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Degeneracy of Relativistic Cyclotron Motion

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The quantum-mechanical problem of the relativistic cyclotron motion of a charged particle in a uniform magnetic field is solved by consideration of the symmetry which the system obeys. It is shown that its symmetry is isomorphic to the Lie group called G(0,1) or G(1,0), and doubly degenerate infinite series of wave functions with a constant energy eigenvalue are labeled by the eigenvalues of the operators $\mathcal{H}^2$, L_z+S_z, and S_z. Here $\mathcal{H}$ is the relativistic Hamiltonian referred to in the present problem, and L_z and S_z are the usual orbital and spin angular momentum operators, respectively.

I. INTRODUCTORY REMARK

In a previous paper[1] it was shown that the nonrelativistic Hamiltonian H^{nr},

$$H^{\text{nr}}=\frac{1}{2m}(\Pi_x{}^2+\Pi_y{}^2)$$
$$\equiv\frac{1}{2m}\left[\left(P_x-\frac{eH}{2c}y\right)^2+\left(P_y+\frac{eH}{2c}x\right)^2\right], \quad (1)$$

which expresses the motion of a free electron in a uniform magnetic field H directed in the z direction, apart from the z component of the space coordinates, has a symmetry of the Lie group G(0, b) generated by the infinitesimal operators $A_\pm$, A_3^{nr}, and E (identity) defined as

$$A_\pm=A_x\pm iA_y\,,$$
$$A_x=-\frac{\partial}{\partial x}-i\frac{eH}{2c\hbar}y\,,$$
$$A_y=-\frac{\partial}{\partial y}+i\frac{eH}{2c\hbar}x\,, \quad (2)$$

and

$$A_3^{\text{nr}}=\frac{1}{i}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\equiv\frac{L_z}{\hbar}\,.$$

They satisfy the following commutation relations:

$$[A_+,A_-]=-\frac{2eH}{c\hbar}E\equiv-bE\,,$$
$$[A_3^{\text{nr}},A_+]=A_+\,, \quad (3)$$

and

$$[A_3^{\text{nr}},A_-]=-A_-\,.$$

Each operator commutes with H^{nr}, and all degenerate eigenfunctions $\psi_n,\psi_n',\ldots$ of semi-infinite numbers with a constant eigenvalue $(n+\frac{1}{2})\hbar\omega_c$ can be obtained by operating with the $A_\pm$ operators on any eigenfunction with the same eigenvalue; namely,

$$A_+\psi_n=\psi_n'$$
or $\quad (4)$
$$A_-\psi_n=\psi_n''\,.$$

These functions are given explicitly in Ref. 1, and each function is labeled by the eigenvalue of A_3^{nr}. Here, A_+ or A_- is nothing but the raising or lowering operator for angular momentum (L_+ or L_-), respectively. Further, when we define operators $B_\pm$ and B_3 as

$$B_+=\left(\frac{\hbar}{2m\omega_c}\right)^{1/2}A_++\left(\frac{m\omega_c}{2\hbar}\right)^{1/2}(x+iy)$$
$$\equiv-i\left(\frac{c}{2e\hbar H}\right)^{1/2}(\Pi_x+i\Pi_y)\,,$$
$$B_-=-\left(\frac{\hbar}{2m\omega_c}\right)^{1/2}A_-+\left(\frac{m\omega_c}{2\hbar}\right)^{1/2}(x-iy) \quad (5)$$
$$\equiv i\left(\frac{c}{2e\hbar H}\right)^{1/2}(\Pi_x-i\Pi_y)\,,$$

and

$$B_3 = B_+ B_- \,,$$

it can easily be seen that the operators B_+ and B_- correspond to creation and annihilation operators of a boson particle, and they (with the operators B_3 and E) form the Lie group G(0, 1) as shown by the commutation relations

$$[B_+, B_-] = -E \,,$$
$$[B_3, B_+] = B_+ \,, \tag{6}$$
and
$$[B_3, B_-] = -B_- \,.$$

The Hamiltonian H^{nr} may be written in terms of these operators as

$$H^{\text{nr}} = \hbar\,\omega_c (B_+ B_- + \tfrac{1}{2}) \,. \tag{7}$$

It is to be noted that the operator B_+ or B_- does not commute with H^{nr}, and the role of these operators, unlike that of A_+ or A_-, is one of getting a new function by raising or lowering the eigenvalue of ψ_n, namely,

$$B_\pm \psi_n = \psi_{n\pm 1} \,. \tag{8}$$

II. RELATIVISTIC CYCLOTRON MOTION

Now we treat the relativistic Hamiltonian

$$\mathcal{H} = c(\alpha_x \Pi_x + \alpha_y \Pi_y) + \beta m c^2 \tag{9}$$

from the same standpoint. It is more convenient to use $\mathcal{H}^2$ than $\mathcal{H}$, because the eigenvalue problem for the eigenfunction Φ with four components (ϕ_1, ϕ_2, ϕ_3, ϕ_4),

$$\mathcal{H}\Phi = E\Phi \,, \tag{10}$$

is equivalent to the following one for the function with two components (ϕ_1, ϕ_2), defining $S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$:

$$\mathcal{H}^2 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \equiv (2mc^2 H^{\text{nr}} + e\hbar c H S_z + m^2 c^4) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = E^2 \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} . \tag{11}$$

ϕ_3 and ϕ_4 are obtained from the relations

$$\phi_3 = \frac{c}{E + mc^2} (\Pi_x - i\Pi_y)\phi_2$$
and
$$\phi_4 = \frac{c}{E + mc^2} (\Pi_x + i\Pi_y)\phi_1 \,. \tag{12}$$

Here Eqs. (12) are to be compared with Eqs. (5) and (8). From Eq. (11) we get for E^2

$$E^2 = 2mc^2 (n - \tfrac{1}{2})\hbar\,\omega_c + e\hbar c H + m^2 c^4 , \tag{13a}$$

$$= 2mc^2 (n + \tfrac{1}{2})\hbar\,\omega_c - e\hbar c H + m^2 c^4 \,. \tag{13b}$$

Both give the same eigenvalue

$$2ne\hbar\, cH + m^2 c^4 \,. \tag{14}$$

We write the eigenfunction for (13a) as $\Phi_n(S_z = 1)$, whose first component, ϕ_1, is ψ_{n-1}. The eigenfunction for (13b) will be written as $\Phi_n(S_z = -1)$, where the second component, ϕ_2, is ψ_n. The third component, ϕ_3, and the fourth one, ϕ_4, are given by Eqs. (12).

Because of the infinite degeneracy of the functions ψ_n or ψ_{n-1} as shown for the nonrelativistic case, we have a doubly degenerate series of eigenfunctions: $\{\Phi_n(S_z = 1)\}$ and $\{\Phi_n(S_z = -1)\}$.

Now when we take A_3 as

$$A_3 = A_3^{\text{nr}} + \tfrac{1}{2}\sigma_z \tag{15}$$

in place of A_3^{nr} in Eq. (3), it is easily seen that the operators $A_\pm$ and A_3 commute with $\mathcal{H}$ and then naturally also with $\mathcal{H}^2$. (In treating $\mathcal{H}^2$, one works in a two-dimensional space, $\frac{1}{2}\sigma_z \to S_z$.) The same commutation relations hold as in the nonrelativistic case, namely, they form G(0, b) again. Then we can say that the operator A_+ or A_- plays the same role for the relativistic eigenfunction $\Phi_n(S_z = 1)$ or $\Phi_n(S_z = -1)$, respectively, as the operator A_+ or A_- does for the nonrelativistic eigenfunction ψ_{n-1} or ψ_n. On the other hand the operator $B_\pm$ [Eq. (6)] introduced before translates the one series of functions with a constant eigenvalue of S_z and an energy quantum number n to the corresponding series of functions with the same eigenvalue of S_z and the energy quantum number $n \pm 1$. But the doubly degenerate series cannot be mixed up by these operators in either case.

Let us then introduce the operators in the constant-E space satisfying the commutation relations

$$[Q_+, Q_-] = \frac{\mathcal{H}^2 - m^2 c^4}{e\hbar\, cH} Q_3 \equiv 2a^2 Q_3 \,,$$
$$[Q_3, Q_+] = Q_+ \,, \tag{16}$$
$$[Q_3, Q_-] = -Q_- \,.$$

They are

$$Q_+ = B_- \tfrac{1}{2}\sigma_+ \,,$$
$$Q_- = B_+ \tfrac{1}{2}\sigma_- \,, \tag{17}$$
$$Q_3 = \tfrac{1}{2}\sigma_z \,.$$

It is well known that the relations in Eq. (16) are identical to those which angular momentum operators satisfy; namely, they form the Lie group G(a, 0), isomorphic to the three-dimensional rotation group or SU(2). They do not commute with the Hamiltonian $\mathcal{H}$, but they do commute with $\mathcal{H}^2$. However, as stated before, it is not necessary to consider the symmetry which keeps the Hamiltonian $\mathcal{H}$ invariant, but it is sufficient for us to use the symmetry of $\mathcal{H}^2$ for our present system. Keeping in mind the commutativity with $\mathcal{H}^2$, let us operate with B_+S_- on the functions of the first series, and with B_-S_+ on the functions of the second series, in which we define

$$\tfrac{1}{2}\sigma_\pm = \begin{pmatrix} S_\pm & 0 \\ 0 & S_\pm \end{pmatrix} .$$

The resulting functions will be the functions of the second and the first series, respectively. Thus we see that the first series $\{\Phi_n(S_z=1)\}$ is connected to the second series $\{\Phi_n(S_z=-1)\}$ by the operators B_+S_-, and vice versa by the operator B_-S_+.

Each Casimir operator of G(0, b) or G(a, 0) has no difference from $\mathcal{H}^2$ apart from some constants. Thus we can conclude that the system has the invariance symmetry of G(a, 0) or G(0, b), and each degenerate function is to be labeled by the eigenvalues of the operator $\mathcal{H}^2$, A_3, and Q_3. By the way, if we construct the operator

$$B_-\tfrac{1}{2}\sigma_+ - B_+\tfrac{1}{2}\sigma_-$$

we find that it commutes (fortunately) with $\mathcal{H}$ and A_3. This may be quite accidental within our present knowledge. But this operator is just the operator that transforms the function $\Phi_n(S_z=1)$ into the function $\Phi_n(S_z=-1)$, and vice versa, in four-dimensional Dirac space.

Thus the whole space of degenerate eigenfunctions is reduced to a 4-fold-degenerate irreducible-representation space of G(0, b). On the other hand, the same space is reduced to infinitely degenerate two-dimensional irreducible-representation spaces of G(a, 0).

We have a relation similar to the relation between the three-dimensional rotation group and the permutation group in the theory of many-electron systems.

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Conformal Invariance and Field Theory in Two Dimensions*

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The relevance of the representation theory of the conformal group to quantum field theory is illustrated in two dimensions by the Thirring model. Space-time position operators and their complex extensions are defined as operator-valued functions of the generators. These complex position variables coincide with the Gel'fand-Naimark labels and can be interpreted as labels for nonorthogonal coherent states. A Hilbert-space metric then becomes necessary. It is given by the matrix elements of the metric operator G and is nontrivial for the nonanalytic supplementary series and the analytic representations. In this case G^{-1} gives the two-point function for the Thirring model. For nonanalytic representations only weak (infinitesimal) conformal invariance holds for interacting fields if causal and spectral properties are imposed, while those properties become compatible with strong (global) conformal invariance in the case of analytic representations which lead either to free fields or to interacting fields with a quantized value of the coupling constant.

I. INTRODUCTION

In the axiomatic approach to quantum field theory or S-matrix theory, the symmetries and the analyticity properties of n-point functions were investigated at first, using respectively the methods of group theory and functional analysis, in a quite separate manner. The use of group theory was largely restricted to the finite-dimensional representations of the Lorentz group in space-time or those of the internal symmetry groups in charge space. The deeper methods of unitary representations of groups that unify invariance principles with Hilbert-space construc-