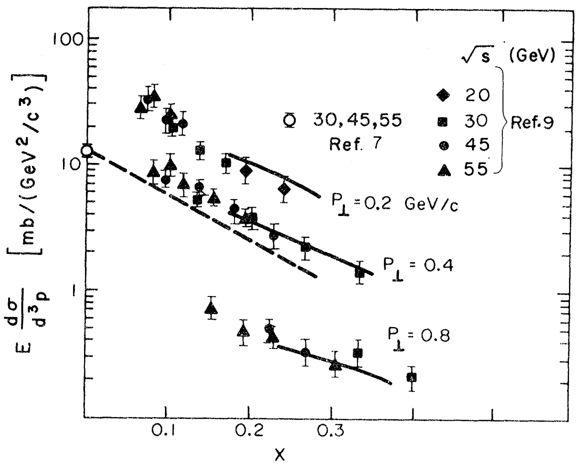

FIG. 8. The data of Refs. 7 and 9 for the invariant cross section for $(p:\pi^-|p)$. The solid curves are fits to lower-energy data for $(p:\pi^-|p)$. The dashed curve is the value of the invariant cross section for $(p:\pi^0|p)$ at $p_\perp = 0.4$ GeV/c derived with the Sternheimer approximation, Eq. (72). The data of Ref. 10 for $(p:\pi^+|p)$ are slightly higher than those shown for $(p:\pi^-|p)$. Isospin invariance together with scaling requires that $(p:\pi^0|p) = \frac{1}{2}[(p:\pi^+|p) + (p:\pi^-|p)]$.

For $x > 0$, isospin invariance requires for scaled fragmentation of protons[8]

$$(p:\pi^0|p) = \tfrac{1}{2}[(p:\pi^+|p) + (p:\pi^-|p)]. \quad (73)$$

A comparison is shown in Fig. 8. The agreement is less satisfactory for $x > 0$ than for $x = 0$.

The data shown are for $(p:\pi^-|p)$ and are taken from Refs. 7 and 9. For $x > 0$, we would expect the π^0 curve to lie above the data, since the π^+ data of Ref. 10 are slightly higher. While the relation in Eq. (73) does not appear to be satisfied for the data of Refs. 9 and 10, we should like to emphasize that this relation is on very good footing. It requires only the dominance of an $I = 0$ amplitude in the $b\bar{b}$ channel: Factorization of the Pomeranchukon is not required. Thus the good agreement with the data of Ref. 7 is reassuring and we expect to see similar agreement for $x > 0$ as the accuracy of the experiments improves. As a result of η production, the number of protons produced should be slightly greater than the number of charged pions produced. With very precise measurements, this discrepancy could be used to deduce the magnitude of the η production.

A comparison similar to those made above has been performed by Charlton and Thomas who found good agreement between the charged- and neutral-pion data.[11] In their comparison, however, they treated the data of Refs. 9 and 10 as if they were taken at 90°, and compared them with the inferred π^0 spectrum at 90°. Actually, the charged-pion data they used were for $x > 0.05$ and should have been compared with the inferred pion spectrum for the same x values. Had this been done, the inferred π^0 spectrum would lie below the charged-pion data as in Fig. 8. This is a strong reminder that in many instances a value of $x = 0.05$ is not necessarily small.

The parametrization of Neuhofer *et al.* gives us a means of evaluating the photon multiplicity and implicitly the charged-pion multiplicity. From Eq. (66) we find

$$\langle n_\gamma \rangle \simeq 2\pi A k_0 \ln\left(\frac{x_0^2}{k_0^2} s\right) \qquad (74)$$

from which Neuhofer *et al.* deduced a photon multiplicity at $s = 2000 \text{ GeV}^2$ of 9.4. By our isospin equality, Eq. (73), this means that the charged-pion multiplicity is also 9.4. The photon data from $s = 900$ to 2800 GeV^2 showed no clear energy dependence while data on the production of charged particles appear to show energy dependence.[12] A good deal of caution is called for under these circumstances. The Saclay-Strasbourg Collaboration found the charged-pion production at 90° to be energy independent from $s = 910$ to 2800 GeV^2. This is consistent with the constancy of the photon spectrum, but it would require that the energy dependence be due entirely to nonpion sources.

VIII. SUMMARY

We have shown how the scaled photon spectrum can be derived from a given scaled π^0 spectrum. The inverse procedure can be accomplished approximately by a modified version of Sternheimer's prescription. These procedures may be useful in testing symmetry relations in inclusive reactions.

In the central region ($x = 0$) the photon distribution at low transverse momentum consists of two parts. One component comes from π^0 decays and its magnitude is related to the integral of the spectrum over all values of the transverse momentum at $x = 0$. The second component is due to bremsstrahlung, primarily from the creation of charged particles in the collision. This component behaves as $1/k_\perp^2$ and is nearly proportional to the average charged multiplicity.

The analytic structure of the scaled photon distributions has been examined. The distribution is continuous as $x \to 0$ except for $k_\perp = 0$.

Data from the CERN ISR is consistent with the theoretical predictions presented. More precise measurements of the very low transverse momentum region would be useful and could in principle provide a means for measuring the average charged multiplicity.

Note added in proof. The incoherence assumption used in Sec. VI may not be appropriate at high energies since produced particles are concentrated in the extreme forward and backward directions. If we assume that all charged particles are relativistic and moving along the beam direction, either forward or backward, we find for the bremsstrahlung spectrum

$$\frac{k d\sigma}{\sigma_{\text{tot}}\, d^3k} = \frac{\alpha}{\pi^2 k_\perp^2} \langle (\Delta Q_R)^2 \rangle ,$$

where $\langle (\Delta Q_R)^2 \rangle$ is the mean value of the square of the final right-moving charge minus the initial right-moving charge. If there are no correlations, $\langle (\Delta Q_R)^2 \rangle = \langle n^{\text{ch}} \rangle / 2$ (Ref. 13).

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APPENDIX: DERIVATION OF EQ. (41)

For small $k_\perp/m$, with am^2 fixed, $p_0(k_\perp)$ can be approximated by $-m^2/4k_\perp$ so that (39) becomes

$$f_\gamma(0, k_\perp^2 \text{ small}) \simeq \frac{4}{am^2} \int_0^\infty dz \exp\left[-a\left(\frac{2k_\perp z}{am^2} - \frac{m^2}{4k_\perp}\right)^2\right] e^{-z} I_0(z) . \qquad (\text{A1})$$

The final factor, $e^{-z} I_0(z)$, is well behaved for all z in the range of integration. The other factor of the integrand is largest near $z = am^4/(8k_\perp^2)$. For z of order $m/k_\perp$, this factor is itself of order $\exp[-am^4/(16k_\perp^2)]$. Thus we can replace our lower limit of integration by one of order $m/k_\perp$. We have then

$$f_\gamma(0, k_\perp^2 \text{ small}) \simeq \frac{4}{am^2} \int_{m/k_\perp}^\infty dz \exp\left[-\left(\frac{2k_\perp z}{am^2} - \frac{m^2}{4k_\perp}\right)^2\right] e^{-z} I_0(z) . \qquad (\text{A2})$$

Again using the dominance of the region where z is of order $m^2/k_\perp^2$, we can replace $e^{-z}I_0(z)$ by its asymptotic value, $(2\pi z)^{-1/2}$. Changing variables, with $u=2k_\perp z/am^2-m^2/4k_\perp$, we find

$$f_\gamma(0, k_\perp^2 \text{ small}) \simeq 2(\pi a m^2 k_\perp)^{-1/2}\int_{2/am-m^2/4k_\perp}^{\infty} du\, e^{-au^2}\left(u+\frac{m^2}{4k_\perp}\right)^{-1/2}. \tag{A3}$$

Now over the region of integration, we see that $(u+m^2/4k_\perp) \geq 2/am$, so that this factor is well behaved. The integration is dominated by values of u of order $a^{-1/2}$, so we can replace the final factor by $(m^2/[4k_\perp])^{-1/2}$ and finally obtain

$$\begin{aligned} f_\gamma(0, k_\perp^2 \text{ small}) &\simeq 2(\pi a m^2 k_\perp)^{-1/2}[m^2/(4k_\perp)]^{-1/2}a^{-1/2}\pi^{1/2} \\ &\simeq 4(am^2)^{-1}. \end{aligned} \tag{A4}$$

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