Relativistic Algebraic Analogs of the Charged Harmonic Oscillator*

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A new relativistic algebraic analog of the harmonic oscillator is constructed, based on the algebra of the pseudo-orthogonal group SO(3, 2) and a certain family of Hermitian representations which have the level and multiplet structure of the harmonic oscillator. This SO(3, 2) oscillator approaches the nonrelativistic oscillator in the limit in which the parameter Ω which labels the representations tends to infinity. The relativistic character of the SO(3, 2) oscillator depends on the fact that the generators include a Lorentz subalgebra together with a set which transforms as a four-vector under that subalgebra and permits the construction of a conserved covariant electromagnetic current. It is found that the "minimal" such current coincides with the electromagnetic current of the harmonic oscillator in the nonrelativistic limit. The SU(3, 1) oscillator model previously described by Cocho *et al.* is also investigated with regard to the electromagnetic current, its conservation and its nonrelativistic limit. The two models are qualitatively very similar, except that the SO(3, 2) oscillator with 10 generators is somewhat simpler than the SU(3, 1) oscillator with 15.

I. INTRODUCTION

In this paper we compare two relativistic analogs of the charged harmonic oscillator based on representations of the algebras of SO(3, 2) and SU(3, 1).[1,2] The motivation for investigating these models arises from the successes of the symmetric oscillator quark model in predicting particle properties,[3] and the evident desirability of recasting it into a relativistic framework,[4] together with the considerable degree of success that has been achieved in applying algebraic methods to the hydrogen atom,[5] to nonquark models for elementary particles,[6,7] and to nuclear physics.[8]

We are primarily interested in investigating the new SO(3, 2) oscillator model, but feel that it is useful to present a more detailed treatment of the SU(3, 1) oscillator model at the same time. Although the SU(3, 1) oscillator (with 15 generators) is more complicated than the SO(3, 2) oscillator (with 10 generators), the two models are really quite similar, and so far as we know are the simplest examples of relativistically covariant algebraic models having a familiar system as a nonrelativistic limiting form.

After considering the one-dimensional analog of these models as an illustrative example in Sec. II we develop the relationship between the dynamical Lie algebra of p_i, ξ_i, L_i, and n for the nonrelativistic oscillator, and the Lie algebras of SO(3, 2) and SU(3, 1), in Sec. III. The construction of the electromagnetic vertices and the implications of current conservation in these models are investigated in Sec. IV. From these considerations, we relate the parameter Ω which occurs in the algebraic models to the masses and "spring constant" of the nonrelativistic oscillator and show that the minimal electromagnetic currents have the correct nonrelativistic limit (i.e., that of the nonrelativistic harmonic oscillator).

II. A ONE-DIMENSIONAL EXAMPLE

The simplified case of one space dimension will be considered first to illustrate the correspondence between the algebras and the limit that will be used in later sections. As usual, the states of the one-dimensional oscillator will be labeled by the eigenvalues of $\hat{n} \equiv a^\dagger a$:

$$\hat{n}|n\rangle = n|n\rangle , \tag{2.1}$$

where $n = 0, 1, 2, \ldots$. (The caret will be dropped in the equations below; n will denote either the operator $\hat{n}$ or its eigenvalue, as the context requires.) The actions of a and $a^\dagger$ on these states are given by

$$a^\dagger|n\rangle = (n+1)^{1/2}|n+1\rangle, \quad a|n+1\rangle = (n+1)^{1/2}|n\rangle . \tag{2.2}$$

These states and matrix elements can be diagrammed as in Fig. 1. The groups SO(1, 2) and SU(1, 1) share the same Lie algebra (the noncompact real form of A_1 in the standard notation)[9]; this algebra has three generators $E_{\pm\gamma}$ and H_γ which can be chosen to satisfy

$$[E_{+\gamma}, E_{-\gamma}] = H_\gamma, \quad [H_\gamma, E_{\pm\gamma}] = \pm 2E_{\pm\gamma} ,$$
$$E^\dagger_{\pm\gamma} = -E_{\mp\gamma}, \quad H^\dagger_\gamma = H_\gamma . \tag{2.3}$$

The states of the family of Hermitian representations of this algebra[10] which will be used here are

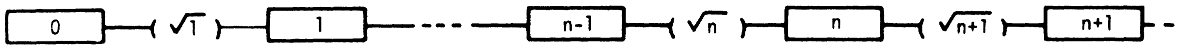

FIG. 1. States and matrix elements for the one-dimensional oscillator. The numbers in the boxes are eigenvalues of $n = a^\dagger a$; the numbers in the lines are matrix elements of $a^\dagger$ and a, which act toward the right and left, respectively.

completely labeled by the eigenvalues of $N \equiv -\frac{1}{2}H_\gamma$:

$$N|n'\rangle = (n' + \tfrac{1}{2}\Omega)|n'\rangle, \tag{2.4}$$

where $n' = 0, 1, 2, \ldots,$ and $\frac{1}{2}\Omega$, the minimum eigenvalue of N in the representation, labels the members of the family of representations. The actions of $E_{\pm\gamma}$ are given by

$$\begin{aligned} E_{-\gamma}|n'\rangle &= [(n'+1)(n'+\Omega)]^{1/2}|n'+1\rangle, \\ E_{+\gamma}|n'+1\rangle &= -[(n'+1)(n'+\Omega)]^{1/2}|n'\rangle. \end{aligned} \tag{2.5}$$

The states and matrix elements are diagrammed in Fig. 2. Comparison of Figs. 1 and 2, or of Eqs. (2.1), (2.2), (2.4), and (2.5), shows that if the states with $n = n'$ are identified, then

$$\left.\begin{aligned} \Omega^{-1/2}E_{-\gamma} &\to a^\dagger \\ -\Omega^{-1/2}E_{+\gamma} &\to a \end{aligned}\right\} \text{ in the limit } \Omega \to \infty. \tag{2.6}$$

Thus the momentum $p = i\sqrt{\frac{1}{2}}(a^\dagger - a)$ and the coordinate $\xi = \sqrt{\frac{1}{2}}(a^\dagger + a)$, though not the Hamiltonian, can be expressed in terms of the operators $E_{\pm\gamma}$, within the representation of Fig. 2, in the limit $\Omega \to \infty$. The number operator n does not have an exact analog in the SO(1, 2)-SU(1, 1) Lie algebra; but $N = n + \frac{1}{2}\Omega$ labels the energy levels in a similar way, and is an adequate substitute.[11] At the same time, for arbitrary Ω, the SO(1, 2)-SU(1, 1) algebra contains the algebra of SO(1, 1), the Lorentz group for one spacelike dimension and one timelike dimension; thus "Lorentz-covariant" vertices can be defined.

This suggests that in general a correspondence can be set up between the m-dimensional oscillator and the algebra of SO(m, 2) or SU(m, 1). (The latter two algebras are, however, distinct for $m \geq 2$.) This correspondence will be studied below for the physically interesting case $m = 3$.

III. THE ALGEBRAS OF THE OSCILLATOR, OF SO(3,2), AND OF SU(3,1)

In this section the Lie algebra of the dynamical variables of the nonrelativistic oscillator will be compared with the Lie algebras of SO(3, 2) and SU(3, 1), with special regard for the limiting forms of the latter as Ω (defined below) approaches infinity. As a common meeting ground for these algebras we shall use a form in which the generators appear either as diagonal operators or as raising and lowering operators for those diagonal operators.[9] This form is convenient for the construction of diagrams (such as Figs. 3–5 below) which clearly exhibit the approach of the matrix elements to their limiting values.

A. The Algebra of the Nonrelativistic Harmonic Oscillator

The Hamiltonian for the three-dimensional harmonic oscillator can be written in the form

$$\begin{aligned} H &= \tfrac{1}{2}\hbar\omega(\vec{p}^2 + \vec{\xi}^2) \\ &= \hbar\omega(n + \tfrac{3}{2}), \end{aligned} \tag{3.1}$$

where

$$[\xi_i, p_j] = i\delta_{ij} \tag{3.2}$$

and, as in the last section, n will be used to denote either the level-number operator or its eigenvalue. Because of the presence of angular momentum operators in the algebra, the usual annihilation and creation operators are not as appropriate here as in the one-dimensional example. Rather, we shall introduce operators $F_{\pm\gamma}$ and deal with the following set:

$$L_\pm \equiv L_1 \pm iL_2 = \xi_2 p_3 - \xi_3 p_2 \pm i(\xi_3 p_1 - \xi_1 p_3), \tag{3.3a}$$

$$L_3 = \xi_1 p_2 - \xi_2 p_1, \tag{3.3b}$$

$$F_{\pm\gamma} \equiv \pm\tfrac{1}{2}[\xi_1 + p_2 \pm i(p_1 - \xi_2)], \tag{3.3c}$$

$$n = \tfrac{1}{2}(\vec{p}^2 + \vec{\xi}^2 - 1). \tag{3.3d}$$

These operators have the Hermiticity properties

$$L_\pm^\dagger = L_\mp, \quad L_3^\dagger = L_3, \quad F_{\pm\gamma}^\dagger = -F_{\mp\gamma}, \quad n^\dagger = n, \tag{3.4a}$$

and satisfy a number of commutation relations derivable from (3.2). We shall be primarily interested in the set

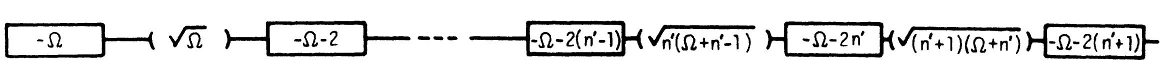

FIG. 2. States and matrix elements for a family of representations of the algebra of SO(1, 2) and SU(1, 1). The numbers in the boxes are eigenvalues of $H_\gamma = -2N$; the numbers in the lines are matrix elements of $E_{-\gamma}$ and $-E_{+\gamma}$, which act toward the right and left, respectively.

$$[L_3, L_\pm] = \pm L_\pm, \quad [n, L_\pm] = 0, \tag{3.4b}$$

$$[L_3, F_{\pm\gamma}] = \mp F_{\pm\gamma}, \quad [n, F_{\pm\gamma}] = \mp F_{\pm\gamma}, \tag{3.4c}$$

$$[L_+, L_-] = 2L_3, \quad [F_{+\gamma}, F_{-\gamma}] = -1, \tag{3.4d}$$

$$[L_3, n] = 0, \quad [L_\pm, F_{\mp\gamma}] = 0. \tag{3.4e}$$

In addition, there are some commutators that serve to define new operators,

$$\begin{aligned} F_{\pm(\alpha+\gamma)} &\equiv [L_\pm, F_{\pm\gamma}] = \xi_3 \pm i p_3, \\ F_{\pm(2\alpha+\gamma)} &\equiv [L_\pm, F_{\pm(\alpha+\gamma)}] = \mp \xi_1 - i\xi_2 - ip_1 \pm p_2, \end{aligned} \tag{3.5}$$

which satisfy

$$\begin{aligned} [L_\pm, F_{\pm(2\alpha+\gamma)}] &= [F_{\pm\gamma}, F_{\pm(\alpha+\gamma)}] \\ &= [F_{\pm\gamma}, F_{\pm(2\alpha+\gamma)}] \\ &= [F_{\pm(\alpha+\gamma)}, F_{\pm(2\alpha+\gamma)}] \\ &= 0. \end{aligned} \tag{3.6}$$

The remaining commutators are expressible in terms of the commutators (3.4b)–(3.4e) by repeated use of the Jacobi identity $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$.

Equations (3.4b) and (3.4c) show that $L_\pm$ and $F_{\pm\gamma}$ act as raising and lowering operators for the two commuting Hermitian operators L_3 and n, while (3.4d) together with (3.4a) and the second of (3.4e) give information about off-diagonal matrix elements. Choosing the eigenvalues of L_3 and n as labels for basis states, and starting from the ground state with $n=0$, $l=0$, it is a straightforward matter to construct the diagram of states shown in Fig. 3. The exact method of construction can be inferred by a careful comparison of Eqs. (3.4) and Fig. 3. It is quite similar to the usual construction of representations of the angular momentum algebra, with the exception of the use of the second of (3.4e); this relation [together with (3.4d) and the Hermiticity relations] determines when degeneracy occurs, and the values of the matrix elements when it does.

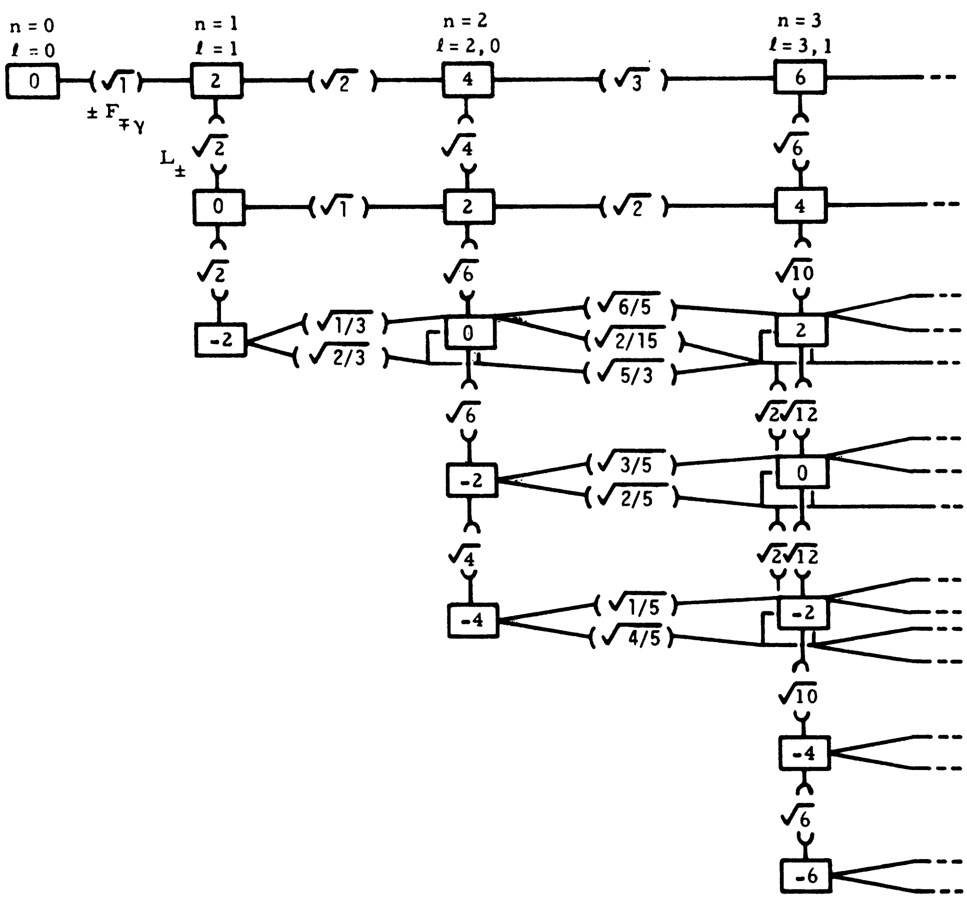

FIG. 3. States and matrix elements for the three-dimensional oscillator. The numbers in the boxes are eigenvalues of $2L_3$; the numbers in the vertical lines are matrix elements of $L_\pm$ (which act upward and downward, respectively); and the numbers in the horizontal lines are matrix elements of $\pm F_{\mp\gamma}$ (which act toward the right and left, respectively).

It should be noted that only Eqs. (3.4) need to be explicitly satisfied in the construction of such diagrams. Direct calculation, using the Jacobi identity and Eqs. (3.4)–(3.5), shows that all the commutators in (3.6) with "+" signs commute with the lowering operators L_- and $E_{-\gamma}$; and, by Hermitian conjugation, all those with "-" signs commute with the raising operators L_+ and $E_{+\gamma}$.[12] This is sufficient to ensure the vanishing of those commutators when there is a "highest" state[13]; this is the ground state in Fig. 3.

Equations (3.3c) and (3.5) can be solved for the ξ_i and p_i:

$$\begin{aligned} \xi_\pm &\equiv \xi_1 \pm i\xi_2 = \mp(\tfrac{1}{2}F_{\pm(2\alpha+\gamma)} + F_{\mp\gamma}), \\ \xi_3 &= \tfrac{1}{2}(F_{+(\alpha+\gamma)} + F_{-(\alpha+\gamma)}), \\ p_\pm &\equiv p_1 \pm ip_2 = i(\tfrac{1}{2}F_{\pm(2\alpha+\gamma)} - F_{\mp\gamma}), \\ p_3 &= -\tfrac{1}{2}i(F_{+(\alpha+\gamma)} - F_{-(\alpha+\gamma)}). \end{aligned} \tag{3.7}$$

Thus Fig. 3 determines the matrix elements of the p_i and ξ_i, as well as n, and hence provides a complete dynamical description of the nonrelativistic harmonic oscillator.

B. The Algebra of SO(3,2) and Its "Oscillator" Representations

In Appendix A it is shown that a Hermitian representation of the algebra of SO(3, 2) is characterized by the Hermiticity and commutation relations (A22) and (A23) for the subset of generators $E_{\pm\alpha}$, $E_{\pm\gamma}$, H_α, and H_γ. As for the nonrelativistic oscillator algebra, these relations can be used to construct the diagram of states, shown in Fig. 4, starting with the ground state $[0, -\Omega]$.[10] Note that this is a family of representations, since the real parameter Ω is restricted only by the condition $\Omega > \frac{1}{2}$. Figures 3 and 4 show a remarkable similarity; in fact, there is an exact correspondence if we let $\Omega \to \infty$ and make the identifications

$$L_\pm = E_{\pm\alpha}, \quad L_3 = \tfrac{1}{2}H_\alpha, \tag{3.8a}$$

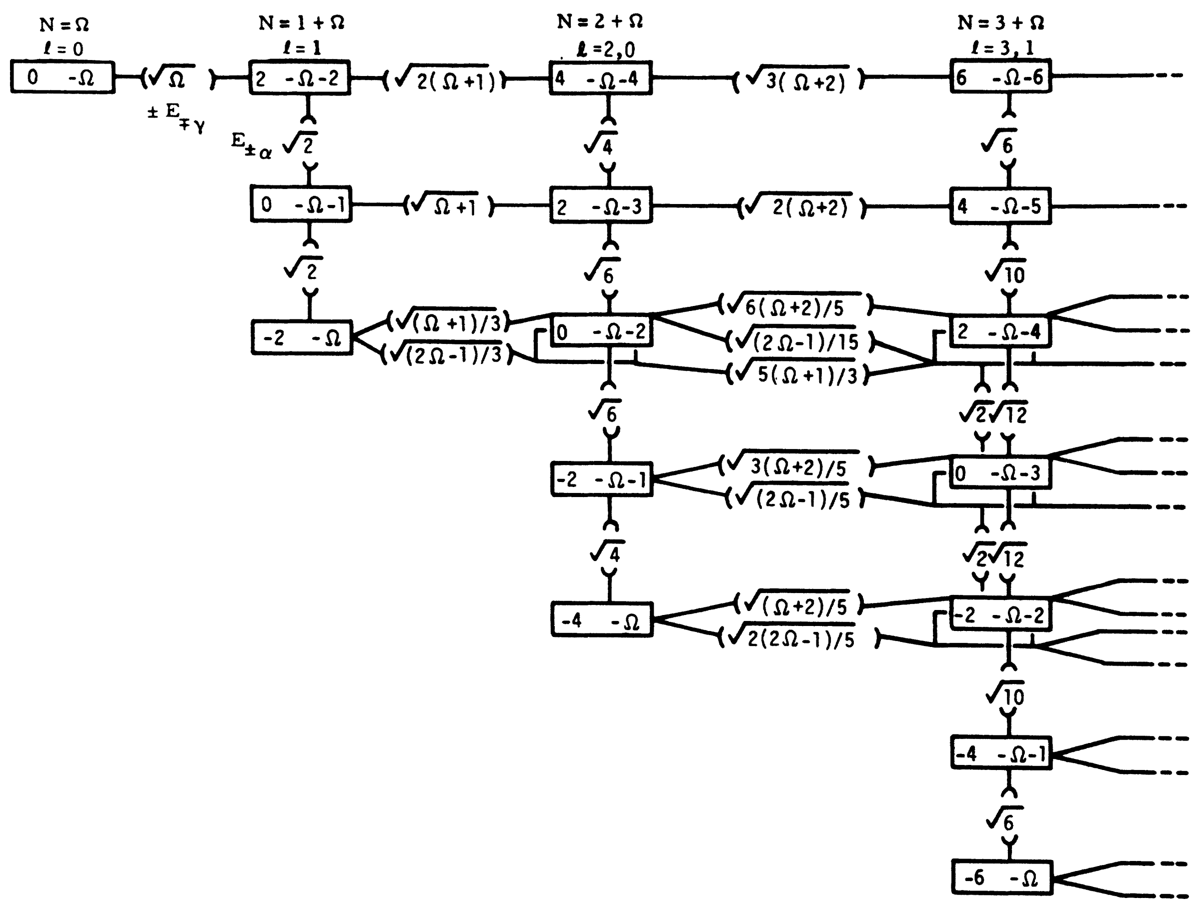

FIG. 4. States and matrix elements for the "oscillator" representations of SO(3, 2). The numbers in the boxes are eigenvalues of $H_\alpha = 2L_3$ and H_γ; the numbers in the vertical lines are matrix elements of $E_{\pm\alpha} = L_\pm$; and the numbers in the horizontal lines are matrix elements of $\pm E_{\mp\gamma}$. The operators $\Omega^{-1/2}E_{\pm\gamma}$ and $N \equiv -\frac{1}{2}H_\alpha - H_\gamma$ are the analogs of the operators $F_{\mp\gamma}$ and n, respectively, which appear in Fig. 3. The parameter Ω is subject to the condition $\Omega > \frac{1}{2}$.

$$F_{\pm\gamma}=\lim_{\Omega\to\infty}(\Omega^{-1/2}E_{\pm\gamma})\,, \tag{3.8b}$$

$$n=N-\Omega\,, \tag{3.8c}$$

where

$$N\equiv-\tfrac{1}{2}H_\alpha-H_\gamma\,. \tag{3.9}$$

Alternatively, we can compare (3.4) with the SO(3, 2) relations (A22) and (A23) written in the following equivalent form:

$$E^\dagger_{\pm\alpha}=E_{\mp\alpha},\quad H^\dagger_\alpha=H_\alpha,\quad E^\dagger_{\pm\gamma}=-E_{\mp\gamma},\quad N^\dagger=N\,, \tag{3.10a}$$

$$[H_\alpha,E_{\pm\alpha}]=\pm2E_{\pm\alpha},\quad [N,E_{\pm\alpha}]=0\,, \tag{3.10b}$$

$$[H_\alpha,E_{\pm\gamma}]=\mp2E_{\pm\gamma},\quad [N,E_{\pm\gamma}]=\mp E_{\pm\gamma}\,, \tag{3.10c}$$

$$[E_{+\alpha},E_{-\alpha}]=H_\alpha,\quad [E_{+\gamma},E_{-\gamma}]=H_\gamma\,, \tag{3.10d}$$

$$[H_\alpha,N]=0,\quad [E_{\pm\alpha},E_{\mp\gamma}]=0\,, \tag{3.10e}$$

where N is defined by (3.9). These relations become identical to (3.4) in the limit $\Omega\to\infty$ when the identifications (3.8) are made, and it is noted from Fig. 4 that H_γ has eigenvalues of the form $-\Omega-a$, where a is constant as $\Omega\to\infty$.

It is interesting to note from Fig. 4 that for large but finite Ω the correspondence between the nonrelativistic oscillator and the SO(3, 2) oscillator becomes poorer as the level number increases. This is what one would expect if the nonrelativistic oscillator is an approximation to a relativistically correct system; the nonrelativistic oscillator must fail as the excitation energy becomes comparable to the rest masses of the constituent particles.

As with the nonrelativistic oscillator algebra, there are further commutation relations which define new operators:

$$E_{\pm(\alpha+\gamma)}\equiv[E_{\pm\alpha},E_{\pm\gamma}]\,,\qquad E_{\pm(2\alpha+\gamma)}\equiv[E_{\pm\alpha},E_{\pm(\alpha+\gamma)}]\,. \tag{3.11a}$$

And again, the commutators involving these operators either vanish,

$$[E_{\pm\alpha},E_{\pm(2\alpha+\gamma)}]=[E_{\pm\gamma},E_{\pm(\alpha+\gamma)}]=[E_{\pm\gamma},E_{\pm(2\alpha+\gamma)}]=[E_{\pm(\alpha+\gamma)},E_{\pm(2\alpha+\gamma)}]=0\,, \tag{3.11b}$$

or can be expressed in terms of the commutators (3.10b)–(3.10e) by repeated use of the Jacobi identity and (3.9).

From comparison of (3.7) with (A26) and (A27) [noting the definitions in (3.5) and (3.11a), and the identifications (3.8a) and (3.8b)], we see that the coordinates and momenta are very simply related to the usual antisymmetric generators of SO(3, 2), namely,

$$\xi_i=\lim_{\Omega\to\infty}(\Omega^{-1/2}L_{i4})\,, \tag{3.12a}$$

$$p_i=\lim_{\Omega\to\infty}(\Omega^{-1/2}L_{i5})\,. \tag{3.12b}$$

The remaining generators of SO(3, 2) are the angular momenta L_{ij} and, from (A27), the number operator $N=L_{45}$.

C. The Algebra of SU(3,1) and Its "Oscillator" Representations

This model is very similar to the SO(3, 2) oscillator. The representation to be used here, shown in Fig. 5, was constructed by the use of relations (B18) and (B19) of Appendix B, starting from the ground state $[0,0,-\Omega]$.[10] Note that in this case there are three commuting generators (H_α, H_β, and H_γ) which can be used to label the states; there is no degeneracy of the states of Fig. 5 with respect to these labels. However, the correspondence with Fig. 3 is not immediately apparent, for the reason that the states are not separated into angular momentum multiplets. From Fig. 5 it is apparent that the identifications

$$L_\pm=\sqrt{2}\,(E_{\pm\alpha}+E_{\pm\beta}),\quad L_3=H_\alpha+H_\beta\,, \tag{3.13}$$

$$n=N-\tfrac{3}{4}\Omega,\quad \text{where}\quad N\equiv-\tfrac{1}{4}H_\alpha-\tfrac{1}{2}H_\beta-\tfrac{3}{4}H_\gamma\,, \tag{3.14}$$

are necessary to obtain the correct multiplet structure and level numbering. When the states of Fig. 5 are combined to form angular momentum multiplets, a diagram very similar to Fig. 4 is obtained. The eigenvalues of $2L_3$ and $L_\pm$ are, of course, the same; and the matrix elements of $\pm E_{\mp\gamma}$ differ only in the constant terms under the square-root signs. From (3.13), (3.14), (B18), and (B19), one obtains the relations

$$L^\dagger_\pm=L_\mp,\quad L^\dagger_3=L_3,\quad E^\dagger_{\pm\gamma}=-E_{\mp\gamma},\quad N^\dagger=N\,, \tag{3.15a}$$

$$[L_3,L_\pm]=\pm L_\pm,\quad [N,L_\pm]=0\,, \tag{3.15b}$$

$$[L_3,E_{\pm\gamma}]=\mp E_{\pm\gamma},\quad [N,E_{\pm\gamma}]=\mp E_{\pm\gamma}\,, \tag{3.15c}$$

$$[L_+,L_-]=2L_3,\quad [E_{+\gamma},E_{-\gamma}]=H_\gamma, \tag{3.15d}$$

$$[L_3,N]=0,\quad [L_\pm,E_{\mp\gamma}]=0\,. \tag{3.15e}$$

Comparison with (3.4) and the observation from Fig. 5 that $H_\gamma\to-\Omega$ as $\Omega\to\infty$ leads to the final identification

$$F_{\pm\gamma}=\lim_{\Omega\to\infty}(\Omega^{-1/2}E_{\pm\gamma})\,. \tag{3.16}$$

The additional operators of interest are

$$\sqrt{2}\,E_{\pm(\beta+\alpha)} \equiv \sqrt{2}\,[E_{\pm\beta}, E_{\pm\gamma}] = [L_{\pm}, E_{\pm\gamma}], \tag{3.17}$$

$$2E_{\pm(\alpha+\beta+\gamma)} \equiv 2[E_{\pm\alpha}, E_{\pm(\beta+\gamma)}] = [L_{\pm}, \sqrt{2}\,E_{\pm(\beta+\gamma)}],$$

with commutators which either vanish,

$$\begin{aligned}[L_{\pm}, E_{\pm(\alpha+\beta+\gamma)}] &= [E_{\pm\gamma}, E_{\pm(\beta+\gamma)}] \\ &= [E_{\pm\gamma}, E_{\pm(\alpha+\beta+\gamma)}] \\ &= [E_{\pm(\beta+\gamma)}, E_{\pm(\alpha+\beta+\gamma)}] \\ &= 0,\end{aligned} \tag{3.18}$$

or can be evaluated in the limit $\Omega\to\infty$ by repeated use of the Jacobi identity and (3.15), noting that the second of (3.15d) reduces to $[F_{+\gamma}, F_{-\gamma}] = -1$ in the limit (3.16). Thus the nonrelativistic limit (3.4)–(3.6) is again regained as $\Omega\to\infty$.

Comparison of (B22) and (B23) with (3.7), taking careful note of the numerical factors in (3.17), yields

$$\xi_i = \lim_{\Omega\to\infty}[(2\Omega)^{-1/2} L_{i4}], \tag{3.19a}$$

$$p_i = \lim_{\Omega\to\infty}[(2\Omega)^{-1/2} T_{i4}], \tag{3.19b}$$

where the $T_{\mu\nu}$ and $L_{\mu\nu}$ are the symmetric and antisymmetric generators of SU(3, 1). The L_{ij} (with $i, j = 1, 2, 3$) are, of course, the angular momentum operators; and $T_{44} = 2N$, from (B23). The remaining generators T_{ij} have no dynamical significance here, but will be needed for the construction of Lorentz-covariant vertices in Sec. IV.

IV. ELECTROMAGNETIC VERTICES

While the last section has shown a close correspondence between the Lie algebra of the nonrelativistic oscillator and those of the algebraic oscillators, this result is really vacuous from a physical point of view until it is shown that there are consistent interactions, in particular, with the electromagnetic field. The construction of covariant electromagnetic vertices for the SO(3, 2) and SU(3, 1) oscillators will be investigated in this section; and it will be shown that the "minimal"

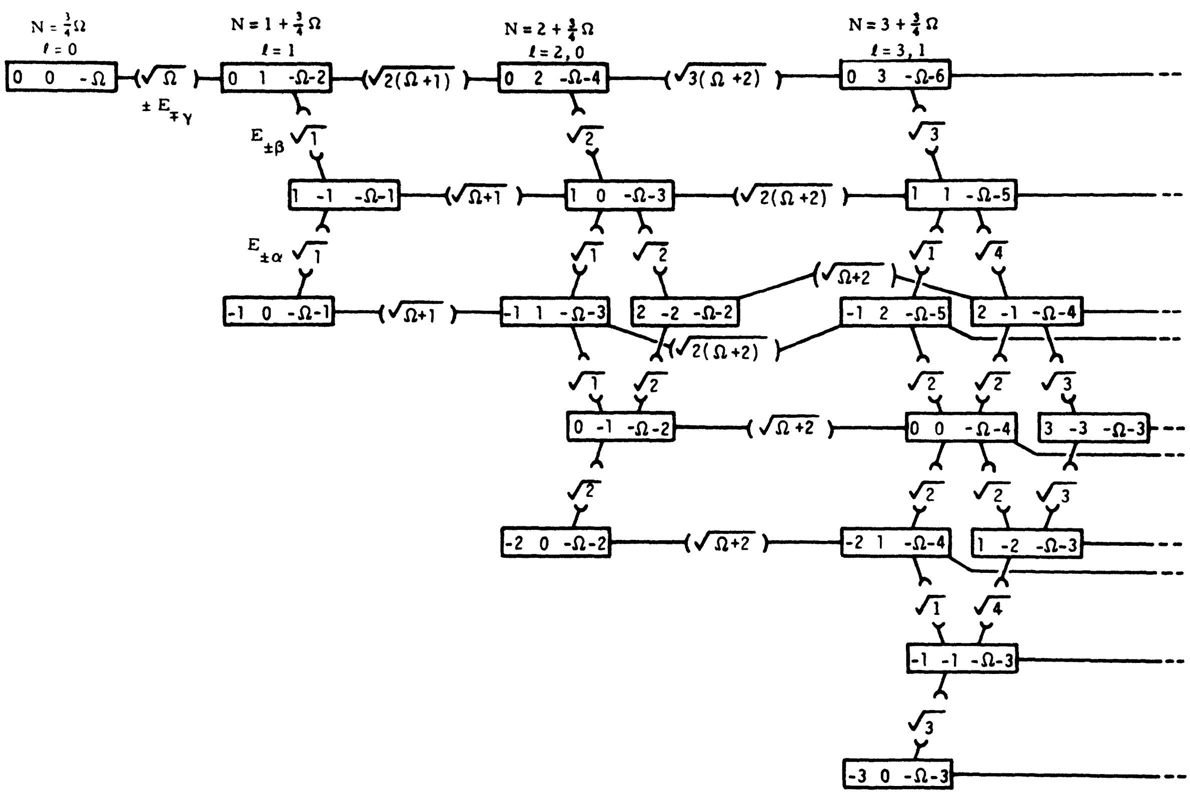

FIG. 5. States and matrix elements for the "oscillator" representations of SU(3, 1). The numbers in the boxes are eigenvalues of H_α, H_β, and H_γ; the numbers in the lines sloping downward to the left are matrix elements of $E_{\pm\alpha}$; the numbers in the lines sloping downward to the right are matrix elements of $E_{\pm\beta}$; and the numbers in the horizontal lines are matrix elements of $\pm E_{\mp\gamma}$. For this algebra, $L_{\pm} = \sqrt{2}(E_{\pm\alpha} + E_{\pm\beta})$ and $L_3 = H_\alpha + H_\beta$. $\Omega^{-1/2}E_{\mp\gamma}$ and $N = -\frac{1}{4}H_\alpha - \frac{1}{2}H_\beta - \frac{3}{4}H_\gamma$ are the analogs of the operators $F_{\mp\gamma}$ and n which appear in Fig. 3. For this representation, $\Omega > 0$.

currents for these oscillators do indeed approach that of the nonrelativistic oscillator in the appropriate limit.

A. The Electromagnetic Vertex of the Nonrelativistic Oscillator

The nonrelativistic Hamiltonian for a system of two harmonically bound masses is

$$\mathcal{H} = \tfrac{1}{2}\vec{p}_1^{\,2}/m_1 + \tfrac{1}{2}\vec{p}_2^{\,2}/m_2 + \tfrac{1}{2}\kappa(\vec{x}_1 - \vec{x}_2)^2 = \tfrac{1}{2}\vec{P}_R^{\,2}/(m_1+m_2) + \tfrac{1}{2}\hbar\omega(\vec{p}^{\,2} + \vec{\xi}^{\,2}) \tag{4.1}$$

with

$$\vec{R} \equiv \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1+m_2}, \quad \vec{P}_R \equiv \vec{p}_1 + \vec{p}_2, \tag{4.2a}$$

$$\vec{x} \equiv -\vec{x}_1 + \vec{x}_2 \equiv \left(\frac{\hbar}{\mu\omega}\right)^{1/2}\vec{\xi}, \qquad \vec{p}_x \equiv \frac{-\mu\vec{p}_1}{m_1} + \frac{\mu\vec{p}_2}{m_2} \equiv (\mu\hbar\omega)^{1/2}\vec{p}, \tag{4.2b}$$

$$\mu \equiv \frac{m_1 m_2}{m_1+m_2}, \quad \omega \equiv \left(\frac{\kappa}{\mu}\right)^{1/2}. \tag{4.2c}$$

The wave functions for this system can be written in the form

$$\psi_a(\vec{R}, \vec{x}) \equiv \langle \vec{R}, \vec{x} \,|\, a, \vec{P}_a\rangle = e^{i\vec{P}_a\cdot\vec{R}/\hbar}\,\phi_a(\vec{x}) \equiv e^{i\vec{P}_a\cdot\vec{R}/\hbar}(\vec{x}\,|\,a, \vec{0}), \tag{4.3}$$

where $\vec{P}_a$ is the total momentum [i.e., the eigenvalue of $\vec{P}_R$ for the state $|\,a, \vec{P}_a)$] and $\phi_a(\vec{x}) \equiv (\vec{x}\,|\,a, \vec{0})$ is the internal wave function describing the relative motion of the two particles.

If the oscillator potential is not to be perturbed by a Coulomb potential, only one of the particles can be charged. If e_1 and m_1 are the charge and mass of this particle, the requirement of minimal coupling applied to (4.1) yields

$$\mathcal{H}_I^{\text{em}}(\vec{x}_1) = -\frac{e_1}{2m_1}[\vec{p}_1\cdot\vec{A}(\vec{x}_1) + \vec{A}(\vec{x}_1)\cdot\vec{p}_1] \tag{4.4}$$

to lowest order in e_1. With the change of variables

$$\vec{R} \to \vec{R} + \mu\vec{x}/m_1, \quad \vec{P}_R \to \vec{P}_R, \qquad \vec{x} \to \vec{x}, \quad \vec{p}_x \to \vec{p}_x - \mu\vec{P}_R/m_1, \tag{4.5}$$

which means that $\vec{p}_1 \to \vec{P}_R - \vec{p}_x$ and $\vec{x}_1 \to \vec{R}$, we find

$$\int d^3R\, d^3x\, \psi_b^* \mathcal{H}_I^{\text{em}} \psi_a = -\frac{e_1}{2m_1}\int d^3R\, d^3x\, e^{-i\vec{P}_b\cdot(\vec{R}+\mu\vec{x}/m_1)/\hbar}\phi_b^*(\vec{x})[(\vec{P}_R - \vec{p}_x)\cdot\vec{A}(\vec{R}) + \vec{A}(\vec{R})\cdot(\vec{P}_R - \vec{p}_x)]\, e^{i\vec{P}_a\cdot(\vec{R}+\mu\vec{x}/m_1)/\hbar}\phi_a(\vec{x}). \tag{4.6}$$

When $\vec{P}_R$ is replaced by its eigenvalues ($\vec{P}_b$ in the first instance and $\vec{P}_a$ in the second), Eq. (4.6) can be put into the form

$$\int d^3R\, d^3x\, \psi_b^* \mathcal{H}_I^{\text{em}} \psi_a = -\int d^3R\, \vec{A}(\vec{R})\cdot\langle b, \vec{P}_b\,|\,\vec{\mathcal{J}}(\vec{R}, 0)\,|\,a, \vec{P}_a\rangle, \tag{4.7}$$

where we have introduced the notation

$$\langle b, \vec{P}_b|\vec{\mathcal{J}}(\vec{R}, 0)|\,a, \vec{P}_a\rangle \equiv e^{i(\vec{P}_a - \vec{P}_b)\cdot\vec{R}/\hbar}\langle b, \vec{P}_b|\vec{\mathcal{J}}(0)|\,a, \vec{P}_a\rangle \tag{4.8}$$

with

$$\langle b, \vec{P}_b|\vec{\mathcal{J}}(0)|\,a, \vec{P}_a\rangle \equiv (b, \vec{0}\,|\,\Lambda^\dagger(m_1, \vec{P}_b)\vec{J}_1\Lambda(m_1, \vec{P}_a)\,|\,a, \vec{0}), \tag{4.9}$$

where

$$\vec{J}_1 \equiv \frac{e_1}{m_1}[\tfrac{1}{2}(\vec{P}_a + \vec{P}_b) - (\mu\hbar\omega)^{1/2}\vec{p}], \tag{4.10}$$

$$\Lambda(m_1, \vec{P}_a) \equiv \exp[im_2(\mu\hbar\omega)^{-1/2}\vec{V}_a\cdot\vec{\xi}], \tag{4.11}$$

and, of course, a similar expression for $\Lambda(m_1, \vec{P}_b)$. The internal states $|\,a, \vec{0})$ are defined by (4.3). In (4.10) and (4.11), $\vec{p}_x$ and $\vec{x}$ have been replaced by the dimensionless variables $\vec{p}$ and $\vec{\xi}$ defined in (4.2b). The velocity $\vec{V}_a$ is given by $\vec{P}_a = (m_1+m_2)\vec{V}_a$. (Note that all states of a nonrelativistic system have the same mass, that of the constituent particles.) From the form of the definitions above, the kets $|a, \vec{P}_a\rangle$ and $|b, \vec{P}_b\rangle$ are evidently time-independent state vectors for the composite system.

Equations (4.9)–(4.11) give us the matrix elements which will be compared below with the matrix elements of the relativistic current. Note that the right-hand side of (4.9) has the form of an operator $\vec{J}_1$ (constructed from $\vec{P}_a$, $\vec{P}_b$, and the internal operator $\vec{p}_x$) acting between internal states which have been "boosted" by the operator Λ.[14] It will be observed that the form (4.11) for the boost operator Λ is not restricted to electromagnetic interactions; the same form would be obtained for any interaction expressible in terms of a local field and its derivatives.

It may happen, and does for the quark model, that one wishes to consider a system in which both particles are charged, and neglect the perturbation of the oscillator potential by the Coulomb potential. If the second particle had been charged instead of the first, Eqs. (4.9)–(4.11) would have been modified by the replacements

$$\vec{J}_1 \to \vec{J}_2, \quad e_1 \to e_2, \quad m_1 \to m_2, \quad \vec{p} \to -\vec{p}, \quad \vec{\xi} \to -\vec{\xi}, \tag{4.12}$$

so when both particles are charged, the matrix element of the electromagnetic current is a sum of two terms of the form (4.9)–(4.11). Of course, one term can generally be neglected if $(e_1/m_1) \gg (e_2/m_2)$ or vice versa. A simplification results also if $m_1 = m_2$, provided ϕ_a and ϕ_b are parity eigenstates. In this case, the signs of $\vec{\xi}$ and $\vec{p}$ can be inverted in the second term to obtain

$$\langle b, \vec{P}_b | \vec{\mathcal{J}}(\vec{0}) | a, \vec{P}_a \rangle = (b, \vec{0} | \Lambda^\dagger(m, \vec{P}_b) \vec{J} \Lambda(m, \vec{P}_a) | a, \vec{0}) \tag{4.13}$$

with

$$\vec{J} = \frac{\mathcal{Q}}{m}[\tfrac{1}{2}(\vec{P}_a + \vec{P}_b) - (\mu\hbar\omega)^{1/2}\vec{p}], \tag{4.14}$$

where

$$m \equiv m_1 = m_2, \quad \mathcal{Q} \equiv e_1 + \pi_a \pi_b e_2 . \tag{4.15}$$

π_a and π_b are the orbital parities of the states $|a, \vec{0})$ and $|b, \vec{0})$.

B. Relativistic Vertices and Current Conservation

It will be assumed that a general Lorentz-covariant vertex can be written in the form

$$\begin{aligned} W_{\mu\nu\cdots}(X) &\equiv e^{i(P_a - P_b)\cdot X/\hbar} \langle b, \vec{P}_b | \mathcal{J}_{\mu\nu\cdots}(0) | a, \vec{P}_a \rangle \\ &= e^{i(P_a - P_b)\cdot X/\hbar} (b, \vec{P}_b | J_{\mu\nu\cdots} | a, \vec{P}_a) \end{aligned} \tag{4.16}$$

(or as a sum of such terms), where $|a, \vec{P}_a\rangle$ and $|b, \vec{P}_b\rangle$ denote time-independent state vectors of the system, the "internal states" $|a, \vec{P}_a)$ and $|b, \vec{P}_b)$ belong to an infinite-dimensional Hermitian irreducible representation of the appropriate algebra, and the operators $J_{\mu\nu\cdots}$ are constructed from the generators of that algebra and the total four-momenta of the states.[14] The moving states $|a, \vec{P}_a\rangle$ or $|a, \vec{P}_a)$ are obtained in the usual manner by applying Lorentz boosts to the rest states, e.g.,

$$|a, \vec{P}_a) = \Lambda_a |a, \vec{0}) \equiv e^{i\vec{\zeta}_a \cdot \vec{B}} |a, \vec{0}), \tag{4.17}$$

where $\vec{\zeta}_a$ is defined by

$$P_{a\mu} = M_a c(\hat{\zeta}_a \sinh\zeta_a, \cosh\zeta_a) \tag{4.18}$$

and the $B_i \equiv L_{i4}$ are the generators of Lorentz boosts. With (4.17), Eq. (4.16) prescribes a vector vertex of the same form as (4.9); the bras, kets, and current operators have the same significance here as in (4.7)–(4.9). Our task for the remainder of this paper is to examine the relativistic electromagnetic current matrix elements $(b, \vec{P}_b | J_\mu | a, \vec{P}_a)$ and the nonrelativistic limits of the $\mu = 1, 2, 3$ components; the latter should coincide with (4.7)–(4.9).

For SO(3, 2), the generators can be divided into a set $L_{\mu\nu}$ which generates a Lorentz subgroup, and a second set $\Gamma_\mu \equiv L_{\mu 5}$ which transforms as a four-vector under that Lorentz subgroup. (See Appendix A.) For SU(3, 1), the generators can be taken to be the Lorentz generators $L_{\mu\nu}$ together with a set $T_{\mu\nu}$ which transforms as a symmetric traceless tensor under the Lorentz subgroup. (See Appendix B.) Thus, if we limit ourselves to vertex operators which are at most linear in the generators, the vector vertex operators are

$$J_\mu = A_1 P_\mu + A_2 Q_\mu + A_3 L_{\mu\nu} P^\nu + A_4 L_{\mu\nu} Q^\nu + A_5 \Gamma_\mu \tag{4.19}$$

for SO(3, 2) and

$$\begin{aligned} J_\mu = {} & B_1 P_\mu + B_2 Q_\mu + B_3 L_{\mu\nu} P^\nu \\ & + B_4 L_{\mu\nu} Q^\nu + B_5 T_{\mu\nu} P^\nu + B_6 T_{\mu\nu} Q^\nu \end{aligned} \tag{4.20}$$

for SU(3, 1), where $P_\mu \equiv \frac{1}{2}(P_{a\mu} + P_{b\mu})$ and $Q_\mu \equiv P_{a\mu} - P_{b\mu}$. The A_i and B_i are invariant functions of $P_{a\mu}$ and $P_{b\mu}$.

For the electromagnetic vertex, there is the additional condition of current conservation

$$\partial^\mu \langle b, \vec{P}_b | \mathcal{J}_\mu(X) | a, \vec{P}_a \rangle = 0, \tag{4.21}$$

or, from (4.16),

$$(P_a - P_b)^\mu (b, \vec{P}_b | J_\mu | a, \vec{P}_a) = 0 . \tag{4.22}$$

The contributions from the first two terms in (4.19) or (4.20) follow immediately; the remaining contributions can be evaluated by the method of Ref. 7. Taking the last term of (4.19) as an example, we note that (4.17) and the four-vector character of Γ_μ imply

$$\Lambda_a^{-1} \Gamma_\mu \Lambda_a = \{\Lambda_a\}_\mu{}^\nu \Gamma_\nu , \tag{4.23}$$

where $\{\Lambda_a\}$ is the 4×4 matrix transformation from the rest frame of particle a, so that

$$P_{a\mu} = \{\Lambda_a\}_\mu{}^\nu P_{a\nu}(0) \tag{4.24}$$

with $P_{a\nu}(0) \equiv (\vec{0}, M_a c)$. The invariance of the Lorentz scalar product now yields

$$\begin{aligned} P_a^\mu \Lambda_a^{-1} \Gamma_\mu \Lambda_a &= P_a^\mu(0) \Gamma_\mu \\ &= -M_a c \Gamma_4 \end{aligned} \tag{4.25}$$

(we are using the metric $+++-$), and thus

$$\begin{aligned} (P_a - P_b)^\mu (b, \vec{P}_b | \Gamma_\mu | a, \vec{P}_a) &= -M_a c (b, \vec{P}_b | \Lambda_a \Gamma_4 | a, \vec{0}) + M_b c (b, \vec{0} | \Gamma_4 \Lambda_b^{-1} | a, \vec{P}_a) \\ &= [-M_a c(n_a + \Omega) + M_b c(n_b + \Omega)] (b, \vec{P}_b | a, \vec{P}_a), \end{aligned} \tag{4.26}$$

from the last of Eqs. (A27) and Fig. 4. The various contributions to the current-conservation equation (4.22) are listed in the second column of Table I, evaluated in the rest frame of state a and with $\vec{P}_b$ along the third axis to simplify the off-diagonal terms.

It will be assumed that the coefficients A_i and B_i, which can be functions of M_a^2, M_b^2, and $Q^2 \equiv Q^\mu Q_\mu$, do not vary violently from one level to the next. With this assumption, it can be shown by taking $\vec{\zeta}_b$ infinitesimal and $\vec{\zeta}_a = 0$ that the terms involving off-diagonal SO(3, 2) or SU(3, 1) generators must vanish individually. Thus the electromagnetic currents reduce to

$$J_\mu = A_1 P_\mu + A_2 Q_\mu + A_4 L_{\mu\nu} Q^\nu + A_5 \Gamma_\mu , \tag{4.27}$$

$$J_\mu = B_1 P_\mu + B_2 Q_\mu + B_4 L_{\mu\nu} Q^\nu + B_5 T_{\mu\nu} P^\nu \tag{4.28}$$

for the SO(3, 2) and SU(3, 1) algebras, respectively.

For the particular value $Q^2 = 0$, the current conservation equation can be written (by reference to Table I) in the form

$$\tfrac{1}{2}(M_a^2 - M_b^2) c F(M_a, M_b) = M_a(n_a + \Omega) - M_b(n_b + \Omega) , \tag{4.29}$$

where

$$F(M_a, M_b) \equiv -A_1(M_a, M_b, 0)/A_5(M_a, M_b, 0) , \tag{4.30}$$

for the SO(3, 2) current (4.27), and in the form

$$\tfrac{1}{2}(M_a^2 - M_b^2) G(M_a, M_b) = M_a^2(n_a + \tfrac{3}{4}\Omega) - M_b^2(n_b + \tfrac{3}{4}\Omega) , \tag{4.31}$$

where

$$G(M_a, M_b) \equiv B_1(M_a, M_b, 0)/B_5(M_a, M_b, 0) , \tag{4.32}$$

for the SU(3, 1) current (4.28). If it is assumed that the inverse dependence of n on M can be expressed as an analytic function, then the same is true of

$$f(M) \equiv M(n + \Omega) . \tag{4.33}$$

Expanding $f(M_a)$ about M_b, inserting the result into (4.29), and setting $M_a = M_b = M$ yields

$$F(M, M) = f'(M)/Mc . \tag{4.34}$$

Hence

$$n = \frac{c}{M} \int_{M_0}^{M} x F(x, x) dx + \left(\frac{M_0}{M} - 1\right)\Omega \tag{4.35}$$

for the SO(3, 2) current, where M_0 denotes the mass of the ground state ($n = 0$). Similarly, for the SU(3, 1) current one finds

$$n = \frac{1}{M^2} \int_{M_0}^{M} x G(x, x) dx + \tfrac{3}{4}\left(\frac{M_0^2}{M^2} - 1\right)\Omega . \tag{4.36}$$

Since $0 < dn/dM < \infty$ for a physically reasonable mass spectrum, Eqs. (4.34) and (4.33) show that neither A_1 nor A_5 can vanish. (The possibility $A_1 = A_5 = 0$ is excluded for a charged system; see below.) The same is true of B_1 and B_5. Thus the "minimal" electromagnetic currents are of the forms

$$J_\mu = A_1 P_\mu + A_5 \Gamma_\mu , \tag{4.37}$$

$$J_\mu = B_1 P_\mu + B_5 T_{\mu\nu} P^\nu \tag{4.38}$$

TABLE I. Contributions to the current conservation equation.

Term in current[a]	Contribution to $(b, \vec{P}_b \vert Q^\mu J_\mu \vert a, \vec{0})$	Comments
P_μ	$-\frac{1}{2}(M_a^2 - M_b^2)c^2(b, \vec{P}_b \vert a, \vec{0})$	
Q_μ	$Q^2(b, \vec{P}_b \vert a, \vec{0})$	Not needed for current conservation.
$L_{\mu\nu}P^\nu$	$M_a c \vert \vec{P}_b \vert (b, \vec{P}_b \vert L_{34} \vert a, \vec{0})$	Cannot be conserved.[b]
$L_{\mu\nu}Q^\nu$	0	Always conserved.
Γ_μ	$[-M_a c(n_a + \Omega) + M_b c(n_b + \Omega)] (b, \vec{P}_b \vert a, \vec{0})$	
$T_{\mu\nu}P^\nu$	$[M_a^2 c^2 (n_a + \frac{3}{4}\Omega) - M_b^2 c^2 (n_b + \frac{3}{4}\Omega)] (b, \vec{P}_b \vert a, \vec{0})$	
$T_{\mu\nu}Q^\nu$	$2[M_a(M_a c^2 - 2E_b)(n_a + \frac{3}{4}\Omega) + M_b^2 c^2 (n_b + \frac{3}{4}\Omega)] (b, \vec{P}_b \vert a, \vec{0}) + 2M_a c \vert \vec{P}_b \vert (b, \vec{P}_b \vert T_{34} \vert a, \vec{0})$	Cannot be conserved.[b]

[a] The term Γ_μ occurs only for SO(3, 2), while $T_{\mu\nu}P^\nu$ and $T_{\mu\nu}Q^\nu$ occur only for SU(3, 1). Recall the definitions of P_μ and Q_μ following Eq. (4.20).

[b] With reasonable behavior of the coefficients.

for the SO(3, 2) and SU(3, 1) cases, respectively. Equations (4.29), (4.31), (4.35), and (4.36) express the mutual relationship between the mass spectrum and the ratio of the coefficients in these minimal conserved currents, in partial analogy to the usual relation between the form of the Hamiltonian and the minimal electromagnetic current.

Up to this point, the total charge has not been specified for the algebraic oscillators. This can be done by setting

$$(a, \vec{0}|\,J_4\,|\,a, \vec{0}) = ec\,, \tag{4.39}$$

where e denotes the total charge; e.g.,

$$McA_1(M, M, 0) + (n+\Omega)A_5(M, M, 0) = ec \tag{4.40}$$

for the SO(3, 2) current (4.27). Substitution of (4.34) and (4.33) into (4.40) yields

$$A_5(M, M, 0) = -(ec/M)dM/dn\,. \tag{4.41}$$

A similar calculation for the SU(3, 1) current yields

$$B_5(M, M, 0) = (e/M^2)dM/dn\,. \tag{4.42}$$

Comparison of (4.41) with (4.34) and (4.33) shows that the coefficients A_1 and A_5 cannot both be constant; the same is true of B_1 and B_5. Furthermore, constancy of A_1, B_1, or B_5 would imply $M \to 0$ as $n \to \infty$, while constancy of A_5 would imply an exponential dependence of M on n.

C. The Nonrelativistic Limit of the SO(3,2) Vertex

In order that the SO(3, 2) electromagnetic (or any other) vertex have the nonrelativistic oscillator vertex as its limiting form, it is necessary first of all that the boost operator in (4.17) have the correct limit. From (4.18) and (3.12a), one finds that

$$\lim \exp(i\vec{\zeta}_a \cdot \vec{\mathrm{B}}) = \exp(i\Omega^{1/2}c^{-1}\vec{\mathrm{V}}_a \cdot \vec{\xi}) \tag{4.43}$$

as $\vec{\mathrm{V}}_a/c \to 0$ and $\Omega \to \infty$. Comparison with (4.11) shows that the correct limit will be attained if

$$\Omega = \frac{m_2}{m_1}\frac{M_0c^2}{\hbar\omega}\,, \tag{4.44}$$

together with the condition (which will be used again below) that the ground-state mass has the limit

$$M_0 \to m_1 + m_2 \tag{4.45}$$

as $\hbar\omega/M_0c^2 \to 0$. (In fact, of course, all the states must approach this mass in the nonrelativistic limit.) We shall take (4.44) as a defining equation of the SO(3, 2) oscillator. It relates the SO(3, 2) parameter Ω to the nonrelativistic oscillator masses and level separation; it also guarantees that $\Omega \to \infty$ in the nonrelativistic limit $\hbar\omega/M_0c^2 \to 0$.

There is an independent item of information contained in (3.1),

$$\left.\frac{dM}{dn}\right|_{n\approx 0} = \frac{\hbar\omega}{c^2}\,, \tag{4.46}$$

which has not yet been used. Inserting this into (4.41) and (4.34), along with (4.33) and (4.44), and setting $e = e_1$ (only one particle charged) yields

$$\begin{aligned} A_5(0) &= -e_1\hbar\omega/M_0c\,,\\ A_1(0) &= -F(0)A_5(0)\\ &= e_1(m_1+m_2)/m_1M_0\,. \end{aligned} \tag{4.47}$$

In the above, $A_i(0) \equiv A_i(M_0, M_0, 0)$, and $F(0) \equiv F(M_0, M_0)$.

With these values for the coefficients, and with (4.45), the nonrelativistic limit of the minimal SO(3, 2) current (4.37) becomes

$$\vec{\mathrm{J}} = \frac{e_1}{m_1}\left(\vec{\mathrm{P}} - \frac{m_1\hbar\omega}{(m_1+m_2)c}\,\vec{\Gamma}\right). \tag{4.48a}$$

Finally, using the definition $P_\mu \equiv \frac{1}{2}(P_{a\mu} + P_{b\mu})$, the limit (3.12b) (recall that $\Gamma_\mu \equiv L_{\mu 5}$), (4.44) and (4.45) yields the nonrelativistic limit

$$\vec{\mathrm{J}} = \frac{e_1}{m_1}\left[\tfrac{1}{2}(\vec{\mathrm{P}}_a + \vec{\mathrm{P}}_b) - (\mu\hbar\omega)^{1/2}\vec{\mathrm{p}}\right]. \tag{4.48b}$$

This is precisely the desired current (4.10) of the nonrelativistic oscillator. Note that the arguments could equally well have been inverted, to yield the level splitting $\hbar\omega$ starting from the requirement that the SO(3, 2) current have the limit (4.48b).

D. The Nonrelativistic Limit of the SU(3,1) Vertex

The considerations of this subsection parallel those for the SO(3, 2) oscillator. Because of the difference by a factor $\sqrt{2}$ between (3.19a) and (3.12a), Eq. (4.44) is replaced by

$$\Omega = \frac{1}{2}\frac{m_2}{m_1}\frac{M_0c^2}{\hbar\omega}\,. \tag{4.49}$$

Using (4.42), (4.46), and the analogs of (4.33) and (4.34) yields

$$\begin{aligned} B_5(0) &= e_1\hbar\omega/M_0^2c^2\,,\\ B_1(0) &= G(0)B_5(0)\\ &= e_1(4m_1+3m_2)/4m_1M_0\,. \end{aligned} \tag{4.50}$$

From (B23) and Fig. 5, $T_{ij} \to \frac{1}{2}\Omega\delta_{ij}$ as $\Omega \to \infty$, for $i, j \leq 3$. Using the above together with (3.19b) and (4.45) in (4.38) leads to

$$\vec{\mathrm{J}} = \frac{e_1}{m_1}\left[\tfrac{1}{2}(\vec{\mathrm{P}}_a + \vec{\mathrm{P}}_b) - (\mu\hbar\omega)^{1/2}\vec{\mathrm{p}}\right] \tag{4.51}$$

in the limit as $\hbar\omega/M_0c^2 \to 0$. Thus the minimal SU(3, 1) current also yields the nonrelativistic oscillator current (4.10) in this limit.

E. The Scalar Form Factors

It can be expected that the dependence of transition amplitudes and form factors on $t \equiv -Q^\mu Q_\mu$ will be important in some applications. Here we wish merely to compare the two models by considering the simplest form factor, that for the interaction of the ground state with a scalar field,

$$\mathcal{G}_0(t) \equiv (0, \vec{0}\,|\,0, \vec{p}) = (0, \vec{0}\,|\,e^{i\zeta L_{34}}\,|\,0, \vec{0}), \qquad (4.52)$$

evaluated in the rest frame of one of the states for simplicity.

The evaluation of expressions such as (4.52) is straightforward in the present formalism, and has been described in Sec. 7B and Appendix A of Ref. 10. Briefly, the method is to write the boost in the form

$$e^{\alpha(E_+ + E_-)}, \qquad (4.53)$$

where the $E_\pm$ belong to a triplet of operators satisfying

$$[E_+, E_-] = H, \quad [H, E_\pm] = \pm 2E_\pm. \qquad (4.54)$$

The matrix elements of (4.53) can easily be evaluated between the states of each representation of the subalgebra generated by the operators in (4.54). Some values for such matrix elements are given in Eq. (A24) of Ref. 10; in particular, for the ground state $|-\omega)$,

$$(-\omega\,|\,e^{\alpha(E_+ + E_-)}\,|-\omega) = (\cosh\alpha)^{-\omega}, \qquad (4.55)$$

where $-\omega$ here denotes the eigenvalue of H. From (A26), (A23), (B22), and (B19), the appropriate operators are found to be

$$E_\pm = iE_{\pm(\alpha+\gamma)}, \quad H = H_\alpha + 2H_\gamma, \quad \text{with } L_{34} = -\tfrac{1}{2}i(E_+ + E_-) \qquad (4.56a)$$

for SO(3, 2), and

$$E_\pm = iE_{\pm(\beta+\gamma)}, \quad H = H_\beta + H_\gamma, \quad \text{with } L_{34} = -i(E_+ + E_-) \qquad (4.56b)$$

for SU(3, 1). From (4.52), (4.53), and (4.56), we see that $\alpha = \frac{1}{2}\zeta$ for SO(3, 2) and $\alpha = \zeta$ for SU(3, 1). Thus Figs. 4 and 5 yield

$$\mathcal{G}_0(t) = (\cosh\tfrac{1}{2}\zeta)^{-2\Omega} = \left(1 - \frac{t}{4M_0^2c^2}\right)^{-\Omega} \qquad (4.57a)$$

for SO(3, 2) and

$$\mathcal{G}_0(t) = (\cosh\zeta)^{-\Omega} = \left(1 - \frac{t}{2M_0^2c^2}\right)^{-\Omega} \qquad (4.57b)$$

for SU(3, 1), where (4.18) has been used to express $\cosh\zeta$ and $\cosh\frac{1}{2}\zeta$ in terms of $t = -Q^\mu Q_\mu$.[15] As a consistency check, we note that the expressions (4.57) coincide with the nonrelativistic form factor

$$\mathcal{G}_{nr}(-k^2)e^{i\vec{k}\cdot\vec{R}/\hbar} = \int d^3x\, e^{i\vec{k}\cdot\vec{x}_1/\hbar}\, \phi_0^2(x) \qquad (4.58)$$

or

$$\mathcal{G}_{nr}(-k^2) = \text{const}\int d^3x\, e^{i\mu\vec{k}\cdot\vec{x}/m_1\hbar}\, e^{-\xi^2} = \exp\left(-\frac{\mu k^2}{4m_1^2\hbar\omega}\right) \qquad (4.59)$$

in the limit of small $|t|$ (for which $t \approx -k^2$), with Ω given by (4.44) or (4.49), and with $M_0 \to m_1 + m_2$.

Factors of the form (4.57) but to different powers enter in all form factors and transition amplitudes (scalar, electromagnetic, or otherwise). Thus the two models have different characteristic t dependences away from the nonrelativistic limit, though, of course, these can be modified by t dependence of the coefficients in the vertex operators.

V. DISCUSSION

The algebras of SO(3, 2) and SU(3, 1) (and the families of representations depicted in Figs. 4 and 5) have been shown to provide Lorentz-covariant algebraic analogs of the nonrelativistic harmonic oscillator,[16] in the sense that the level structure, matrix elements, and minimal electromagnetic currents of the algebraic oscillators coincide with those of the nonrelativistic oscillator in the limit $\vec{V}/c \to 0$, $\hbar\omega/M_0c^2 \to 0$. The electromagnetic currents have been assumed to be of first order in the algebraic generators; the coefficients necessarily have a mass dependence which shows up for excitation energies comparable to rest-mass energies, and is determined by the functional dependence of the mass on the level number n.

Of the two algebraic oscillators, the one based on SO(3, 2) enjoys the advantage of greater simplicity. It has the minimum number (10) of generators that are needed to correspond with the dynamical variables x_i, p_i, L_{ij}, and n of the nonrelativistic oscillator; this is also the minimum number from which one can form a set of Lorentz generators and a four-vector needed to construct a covariant electromagnetic current. There are also quantitative differences, in the relations between mass spectra and electromagnetic currents, and in the form factors, as has been discussed in Sec. IV.

The SO(3, 2) oscillator provides an interesting example of a system which possesses an apparent symmetry that does not correspond to a subalgebra of the dynamical variables. Although the spin multiplets at each level are appropriate to a family of SU(3) representations (as for the nonrelativistic oscillator), the SO(3, 2) algebra does not contain an SU(3) subalgebra that connects these states.

Since the SO(3, 2) and SU(3, 1) algebraic oscillators are consistent (in the sense of Lorentz covariance) even when the excitation energies and c.m. kinetic energies are comparable to the rest-mass energies, they provide generalizations of the harmonic oscillator which may be useful in situations (such as the quark model) where nonrelativistic approximations are not justified.

APPENDIX A: THE GENERATORS OF SO(3,2)

The group SO(3, 2) consists of those unimodular transformations which preserve the bilinear form

$$\tilde{x}gy \equiv x^1y^1 + x^2y^2 + x^3y^3 - x^4y^4 - x^5y^5 \tag{A1}$$

in a real space, where the metric matrix g has the diagonal elements $g_{rr} = g^{rr} = (1, 1, 1, -1, -1)$. If the transformation matrix in

$$x \to ax, \quad y \to ay \tag{A2}$$

is written as an exponential,

$$a = e^{ib}, \tag{A3}$$

then unimodularity of a requires that

$$\mathrm{Tr}b = 0 \tag{A4}$$

and preservation of (A1) requires

$$g\tilde{b}g = -b \tag{A5}$$

or

$$(bg)^{\sim} = -bg. \tag{A6}$$

The form (A3) also requires that b, and hence bg, be imaginary. Thus we can write the expansion

$$bg = \sum_{r,s} \xi_{rs}\, l_{rs}\, g, \tag{A7}$$

where the ξ_{rs} are real parameters and the $l_{rs}g$ are a complete set of imaginary antisymmetric 5×5 matrices:

$$(l_{rs}g)^{tu} \equiv -i(\delta_r^t\,\delta_s^u - \delta_s^t\,\delta_r^u). \tag{A8}$$

[The elements of the matrices a, b, and l_{rs} will be written with one upper and one lower index, e.g., $(l_{rs})^t{}_u$. Note that multiplication by g either lowers or raises the "adjacent" index, as above.] The matrices (A8) are not all independent, since $l_{rs}g = -l_{sr}g$. Solving (A8) for l_{rs} yields

$$(l_{rs})^t{}_u = -i(\delta_r^t g_{su} - \delta_s^t g_{ru}), \tag{A9}$$

and, of course,

$$b = \sum_{r,s} \xi_{rs}\, l_{rs}. \tag{A10}$$

By direct calculation, the commutation relations are found to be

$$[l_{rs}, l_{tu}] = -i(g_{st}\, l_{ru} - g_{su}\, l_{rt} - g_{rt}\, l_{su} + g_{ru}\, l_{st}). \tag{A11}$$

The group of Lorentz transformations will be identified with the subgroup generated by the $l_{\mu\nu}$ with $1 \leq \mu,\ \nu \leq 4$. According to (A11), the remaining generators $l_{\mu 5}$ satisfy

$$[l_{\mu5}, l_{\rho\sigma}] = -i(g_{\mu\rho}l_{\sigma5} - g_{\mu\sigma}l_{\rho5}) \quad \text{for} \quad 1 \leq (\rho, \sigma) \leq 4. \tag{A12}$$

Thus an infinitesimal Lorentz transformation transforms $l_{\mu 5}$ to

$$\begin{aligned} e^{-i\xi_{\rho\sigma}l_{\rho\sigma}}\, l_{\mu5}\, e^{i\xi_{\rho\sigma}l_{\rho\sigma}} &\simeq l_{\mu5} + i\xi_{\rho\sigma}[l_{\mu5}, l_{\rho\sigma}] \\ &= l_{\mu5} + \xi_{\rho\sigma}(g_{\mu\rho}l_{\sigma5} - g_{\mu\sigma}l_{\rho5}). \end{aligned} \tag{A13}$$

By comparison, a covariant four-vector v_μ should transform to

$$\begin{aligned} (ge^{i\xi_{\rho\sigma}l_{\rho\sigma}}g)_\mu{}^\nu v_\nu &\simeq g_{\mu\alpha}(1 + i\xi_{\rho\sigma}l_{\rho\sigma})^\alpha{}_\beta\, g^{\beta\nu} v_\nu \\ &= v_\mu + \xi_{\rho\sigma}(g_{\mu\rho}v_\sigma - g_{\mu\sigma}v_\rho). \end{aligned} \tag{A14}$$

Equations (A13) and (A14) show that $l_{\mu5}$ is a covariant four-vector under Lorentz transformations, the same as the physical momentum (when coordinate vectors are assumed to transform contravariantly).

The set of generators

$$\begin{aligned} &e_{\pm\alpha} \equiv l_{23} \pm il_{31}, \quad h_\alpha \equiv 2l_{12}, \\ &e_{\pm\gamma} \equiv \tfrac{1}{2}[\pm(l_{14} + l_{25}) + i(l_{15} - l_{24})], \quad h_\gamma \equiv -l_{45} - l_{12} \end{aligned} \tag{A15}$$

constitute a complete set from which the rest can be obtained by taking sums and commutators; e.g.,

$$l_{35} = -\tfrac{1}{2}i[e_{+\alpha} + e_{-\alpha},\ e_{+\gamma} - e_{-\gamma}]. \tag{A16}$$

Therefore, the expansion (A10) can be written in the alternate form

$$b = \sum_\rho \xi_\rho h_\rho + \sum_{\pm,\rho} \xi_{\pm\rho} e_{\pm\rho} + \cdots, \tag{A17}$$

where the dots denote terms involving commutators of the operators (A15). Comparing (A17) and (A15) with the expansion (A10), in which all the parameters are real, we find that the parameters of (A17) must satisfy

$$\begin{aligned} &\xi_{\pm\alpha}^* = \xi_{\mp\alpha}, \quad \xi_{\pm\gamma}^* = -\xi_{\mp\gamma}, \\ &\xi_\rho^* = \xi_\rho \quad \text{for} \quad \rho = \alpha \quad \text{or} \quad \gamma. \end{aligned} \tag{A18}$$

In a unitary representation, the transformation will be written

$$A = e^{iB}, \tag{A19}$$

with capital letters to distinguish these operators from those in the fundamental representation. An expansion of the form (A17) must hold here also,

$$B = \sum_\rho \xi_\rho H_\rho + \sum_{\pm,\rho} \xi_{\pm\rho} E_{\pm\rho} + \cdots, \tag{A20}$$

and unitarity of A requires Hermiticity of B,

$$B^\dagger = B, \tag{A21}$$

or, from (A18), that

$$E^\dagger_{\pm\alpha} = E_{\mp\alpha}, \quad H^\dagger_\alpha = H_\alpha, \quad E^\dagger_{\pm\gamma} = -E_{\mp\gamma}, \quad H^\dagger_\gamma = H_\gamma. \tag{A22}$$

The commutation relations are, of course, the same as in the fundamental representation. From (A15) it follows that

$$[E_{+\rho}, E_{-\rho}] = H_\rho, \quad [H_\rho, E_{\pm\rho}] = \pm 2E_{\pm\rho} \quad \text{for } \rho = \alpha \text{ or } \gamma, \quad [H_\alpha, E_{\pm\gamma}] = \mp 2E_{\pm\gamma}, \quad [H_\gamma, E_{\pm\alpha}] = \mp E_{\pm\alpha}, \quad [E_{\pm\alpha}, E_{\mp\gamma}] = [H_\alpha, H_\gamma] = 0. \tag{A23}$$

Four commutators define new generators,

$$[E_{\pm\alpha}, E_{\pm\gamma}] \equiv E_{\pm(\alpha+\gamma)}, \quad [E_{\pm\alpha}, E_{\pm(\alpha+\gamma)}] \equiv E_{\pm(2\alpha+\gamma)}, \tag{A24}$$

and the remaining commutators either vanish or can be expressed in terms of (A23) and (A24) by use of the Jacobi identity.

The generators L_{rs} can be expressed in terms of the E's and H's by use of (A15) and (A11). The results are, for the generators of rotations,

$$L_\pm \equiv (L_{23} \pm iL_{31}) = E_{\pm\alpha}, \quad L_3 \equiv L_{12} = \tfrac{1}{2} H_\alpha. \tag{A25}$$

For the Lorentz boosts the results are

$$L_{\pm 4} \equiv (L_{14} \pm iL_{24}) = \mp(\tfrac{1}{2} E_{\pm(2\alpha+\gamma)} + E_{\mp\gamma}), \quad L_{34} = \tfrac{1}{2}(E_{+(\alpha+\gamma)} + E_{-(\alpha+\gamma)}). \tag{A26}$$

And for the four-vector $\Gamma_\mu \equiv L_{\mu 5}$ the results are

$$\Gamma_\pm \equiv (\Gamma_1 \pm i\Gamma_2) = i(\tfrac{1}{2} E_{\pm(2\alpha+\gamma)} - E_{\mp\gamma}), \quad \Gamma_3 = -\tfrac{1}{2} i (E_{+(\alpha+\gamma)} - E_{-(\alpha+\gamma)}), \quad \Gamma_4 = -\tfrac{1}{2} H_\alpha - H_\gamma. \tag{A27}$$

APPENDIX B: THE GENERATORS OF SU(3,1)

The group SU(3, 1) consists of those unimodular transformations which preserve the scalar product

$$x^\dagger g y \equiv x^{1*}y^1 + x^{2*}y^2 + x^{3*}y^3 - x^{4*}y^4. \tag{B1}$$

Again writing the transformation in the form (A2) and (A3), we find

$$\mathrm{Tr}\, b = 0 \tag{B2}$$

and

$$g b^\dagger g = b \tag{B3}$$

or

$$(bg)^\dagger = bg. \tag{B4}$$

So it is now necessary to expand bg in terms of a complete set of Hermitian matrices. The expansion we shall use is

$$bg = \sum_{\mu,\nu} (\xi_{\mu\nu} l_{\mu\nu} g + \eta_{\mu\nu} t_{\mu\nu} g), \tag{B5}$$

where $\xi_{\mu\nu}$ and $\eta_{\mu\nu}$ are real parameters, $l_{\mu\nu} g$ is defined as in (A8) except that the indices now run from 1 to 4, and

$$(t_{\mu\nu} g)^{\rho\sigma} \equiv \delta^\rho_\mu \delta^\sigma_\nu + \delta^\rho_\nu \delta^\sigma_\mu - \tfrac{1}{2} g^{\rho\sigma} g_{\mu\nu} \tag{B6}$$

or

$$(t_{\mu\nu})^\rho{}_\sigma = \delta^\rho_\mu g_{\nu\sigma} + \delta^\rho_\nu g_{\mu\sigma} - \tfrac{1}{2} \delta^\rho_\sigma g_{\mu\nu}. \tag{B7}$$

We note that $l_{\mu\nu} g$ is antisymmetric, $t_{\mu\nu} g$ is symmetric, and $l_{\mu\nu}$ and $t_{\mu\nu}$ are both traceless. The expansion of b is, of course,

$$b = \sum_{\mu,\nu} (\xi_{\mu\nu} l_{\mu\nu} + \eta_{\mu\nu} t_{\mu\nu}). \tag{B8}$$

The commutation relations

$$[l_{\mu\nu}, l_{\rho\sigma}] = -i(g_{\nu\rho} l_{\mu\sigma} - g_{\nu\sigma} l_{\mu\rho} - g_{\mu\rho} l_{\nu\sigma} + g_{\mu\sigma} l_{\nu\rho}), \tag{B9}$$

$$[t_{\mu\nu}, l_{\rho\sigma}] = [l_{\sigma\rho}, t_{\nu\mu}] = -i(g_{\nu\rho} t_{\mu\sigma} - g_{\nu\sigma} t_{\mu\rho} + g_{\mu\rho} t_{\nu\sigma} - g_{\mu\sigma} t_{\nu\rho}), \tag{B10}$$

$$[t_{\mu\nu}, t_{\rho\sigma}] = +i(g_{\nu\rho} l_{\mu\sigma} + g_{\nu\sigma} l_{\mu\rho} + g_{\mu\rho} l_{\nu\sigma} + g_{\mu\sigma} l_{\nu\rho}) \tag{B11}$$

follow by direct calculation from (A9) and (B7).

The complete set of the $l_{\mu\nu}$ are here identified with the Lorentz generators. An infinitesimal Lorentz transformation transforms $t_{\mu\nu}$ to

$$e^{-i\xi_{\rho\sigma} l_{\rho\sigma}} t_{\mu\nu} e^{i\xi_{\rho\sigma} l_{\rho\sigma}} \simeq t_{\mu\nu} + i\xi_{\rho\sigma}[t_{\mu\nu}, l_{\rho\sigma}] = t_{\mu\nu} + \xi_{\rho\sigma}(g_{\nu\rho} t_{\mu\sigma} - g_{\nu\sigma} t_{\mu\rho} + g_{\mu\rho} t_{\nu\sigma} - g_{\mu\sigma} t_{\nu\rho}), \tag{B12}$$

which is precisely the transformation of a covariant second-rank tensor:

$$(ge^{i\xi_{\rho\sigma}l_{\rho\sigma}}g)_\mu{}^\alpha(ge^{i\xi_{\rho\sigma}l_{\rho\sigma}}g)_\nu{}^\beta t_{\alpha\beta}\simeq g_{\mu\gamma}(1+i\xi_{\rho\sigma}l_{\rho\sigma})^\gamma{}_\delta g^{\delta\alpha}g_{\nu\epsilon}(1+i\xi_{\rho\sigma}l_{\rho\sigma})^\epsilon{}_\tau g^{\tau\beta}t_{\alpha\beta}$$
$$=t_{\mu\nu}+\xi_{\rho\sigma}[(\delta^\alpha_\sigma g_{\mu\rho}-\delta^\alpha_\rho g_{\mu\sigma})\delta^\beta_\nu+\delta^\alpha_\mu(\delta^\beta_\sigma g_{\nu\rho}-\delta^\beta_\rho g_{\nu\sigma})]t_{\alpha\beta}$$
$$=t_{\mu\nu}+\xi_{\rho\sigma}(g_{\mu\rho}t_{\sigma\nu}-g_{\mu\sigma}t_{\rho\nu}+g_{\nu\rho}t_{\mu\sigma}-g_{\nu\sigma}t_{\mu\rho}). \quad (B13)$$

The equations corresponding to (A15), (A17), and (A18) are

$$\begin{aligned}
e_{\pm\alpha}&\equiv\sqrt{\tfrac{1}{8}}\,[(l_{23}+t_{13})\pm i(l_{31}+t_{23})],\\
h_\alpha&\equiv\tfrac{1}{2}(l_{12}-\tfrac{1}{2}t_{11}-\tfrac{1}{2}t_{22}+t_{33}),\\
e_{\pm\beta}&\equiv\sqrt{\tfrac{1}{8}}\,[(l_{23}-t_{13})\pm i(l_{31}-t_{23})],\\
h_\beta&\equiv\tfrac{1}{2}(l_{12}+\tfrac{1}{2}t_{11}+\tfrac{1}{2}t_{22}-t_{33}),\\
e_{\pm\gamma}&\equiv\sqrt{\tfrac{1}{8}}\,[i(-l_{24}+t_{14})\pm(l_{14}+t_{24})],\\
h_\gamma&\equiv-\tfrac{1}{2}(l_{12}+\tfrac{1}{2}t_{11}+\tfrac{1}{2}t_{22}+t_{44}),
\end{aligned} \quad (B14)$$

$$b=\sum_\rho\xi_\rho h_\rho+\sum_{\pm,\rho}\xi_{\pm\rho}e_{\pm\rho}+\cdots, \quad (B15)$$

where the dots denote commutators as before, and

$$\xi^*_{\pm\rho}=\xi_{\mp\rho}\ \text{ for }\ \rho=\alpha \text{ or } \beta,\quad \xi^*_{\pm\gamma}=-\xi_{\mp\gamma},$$
$$\xi^*_\rho=\xi_\rho\ \text{ for }\ \rho=\alpha,\ \beta,\ \text{or}\ \gamma. \quad (B16)$$

We must also remember that

$$t_{\mu\nu}=t_{\nu\mu}\ \text{ and }\ t_{44}=t_{11}+t_{22}+t_{33}. \quad (B17)$$

In a unitary representation of the group, or a Hermitian representation of the algebra, it follows from (B16) and the Hermiticity of B that

$$E^\dagger_{\pm\rho}=E_{\mp\rho}\ \text{ for }\ \rho=\alpha \text{ or } \beta,\quad E^\dagger_{\pm\gamma}=-E_{\mp\gamma},$$
and (B18)
$$H^\dagger_\rho=H_\rho\ \text{ for }\ \rho=\alpha,\ \beta,\ \text{or}\ \gamma.$$

Equations (B9)–(B11) and (B14) yield the commutation relations

$$[E_{+\rho},E_{-\rho}]=H_\rho,\quad [H_\rho,E_{\pm\rho}]=\pm2E_{\pm\rho},$$
$$\text{for }\ \rho=\alpha,\ \beta,\ \text{or}\ \gamma,$$
$$[E_{+\rho},E_{-\sigma}]=0,\quad [H_\rho,E_{\pm\sigma}]=\mp E_{\pm\sigma}, \quad (B19)$$
$$\text{for }\ (\rho,\sigma)=(\alpha,\beta),\ (\beta,\alpha),\ (\beta,\gamma),\ \text{or}\ (\gamma,\beta),$$
$$[E_{+\alpha},E_{\pm\gamma}]=[E_{-\alpha},E_{\pm\gamma}]=[H_\alpha,E_{\pm\gamma}]=[H_\gamma,E_{\pm\alpha}]=0,$$
$$[H_\alpha,H_\beta]=[H_\beta,H_\gamma]=[H_\alpha,H_\gamma]=0.$$

There are now six commutators which define new generators,

$$\begin{aligned}
[E_{\pm\alpha},E_{\pm\beta}]&\equiv E_{\pm(\alpha+\beta)},\\
[E_{\pm\beta},E_{\pm\gamma}]&\equiv E_{\pm(\beta+\gamma)},\\
[E_{\pm\alpha},E_{\pm(\beta+\gamma)}]&\equiv E_{\pm(\alpha+\beta+\gamma)},
\end{aligned} \quad (B20)$$

and the remaining commutators either vanish or follow from the Jacobi identity. From (B14), (B17), and the commutation relations of $L_{\mu\nu}$ and $T_{\mu\nu}$ it is straightforward to express the generators of rotations, the Lorentz boosts, and the components of $T_{\mu\nu}$ in terms of the E's and H's:

$$\begin{aligned}
L_\pm&\equiv(L_{23}\pm iL_{31})=\sqrt{2}\,(E_{\pm\alpha}+E_{\pm\beta}),\\
L_3&\equiv L_{12}=H_\alpha+H_\beta,
\end{aligned} \quad (B21)$$

$$\begin{aligned}
L_{\pm4}&\equiv(L_{14}\pm iL_{24})=\mp\sqrt{2}\,(E_{\pm(\alpha+\beta+\gamma)}+E_{\mp\gamma}),\\
L_{34}&=E_{+(\beta+\gamma)}+E_{-(\beta+\gamma)},
\end{aligned} \quad (B22)$$

$$\begin{aligned}
T_{11}&=E_{+(\alpha+\beta)}-E_{-(\alpha+\beta)}-\tfrac{1}{2}H_\alpha-\tfrac{1}{2}H_\gamma,\\
T_{12}&=-i(E_{+(\alpha+\beta)}+E_{-(\alpha+\beta)}),\\
T_{22}&=-E_{+(\alpha+\beta)}+E_{-(\alpha+\beta)}-\tfrac{1}{2}H_\alpha-\tfrac{1}{2}H_\gamma,\\
T_{\pm3}&\equiv(T_{13}\pm iT_{23})=E_{\pm\alpha}-E_{\pm\beta},\\
T_{33}&=\tfrac{1}{2}H_\alpha-H_\beta-\tfrac{1}{2}H_\gamma,\\
T_{\pm4}&\equiv(T_{14}\pm iT_{24})=i\sqrt{2}\,(E_{\pm(\alpha+\beta+\gamma)}-E_{\mp\gamma}),\\
T_{34}&=-i(E_{+(\beta+\gamma)}-E_{-(\beta+\gamma)}),\\
T_{44}&=-\tfrac{1}{2}H_\alpha-H_\beta-\tfrac{3}{2}H_\gamma.
\end{aligned} \quad (B23)$$

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[10]The construction of such representations has been described by D. W. Joseph, J. Math. Phys. 11, 1249 (1970).

[11]Note that the identifications here are quite different from those of S. Goshen and H. J. Lipkin, Ann. Phys. (N.Y.) 6, 301 (1959), and A. O. Barut, Phys. Rev. 139, B1433 (1965). Those authors identified the Hamiltonian, but not p or ξ, with an operator of the algebra.

[12]We use the term "lowering operator" to denote one which moves us down or to the right in the figure, while "raising operator" denotes one which moves us up or to the left. Thus the state at the upper left is the "highest state." In this terminology, it would be natural to use the label $-n$ in place of n; but that would be contrary to the physical interpretation.

[13]For a proof of this statement, see Eqs. (4.28)–(4.31) of Ref. 10.

[14]Such a form has been used in many of Refs. 5–7 for the construction of covariant vertices. It is analogous to the form used when a spin-$\frac{1}{2}$ particle is represented by a four-component spinor field $\psi(X)$; the correspondences are

$$|a, \vec{P}_a\rangle \sim a^\dagger(\vec{P}_a, s_a)|0\rangle,$$

$$\mathcal{J}_\mu(X) \sim \psi^\dagger(X)[A\gamma_\mu + B(P_a^\nu - P_b^\nu)\sigma_{\mu\nu} + \cdots]\psi(X),$$

$$|a, \vec{P}_a\rangle \sim u(\vec{P}_a, s_a),$$

and

$$J_\mu \sim A\gamma_\mu + B(P_a^\nu - P_b^\nu)\sigma_{\mu\nu} + \cdots,$$

where $a^\dagger(\vec{P}_a, s_a)$ denotes a creation operator and $u(\vec{P}_a, s_a)$ is a Dirac spinor. [Of course, there is the difference that the Dirac spinors do not transform according to a unitary representation of SO(3, 1).]

[15]An expression of the form (4.57b) has previously been obtained by the authors of Ref. 1. Their expression (10) for γ^2, which corresponds to our Ω, differs slightly from (4.49) due to an inaccuracy in their nonrelativistic form factor.

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