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Note on the Generalization of the Burnett-Kroll Soft-Photon Theorem to Polarized Cross Sections*

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Burnett and Kroll have shown from the Low soft-photon theorem that an unpolarized radiative cross section is given by an operator acting on the corresponding unpolarized nonradiative cross section. Tarasov has generalized this result to cases when all spins are not summed. We rederive his result here and are able to express it in a substantially simpler and more useful way. Specifically we show that the cross section for a radiative process in which the particles involved have some particular polarization configuration is given through the first two orders in the photon momentum k by a simple operator acting on the cross section for the nonradiative process with arbitrary polarization.

A number of years ago Low[1-4] showed that the amplitude for a radiative process was given through terms of order $1/k$ and k^0 in the photon momentum by the corresponding amplitude for the nonradiative process. This theorem has subsequently been extremely useful in analyzing radiative processes,[5] particularly decay processes where the photon energy is limited by kinematics, and in delimiting the kinds of new information one can expect to obtain from radiative reactions.

More recently the theorem was extended by Burnett and Kroll[6] and Bell and Van Royen.[7] They showed, simply by squaring the Low amplitude and proving that certain cross terms vanished, that the radiative *cross section* (or, more precisely, the square of the radiative amplitude) *summed on all spins* was given through the first two orders in k by an operator acting on the corresponding *unpolarized,* nonradiative *cross section*. Furthermore the operator was very simple, depending only on multiplicative factors and first derivatives, and so can be used as a convenient and practical method for simplifying the actual calculation of a radiative cross section.[5]

The sum on spins was a crucial part of the proof of Burnett and Kroll. However the Low theorem actually contains spin information, since it gives the radiative amplitude for any particular spin configuration in terms of quantities which can be obtained from a complete knowledge of the nonradiative amplitude for the same spin configuration. Thus one is led to ask, can one generalize the Burnett-Kroll theorem and remove the requirement of a sum on spins. In particular, two questions are suggested. First, can one show formally that the cross section for a radiative process for particular polarization states of the particles involved, i.e., the polarized cross section, depends only on the polarized cross sections for the nonradiative process. Second, can one put such a result in a useful form. That is, can one write the polarized radiative cross section as a simple operator acting on the polarized nonradiative cross sections in a fashion analogous to the Burnett-Kroll theorem.

The first of these questions has in fact been considered previously and answered in the affirmative.[8] In the course of our discussion we will obtain this same result by a slightly different method which, we feel, is simpler and more straightforward.

The main purpose of this note however, is to show that the answer to the second question is also affirmative. That is, we extend somewhat the trick employed by Burnett and Kroll and use it to obtain explicitly an operator which, acting on a polarized nonradiative cross section, gives the polarized radiative cross section through the first two orders in k. This operator involves more terms than the Burnett-Kroll operator and contains magnetic moments as well as charges. It is however simple, in that it has at worst first derivatives, and it provides a direct generalization of the Burnett-Kroll theorem to cases where all spins are not summed.

Finally we prove for the new result, just as has been proved for the Low theorem and the Burnett-Kroll theorem,[7,9] that the particular analytic form or the particular kinematic variables chosen for the nonradiative amplitude make no difference in the radiative amplitude through the first two orders in k.[10]

These results are important for two major reasons. In the first place the general result sets limits on the amount of new information which can be obtained by measuring polarizations in radiative processes in the soft-photon region. In particular it means that as long as the soft-photon terms are the only important ones the polarized cross section for a radiative process is completely determined by the corresponding nonradiative polarized cross section, which may often be more accessible experimentally, and by the electromagnetic parameters of the particles involved. However, when polarizations are measured, the cross section for a radiative process contains information about anomalous magnetic moments, which, as a consequence of the Burnett-Kroll theorem, is not the case for unpolarized cross sections. Thus measurements of polarized radiative processes even in the soft-photon region may be a useful way of determining these moments. Secondly, the operator formula to be derived here provides a simple shortcut for actually carrying out the algebra necessary to calculate a radiative cross section, since with it one really only has to calculate the nonradiative cross section, which generally will be much simpler. Thus for example, this theorem should simplify predictions for radiative decays of polarized particles, just as the Burnett-Kroll theorem has been used to simplify the calculation of unpolarized processes.[5,11]

To provide the starting point for our calculation and to establish notation we review briefly the result given by the Low theorem for the amplitude for a radiative process. For details of the derivation the reader is referred to the original papers of Low,[1] Feshbach and Yennie,[3] and Adler and Dothan[4] or to the review contained in Ref. 5. With the exception of metric[12] we will follow the notation of Ref. 5.

Consider the radiative process $a+c\to b+d+\gamma$, where c and d are spin-$\frac{1}{2}$ fermions and a and b stand for any number of spin-0 bosons. For simplicity of notation we have restricted ourselves to a single incident and single final fermion, though the generalization to more than one is straightforward. Let the four-momentum, mass, charge, and anomalous magnetic moment of particle i be p_i, m_i, Q_i, and κ_i, respectively, and let k and ϵ be the four-momentum and polarization vectors of the photon. Take s_i, with $s_i\cdot s_i=-1$, to be a four vector which reduces in the rest frame of particle i to a unit vector along the spin. The Low theorem then gives for the amplitude for the radiative process the following formula (in the notation of Ref.5):

$$\begin{aligned}T_{\rm rad} &\equiv T_{\rm rad}(p_a,p_b,p_c,p_d,k)\\ &=\hat{Q}\bar{u}(p_d,s_d)T_0u(p_c,s_c)+\sum_i Q_iD_\lambda(p_i)\bar{u}(p_d,s_d)\frac{\partial T_0}{\partial p_{i\lambda}}u(p_c,s_c)\\ &\quad+\bar{u}(p_d,s_d)T_0\left(Q_c+\frac{\kappa_c}{2m_c}(\gamma\cdot p_c+m_c)\right)\frac{\gamma\cdot k\gamma\cdot\epsilon}{2k\cdot p_c}u(p_c,s_c)+\bar{u}(p_d,s_d)\frac{\gamma\cdot\epsilon\gamma\cdot k}{2k\cdot p_d}\left(Q_d+\frac{\kappa_d}{2m_d}(\gamma\cdot p_d+m_d)\right)T_0u(p_c,s_c)+O(k)\,.\end{aligned} \tag{1}$$

In this expression

$$\hat{Q}=\sum_i\eta_iQ_i\frac{\epsilon\cdot p_i}{k\cdot p_i} \tag{2}$$

and

$$D_\lambda(p_i)=\frac{\epsilon\cdot p_i}{k\cdot p_i}k_\lambda-\epsilon_\lambda\,, \tag{3}$$

where $\eta_i=+1(-1)$ for outgoing (incoming) particles and where the sum on i is over all particles.

The amplitude $T_0\equiv T_0(p_a,p_b,p_c,p_d)$ has the same analytic form as the on-mass-shell amplitude for the nonradiative process $a+c\to b+d$. Since $p_i^2=m_i^2$ it is, in the simplest case, a function of two independent scalar variables, which however must be evaluated at values of the four-momenta p_i satisfying $p_a+p_c=p_b+p_d+k$. Thus in general T_0 corresponds to the nonradiative amplitude evaluated at a kinematic point which may be slightly unphysical for the nonradiative process. If one wishes of course one may make a further expansion of the p_i about a set p_i' satisfying $p_a'+p_c'=p_b'+p_d'$ (cf. Ref. 6) or equivalently choose an appropriate set of variables (cf. Ref. 1) so that Eq. (1) holds with T_0 interpreted as the actual nonradiative amplitude evaluated at a physical point for the nonradiative process.

In general there may be a number of expressions for T_0 which are equivalent for the nonradiative process. These are related to each other by Dirac algebra or correspond simply to different choices of the independent scalar variables appearing in T_0. It has been shown however[3,7,9] that different choices of T_0 change the radiative amplitude only by terms of order k, and thus from the point of view of the Low theorem are equivalent.

To obtain the Burnett-Kroll result one simply squares Eq. (1), sums on spins, and uses the relation γ_μ

$=\partial(\gamma\cdot p)/\partial p^\mu$ to put the result in simple form. To generalize this result we also begin by squaring Eq. (1), but without the sum on spins. This gives

$$\begin{aligned}|T_{\rm rad}|^2 &= \hat{Q}^2|\bar{u}(p_d, s_d)T_0 u(p_c, s_c)|^2 \\ &+\left(\hat{Q}\bar{u}(p_d, s_d)T_0u(p_c, s_c)\bar{u}(p_c, s_c)\sum_i Q_i D_\lambda(p_i)\frac{\partial \overline{T}_0}{\partial p_{i\lambda}} u(p_d, s_d)+\text{c.c.}\right) \\ &+\left[\hat{Q}\bar{u}(p_d, s_d)T_0u(p_c, s_c)\bar{u}(p_c, s_c)\overline{T}_0\left(Q_d+\frac{\kappa_d}{2m_d}(\gamma\cdot p_d+m_d)\right)\frac{\gamma\cdot k\gamma\cdot\epsilon}{2k\cdot p_d}u(p_d, s_d)+\text{c.c.}\right] \\ &+\left[\hat{Q}\bar{u}(p_d, s_d)T_0u(p_c, s_c)\bar{u}(p_c, s_c)\frac{\gamma\cdot\epsilon\gamma\cdot k}{2k\cdot p_c}\left(Q_c+\frac{\kappa_c}{2m_c}(\gamma\cdot p_c+m_c)\right)\overline{T}_0u(p_d, s_d)+\text{c.c.}\right]+O(k^0)\,, \qquad (4)\end{aligned}$$

where $\overline{T}_0=\gamma_0 T_0^\dagger\gamma_0$. Clearly the first term is in the desired form of an operator $\hat{Q}^2$ acting on the polarized nonradiative cross section $|\bar{u}(p_d, s_d)T_0u(p_c, s_c)|^2$. We now want to simplify the remaining terms and put them in an analogous form.

Consider first the third term and write it out explicitly:

$$\begin{aligned}&\hat{Q}\bar{u}(p_c, s_c)\overline{T}_0\left[\left(Q_d+\frac{\kappa_d}{2m_d}(\gamma\cdot p_d+m_d)\right)\frac{\gamma\cdot k\gamma\cdot\epsilon}{2k\cdot p_d}u(p_d, s_d)\bar{u}(p_d, s_d)\right. \\ &\qquad\left.+u(p_d, s_d)\bar{u}(p_d, s_d)\frac{\gamma\cdot\epsilon\gamma\cdot k}{2k\cdot p_d}\left(Q_d+\frac{\kappa_d}{2m_d}(\gamma\cdot p_d+m_d)\right)\right]T_0u(p_c, s_c)\,. \qquad (5)\end{aligned}$$

Our aim is to express the term in brackets as some operator acting on $u(p_d, s_d)\bar{u}(p_d, s_d)$. To this end we write $u\bar{u}$ in terms of the standard projection operator, i.e.,

$$u(p_d, s_d)\bar{u}(p_d, s_d)=\frac{(1+\gamma_5\gamma\cdot s_d)(\gamma\cdot p_d+m_d)}{4m_d}\,. \qquad (6)$$

It is then perfectly straightforward, albeit tedious, to commute first $u\bar{u}$ and then $\gamma\cdot\epsilon\gamma\cdot k$ to the right in the second half of Eq. (5) above. This gives

$$\begin{aligned}&\frac{\hat{Q}}{4m_d}\bar{u}(p_c, s_c)\overline{T}_0\Big[Q_d(1+\gamma_5\gamma\cdot s_d)\gamma\cdot D(p_d) \\ &\qquad+(Q_d+\kappa_d)\gamma_5\gamma\cdot E(p_d)(\gamma\cdot p_d+m_d)+\frac{\kappa_d}{m_d}s_d\cdot D(p_d)\gamma_5(\gamma\cdot p_d+m_d)\Big]T_0u(p_c, s_c)\,, \qquad (7)\end{aligned}$$

where

$$E_\lambda(p_i)=\frac{\epsilon\cdot s_i}{k\cdot p_i}k_\lambda-\frac{k\cdot s_i}{k\cdot p_i}\epsilon_\lambda\,. \qquad (8)$$

In principle Eq. (7), together with a similar expression for the fourth term of Eq. (4), gives, when substituted into Eq. (4), a result equivalent to that obtained by more formal methods in Ref. 8. To actually show that explicitly however one must carry out some rather complicated and tedious, though perfectly straightforward, algebra and in addition project from the square of the matrix element the polarization tensors used there.

We, however, want to go further and cast the result in a much simpler and more compact form which is analogous to the Burnett-Kroll theorem. The crucial observation necessary to do this is the following. If we consider the spin vectors s_i as independent variables then we can write

$$\gamma_5\gamma_\mu=\frac{\partial}{\partial s^\mu}(1+\gamma_5\gamma\cdot s) \qquad (9)$$

and

$$\gamma_5(\gamma\cdot p+m)=\frac{p_\lambda}{m}\frac{\partial}{\partial s_\lambda}(1+\gamma_5\gamma\cdot s)(\gamma\cdot p+m) \qquad (10)$$

which are simply generalizations of the relation

$$\gamma_\mu=\frac{\partial}{\partial p^\mu}(\gamma\cdot p+m) \qquad (11)$$

used by Burnett and Kroll. Using these relations Eq. (7) becomes

$$\hat{Q}\bar{u}(p_c, s_c)\bar{T}_0\left[\left(Q_d D_\lambda(p_d)\frac{\partial}{\partial p_{d\lambda}} + (Q_d+\kappa_d)E_\lambda(p_d)\frac{\partial}{\partial s_{d\lambda}} + \frac{\kappa_d}{m_d}D(p_d)\cdot s_d \frac{p_{d\lambda}}{m_d}\frac{\partial}{\partial s_{d\lambda}}\right)\frac{(1+\gamma_5\gamma\cdot s_d)(\gamma\cdot p_d+m_d)}{4m_d}\right]T_0 u(p_c, s_c)\,. \tag{12}$$

Observe that, although we have chosen the particular order $(1+\gamma_5\gamma\cdot s)(\gamma\cdot p+m)$ for the derivation, the result must be independent of this order, since the two factors commute. One can show this explicitly simply by starting with $(\gamma\cdot p+m)(1+\gamma_5\gamma\cdot s)$ and using $s\cdot D(p)+p\cdot E(p)=0$ to show that the end results are equivalent.

Now we clearly can sum the operator in large parentheses in Eq. (12) over all particles without changing the result. Furthermore T_0 does not contain the spin vectors so that $\partial T_0/\partial s=0$. Thus this operator can be combined with a similar one obtained from the last term of Eq. (4) and with the $\partial T_0/\partial p_i$ terms of the same equation to give for the entire radiative cross section

$$|T_{\text{rad}}|^2=\left[\hat{Q}^2+\hat{Q}\sum_i\left(Q_i D_\lambda(p_i)\frac{\partial}{\partial p_{i\lambda}} + (Q_i+\kappa_i)E_\lambda(p_i)\frac{\partial}{\partial s_{i\lambda}} + \frac{\kappa_i}{m_i^2}\,s_i\cdot D(p_i)p_{i\lambda}\frac{\partial}{\partial s_{i\lambda}}\right)\right]|\bar{u}(p_d, s_d)T_0 u(p_c, s_c)|^2+O(k^0)\,. \tag{13}$$

This equation provides the desired simple form for the generalization of the Burnett-Kroll theorem, that is, it gives the radiative cross section in terms of an operator involving at most first derivatives acting on the nonradiative cross section. Observe that the derivatives with respect to s, e.g., $A\cdot\partial/\partial s$, just provide a compact way of saying that one should replace the vector s, which appears only linearly, with the vector A .[13] If we now sum on one spin, we get a formula for a partially polarized cross section. If we sum on all spins we get, as we must, just the Burnett-Kroll theorem, since the last two terms are linear in s and so vanish.

As before, the quantity $|\bar{u}(p_d, s_d)T_0 u(p_c, s_c)|^2$ is the on-mass-shell, nonradiative cross section evaluated for values of the momenta satisfying $p_a+p_c=p_b+p_d+k$. In general it can be written as

$$|T_{\text{nonrad}}|^2=|\bar{u}(p_d, s_d)T_0 u(p_c, s_c)|^2=A+s_d^\mu B_\mu+s_c^\mu C_\mu+s_d^\mu s_c^\nu D_{\mu\nu}\,, \tag{14}$$

where A, B, C, and D are the observable (or calculable) quantities and are functions of the various scalar invariants. However there remains the usual ambiguity in that there may be different choices of A, B, C, and D which are identical for the nonradiative process but which differ when evaluated for the radiative variables. We now show that the difference in $|T_{\text{rad}}|^2$ caused by such ambiguities is of order k^0.

Suppose that there are two choices A, B, C, D and A', B', C', D' which give identical results for the nonradiative process. Their difference must then be proportional to $X=p_a+p_c-p_b-p_d$ which vanishes in the kinematic region corresponding to the nonradiative process. Thus we can write

$$|T_{\text{nonrad}}|^2-|T'_{\text{nonrad}}|^2=(\hat{A}_\tau+s_d^\mu\hat{B}_{\mu\tau}+s_c^\mu\hat{C}_{\mu\tau}+s_d^\mu s_c^\nu\hat{D}_{\mu\nu\tau})X^\tau\,, \tag{15}$$

where $\hat{A}\cdot X=A-A'$, etc. Substituting this into our basic result, Eq. (13) gives

$$|T_{\text{rad}}|^2-|T'_{\text{rad}}|^2=\left(\hat{Q}^2X^\tau-\hat{Q}\sum_i\eta_i Q_i D^\tau(p_i)\right)(\hat{A}_\tau+s_d^\mu\hat{B}_{\mu\tau}+s_c^\mu\hat{C}_{\mu\tau}+s_d^\mu s_c^\nu\hat{D}_{\mu\nu\tau})+O(k^0) \tag{16}$$

which is of $O(k^0)$ since

$$\sum_i\eta_i Q_i D^\tau(p_i)=\hat{Q}k^\tau-\sum_i\eta_i Q_i\epsilon^\tau\,, \tag{17}$$

$\sum_i\eta_i Q_i=0$ by charge conservation, and $X^\tau=k^\tau$.

To summarize briefly, the central result of this note is Eq. (13) which generalizes in a simple way the Burnett-Kroll theorem to cases where not all spins are summed. This result gives the first two orders in k of the cross section for a radiative process in terms of a simple operator acting on the nonradiative cross section. It means that to this order, a polarized or partially polarized radiative cross section is completely determined by knowledge of the nonradiative polarized cross sections and by the electromagnetic properties of the particles involved. The result also allows one to calculate a radiative cross section by simply evaluating the nonradiative cross section and applying the operator. Since the operator is relatively simple, this may be significantly easier algebraically than simply squaring the complete radiative amplitude of Eq. (1), especially when several spins are not summed. Finally we have shown that for the new theorem, just as for the Low and Burnett-Kroll theorems, the particular form of the nonradiative amplitude or the particular set of variables chosen makes no difference in the radiative cross section through the first two orders in k.

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[13]It should be noted that because of the explicit appearance of derivatives with respect to polarization vectors in the operator in Eq. (13), knowledge of the nonradiative cross section for arbitrary direction of polarization is required in order to determine the radiative cross section for a particular polarization configuration. That is to say, the components of the functions A, B_μ, C_μ, and $D_{\mu\nu}$ defined in Eq. (14) must be known separately rather than just the linear combination required for a particular polarization configuration. This is in contrast to the situation for the original Burnett-Kroll theorem where the single function A, determined by the unpolarized nonradiative cross section, is sufficient to determine the first two orders in k of the unpolarized radiative cross section.

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Inclusive Photon Distributions: Contributions from π^0's and Bremsstrahlung*

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Photons produced in high-energy hadronic collisions come primarily from π^0 decays. If the π^0 inclusive cross section exhibits Feynman scaling, then so does the derived photon spectrum. The scaled photon-spectrum behavior in the central region ($x = 0$) is examined in detail. The bremsstrahlung contribution can be estimated for not too energetic photons, and provides a means for measuring the mean charged multiplicity at very high energies.

I. INTRODUCTION

The study of high-energy hadronic collisions has been greatly facilitated by advances in the understanding of inclusive processes – processes in which not all the final-state particles are specified.[1] The most extensively studied inclusive processes are the single-particle inclusive reactions of the form $(a : c\,|\,b)$.[2] If particle c is not a hadron, but rather the decay product of a hadron, d, the observed spectrum is an indirect image of the original inclusive process $(a : d\,|\,b)$. In particular, the observation of $(a : \gamma\,|\,b)$ yields information primarily about $(a : \pi^0\,|\,b)$. Of course the information is not