Post-Newtonian Gravitational Effects in Lunar Laser Ranging*

Kenneth Nordtvedt, Jr.†
Montana State University, Bozeman, Montana 59715
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The dominant-range perturbations of the lunar orbit are calculated in relativistic gravitational theory, the results being applicable to all metric theories of gravity (a class of theories which includes general relativity). By searching for the various effects calculated in this paper using lunar laser ranging, the several parametrized post-Newtonian coefficients of the metrical gravitational field can be determined, thus pointing toward specific relativistic, post-Newtonian theories of gravity and empirically ruling out other theories.

I. INTRODUCTION

Efforts to understand the orbital motion of the moon around the earth have historically played a central role in the development of gravitational theory, starting with Newton's success in showing that the moon's motion was a long-range manifestation of the earth's gravity. Precise fitting of the lunar motion later became a testing ground for perturbation-theory techniques and the gravitational many-body problem. In more recent times, since the revolution in gravitational theory brought by Einstein's relativity theories, several calculations have been made to determine whether post-Newtonian (relativistic) gravitational effects might be detectable in the lunar orbit. Concentrating on secular effects such as node and perigee motion, workers concluded that it would not be possible to detect relativistic effects.

In the last few years U.S. Apollo astronauts have positioned several (three at the time of this writing) laser corner reflectors at separate positions on the lunar surface. This permits ranging between earth stations and the lunar surface reflectors by timing the round trips for laser pulses. Accumulation of several years' range data will allow for the detection and elimination of range terms produced by wobble of the earth and the moon, leaving earth-moon range data with an accuracy of about 10 cm or better in measuring the oscillatory contributions to the interbody distance (the mean earth-moon range can be known only to the precision of the knowledge of the speed of light).[1] Baierlein[2] calculated oscillatory contributions to the earth-moon range using Einstein's general relativity theory of gravity; he found at least one term which could possibly be detected by laser ranging: a 100-cm amplitude oscillation with frequency $2\omega - 2\Omega$ (ω is the moon's mean motion about the earth; Ω is the earth's mean motion about the sun). Nordtvedt[3] showed that if Einstein's equivalence principle failed for massive celestial bodies because of the internal gravitational energy of the bodies, the lunar orbit would possess an anomalous range oscillation of about 1000-cm amplitude and frequency $\omega - \Omega$. This perturbation is predicted to vanish in general relativity but not in most other gravitational theories,[4] so this can be considered a test of one of general relativity's null predictions.

There now exists a general method for analyzing gravitational solar-system experimental effects within the framework of all possible metric theories of gravity (general relativity is in this class). The foundation for this approach to experimental solar-system gravity is the parametrized post-Newtonian (PPN) metric, which is given in its most complete form by Will and Nordtvedt.[5] For weak gravitational field (solar-system environment) purposes any metric theory of gravity can be summarized by specifying seven dimensionless coefficients which multiply various gravitational potentials, one theory differing from the other only in the value of these PPN coefficients.

Within this framework the various range oscillations which might result in the lunar orbit will be calculated, and the PPN coefficients which are determinable by measuring the various oscillatory range contributions will be exhibited. A few effects which perhaps can be enahnced by ranging to low earth-orbital satellites will be discussed in a later section.

For simplicity of presentation, we will neglect lunar node (out-of-plane) motion, and a circular lunar orbit will be assumed whenever possible [the results of this paper can be considered the (eccentricity) $e \to 0$ and (inclination) $i \to 0$ limit of the completely general expression for the relativistic range perturbations]. Since $|e| \ll 1$ and $|i| \ll 1$, our results give the most measurable effects. Table I lists the dominant-range oscillations found in this paper's analysis.

Our results disagree with those of Baierlein[2] on

TABLE I. Summary of largest lunar range oscillations.

Range oscillations (in centimeters)	Equation No.
$5\times 10^6 \alpha_3 \cos(\omega t - \theta_1)$	(41b)
$4400\alpha_1 \sin(\omega t - \theta_0)$	(29)
$2700(\alpha_1 - \frac{1}{2}\alpha_2 - \alpha_3)\sin(\Omega t - \theta_0)$	(33b)
$1500\alpha_2 \cos(2\omega t - 2\theta_0)$	(35b)
$1000(4\beta - 3 - \gamma - \alpha_1 + \frac{4}{3}\alpha_2 - \frac{4}{3}\rho_2 + \frac{1}{3}\rho_2)\cos(\omega - \Omega)t$	(18)
$-450\alpha_2 \sin[(2\omega - \Omega)t - \theta_0]$	(33b)
$90\alpha_1 \sin(\Omega t - \theta_0)$	(31b)
$+35\alpha_2 \cos(2\omega - 2\Omega)t$	(13b)
$-40\alpha_1 \cos(\omega - \Omega)t$	(23b)
$33\alpha_2 \cos(2\Omega t - 2\theta_0)$	(38b)
$28\alpha_2 \cos[(2\omega - 4\Omega)t + 2\theta_0]$	(38b)
$28\alpha_2 \cos[(2\omega + 2\Omega)t - 4\theta_0]$	(38b)

TABLE II. Definition of symbols used.

ω	Mean angular frequency of the moon around the earth
ω_0	Natural frequency for radial perturbations of the lunar orbit; also angular frequency of lunar orbit with respect to perigee
Ω	Mean angular frequency of the earth around the sun
e	Eccentricity of an orbit
i	Inclination of an orbit
M	Mass of sun
m	Mass of earth
R	Sun-moon distance (also can be vector $\vec{R}$)
R_0	Sun-earth distance (also can be vector $\vec{R}_0$)
r	Earth-moon distance (also can be vector $\vec{r}$)
G	Newton's gravitational constant
c	Speed of light
γ, β, α_1, α_2, α_3, ρ_1, ρ_2	Coefficients (dimensionless) which appear in the most general gravitational metric potential expression. Observational effects are proportional to some linear combination of these seven PPN coefficients.
δ	A parameter indicating a breakdown of Einstein's equivalence principle ($M_G/M_I \equiv 1+\delta$)
$\vec{w}$	Velocity of sun through universe
$\vec{w}'$	Part of $\vec{w}$ lying in moon orbital plane
$\vec{\omega}$	Spin angular velocity of earth
$\vec{g}_s$	Sun's gravitational acceleration at earth
U_G	Internal gravitational energy of a celestial body
t	Time coordinate measured from occurrence of a new moon
t'	Time coordinate measured from occurrence of earth-orbit perigee
θ_0	Longitude of w' from new moon defining t
θ_1	Longitude of $(\vec{\omega}\times\vec{w}')$ from new moon defining t
$\vec{v}$	Velocity of moon relative to earth
$\vec{v}_s$	Velocity of earth relative to sun

the presence of the 100-cm amplitude, $2(\omega-\Omega)$-frequency term he found in general relativity; we here find this term to be a Lorentz contraction of the lunar orbit when viewed from the heliocentric coordinate system (the earth-moon system moves relative to the sun at 30 km/sec); however, this contraction is unobservable from the inertial frame of the earth, in which the laser ranging experiments are performed.

Table II defines the various symbols used in this paper; also see Fig. 1 for a view of the problem's geometry.

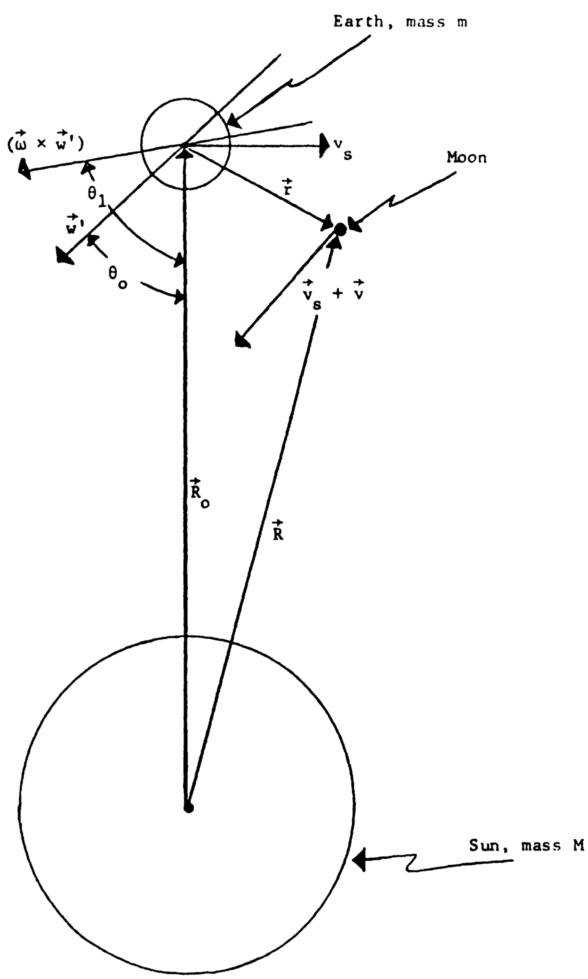

FIG. 1. The sun is at rest and the earth is at position $\vec{R}_0$ relative to the sun and moves at velocity $\vec{v}_s$, while the moon at position $\vec{R}=\vec{R}_0+\vec{r}$ moves at velocity $\vec{v}_s+\vec{v}$, i.e., $\vec{v}$ relative to earth. $\vec{w}'$ is the component of the sun's velocity through the universe lying in the plane of the earth and moon orbits; $\vec{w}'$ makes angle θ_0 with the earth-moon line ($\vec{r}$) at some particular new moon used as a time reference. $\vec{\omega}$ is the earth's spin angular velocity; $(\vec{\omega}\times\vec{w}')$ is the part of this vector cross product lying in the orbital plane; θ_1 is its longitude angle with respect to the same new-moon event.

II. THE LUNAR EQUATION OF MOTION

The moon's equation of motion is determined by the gravitational environment produced jointly by the sun and the earth. For our purpose of obtaining relativistic corrections to the classical motion, the moon is treated as a massless test body. The earth is assumed to move in the gravitational environment of the sun and the earth itself. (We incorporate two possible effects in which the earth's own gravitational self-energy alters its own equation of motion in the sun's gravitational field. The first such effect is the possibility that the earth's gravitational-to-inertial mass ratio differs from unity by an amount proportional to the earth's internal gravitational energy.[4] Another effect is the possibility that the spinning earth self-accelerates due to "preferred inertial frame" gravitational potentials[6]; see Appendix A.)

Assuming a metric field $g_{\mu\nu}(\vec{r},t)$ jointly produced by the sun and the earth, the moon's equation of motion is obtained by requiring the action integral

$$A=\int\left[g_{\mu\nu}(\vec{r},t)\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right]^{1/2}dt \tag{1}$$

to be an extremum for the moon's trajectory $\vec{r}(t)$. This is the geodesic hypothesis for test particles in metrical theories of gravity. The metric field indices μ and ν separately sum over 0, 1, 2, and 3, equal to ct, x, y, and z, respectively. The various metric field components have the forms

$$g_{00}=1-2\psi+h_{00}^{(2)}\ , \tag{2a}$$

$$g_{0x},g_{0y},g_{0z}\equiv\vec{h}\ , \tag{2b}$$

$$g_{xx}=g_{yy}=g_{zz}=-1-2\gamma\psi\,, \tag{2c}$$

and

$$g_{xy},\ \text{etc.}=0\,. \tag{2d}$$

ψ is the Newtonian potential of the sun plus the earth divided by the square of the speed of light;

$$\psi=\frac{G}{c^2}\left(\frac{M}{R}+\frac{m}{r}\right). \tag{3}$$

(See Fig. 1 for definitions of M, m, R, r, etc.) $h_{00}^{(2)}$ are additional potentials proportional to $1/c^4$; $\vec{h}$ is a 3-vector potential proportional to $1/c^3$, while γ is a dimensionless PPN coefficient which varies from one gravitational theory to another and has value unity in general relativity.

The off-diagonal space-space components of $g_{\mu\nu}(\vec{r},t)$ are made zero [Eq. (2d)] to the necessary approximation by choosing an appropriate "gauge" or coordinate system, a freedom available to us in metrical gravitational theories. All four space-time coordinates can be transformed to new coordinates,

$$X'^{\mu}=f^{\mu}(X^{\nu}), \tag{4}$$

which then generate changes in the metric field components,

$$g'_{\mu\nu}=\frac{\partial X^{\lambda}}{\partial X'^{\mu}}\frac{\partial X^{\gamma}}{\partial X'^{\nu}}g_{\lambda\gamma}. \tag{5}$$

Use of three of the coordinate transformations produces the "isotropic coordinate" properties of Eqs. (2c) and (2d). A remaining gauge freedom is at our disposal,

$$X'^{0}=f^{0}(X^{\nu}), \tag{6}$$

to put the metrical field components h_{00} and $\vec{h}$ into a standard or convenient form.

Variation of the action integral given by Eq. (1) generates the equation of motion

$$\vec{a}=\vec{\nabla}c^2\psi+\tfrac{1}{2}\nabla c^2(\psi^2-h_{00}^{(2)})+(\gamma+\tfrac{1}{2})(\vec{v}+\vec{v}_s)^2\nabla\psi-(\vec{v}+\vec{v}_s)\times(\vec{\nabla}\times c\vec{h})$$
$$-\frac{1}{2}\frac{(\vec{v}+\vec{v}_s)^2}{c^2}\vec{a}-\frac{(\vec{v}+\vec{v}_s)\cdot\vec{a}(\vec{v}+\vec{v}_s)}{c^2}+c\frac{\partial\vec{h}}{\partial t}-(2\gamma+1)\frac{d}{dt}\psi(\vec{v}+\vec{v}_s), \tag{7}$$

with $\vec{v}+\vec{v}_s$ being the velocity of the moon, and $\vec{a}$ being the acceleration.

The general form of $g_{\mu\nu}(\vec{r},t)$ in metrical theories has been found in previous work[5]; here we specialize to the case of two sources, the sun and the earth, with the sun at rest and the earth moving at velocity $\vec{v}_s$ relative to the sun, while the earth accelerates at rate $(1+\delta)\vec{g}_s$ ($\delta\neq 0$ is discussed in Appendix A). $\vec{g}_s$ is the sun's gravitational acceleration,

$$\vec{g}_s=-\frac{GM\vec{R}_0}{R_0^{\,3}}. \tag{8}$$

(See Fig. 1.)

Using our final gauge freedom given in Eq. (6), the general metric field expansion can be put into the form

$$g_{00}=1-2\psi+2\beta\psi^2+(4\beta-2-2\rho_2)\frac{G^2Mm}{c^4R_0}\left(\frac{1}{R}+\frac{1}{r}\right)-(2\gamma+1+\rho_1+\alpha_3)\frac{Gm}{c^4r}v_s^{\,2}+\rho_1\frac{Gm}{c^4r^3}(\vec{v}_s\cdot\vec{r})^2$$
$$+\alpha_2\frac{G}{c^4}\left[\frac{m}{r^3}(\vec{W}\cdot\vec{r})^2+\frac{M}{R^3}(\vec{W}\cdot\vec{R})^2\right]+(\alpha_1-\alpha_2-\alpha_3)\psi\frac{W^2}{c^2}+(\alpha_1-2\alpha_3)\frac{G}{c^4}\frac{m}{r}\vec{W}\cdot\vec{v}_s, \tag{9a}$$

$$\vec{h}=(2\gamma+2+\tfrac{1}{2}\alpha_1)\frac{Gm}{c^3r}\vec{v}_s-\tfrac{1}{2}(1+\alpha_2-\rho_1)\vec{\nabla}\frac{Gm}{c^3r}\vec{r}\cdot\vec{v}_s+\tfrac{1}{2}\alpha_1\psi\frac{\vec{W}}{c}-\alpha_2\frac{G}{c^3}\vec{\nabla}\left(\frac{m\vec{r}}{r}+\frac{M\vec{R}}{R}\right)\cdot\vec{W}, \tag{9b}$$

and

$$g_{xx}=g_{yy}=g_{zz}=-1-2\gamma\psi, \tag{9c}$$

with ψ given by Eq. (3). $\vec{W}$ is the velocity of the coordinate frame with respect to a "preferred universe rest frame;" in this case $\vec{W}$ is the sun's velocity through the universe. γ, β, ρ_1, ρ_2, α_1, α_2, and α_3 are dimensionless PPN coefficients which are specified when choosing a particular gravitational theory. (See Refs. 5 and 6 for compilation of the PPN coefficients for several representative theories of gravity, and for discussion of various experimental determinations of the coefficients' values.)

The coefficients can be given physical interpretations: γ determines the degree to which mass distorts the spatial geometry of nearby physical objects; β is a measure of the nonlinearity of the gravitational fields; α_1, α_2, and α_3 measure the extent and manner in which the universe rest frame may enter local gravitational physics, while ρ_1 and ρ_2 signal breakdown of one of the energy-momentum conservation laws usually present in physical theory.[7] In general relativity the parameter values are

$$\gamma=\beta=1, \tag{10a}$$

$$\alpha_1=\alpha_2=\alpha_3=\rho_1=\rho_2=0. \tag{10b}$$

In the next several sections the various oscillatory earth-moon range contributions are calculated, which should be measurable by lunar laser ranging experiments.

III. "LORENTZ CONTRACTION" OF LUNAR ORBIT

Simply from considerations of special relativity the orbit of the moon ought to be flattened or contracted in one dimension when viewed from the heliocentric coordinate system in which the earth-moon system travels at speed v_s. However, when viewed from the inertial frame of the earth, this contraction is not "seen" or measured. The calculation of this section verifies this, although additional *observable* flattening of the lunar orbit exists if the PPN parameters α_i are nonzero.

Using the equation of motion given by Eq. (7) and the metric field given by Eqs. (9a)–(9c) we find acceleration terms of order $Gmv_s{}^2/c^2r^2$ acting on the moon:

$$a_{(r)} = [\tfrac{7}{4} + \tfrac{1}{2}(\alpha_1 - \alpha_2 - \alpha_3) + (\tfrac{3}{4} + \tfrac{1}{4}\alpha_2)\cos 2(\omega - \Omega)t] \times \frac{Gm}{r^2}\frac{v_s{}^2}{c^2} \tag{11a}$$

and

$$a_{(t)} = \tfrac{1}{2}\alpha_2 \frac{Gm}{r^2}\frac{v_s{}^2}{c^2}\sin 2(\omega - \Omega)t\,, \tag{11b}$$

where the subscripts (r) and (t) refer to radial and tangential components. ω is the moon's angular frequency around the earth; Ω is the earth's angular frequency around the sun. A circular Newtonian lunar orbit lying in the earth's orbital plane is assumed as a first approximation. Time is measured from a new-moon lunar phase. Using the basic range perturbation equation of Appendix B, Eqs. (11a) and (11b) lead to an oscillation of the earth-moon range:

$$X(t) \simeq \left(1 + \frac{8}{3}\frac{\Omega}{\omega} + \tfrac{1}{3}\alpha_2\right)\frac{1}{4}\frac{v_s{}^2}{c^2} r\cos 2(\omega - \Omega)t\,. \tag{12}$$

The Ω/ω term in the bracket is canceled by another term in Eq. (26). A Lorentz transformation to the earth's rest frame leaves an *observable* flattening of

$$X_{\text{obs}}(t) \simeq +\tfrac{1}{12}\alpha_2 \frac{v_s{}^2}{c^2} r\cos 2(\omega - \Omega)t \tag{13a}$$

$$\simeq +35\alpha_2\cos 2(\omega - \Omega)t \text{ cm} \tag{13b}$$

when evaluated for the lunar orbit. The nonoscillatory part of the radial perturbation in Eq. (11a) alters the orbital radius-frequency relationship for the orbit. The $\frac{7}{4}$ term properly adjusts according to special relativity the moon-as-a-clock rate when viewed from the heliocentric coordinate frame. The *observable* part of the orbital readjustment is

$$(\omega^2 r^3)_{\text{obs}} = Gm\left[1 - \frac{v_s{}^2}{c^2}\tfrac{1}{4}(2\alpha_1 + 2\alpha_3 - \alpha_2)\right]. \tag{14}$$

IV. NONLINEAR GRAVITY EFFECTS

The moon's equation of motion includes some accelerations nonlinear in the source masses, in particular terms proportional to mM. We collect in this section all perturbations proportional to $(Gm/c^2r)\vec{g}_s$; terms functionally proportional to $(v^2/c^2)\vec{g}_s$ ($\vec{v}$ is the moon's velocity relative to earth) will be included also as $v^2/c^2 = Gm/c^2r$ for the lunar orbit. Also included are accelerations of the moon relative to the earth proportional to $-\delta\vec{g}_s$, as discussed in Appendix A. The resulting perturbing acceleration components of the above form are

$$a_{(r)} = \left[(3\gamma + 2 + \tfrac{1}{2}\alpha_1)\frac{Gm}{c^2r} - \delta\right] g_s\cos(\omega - \Omega)t \tag{15a}$$

and

$$a_{(t)} = \left\{[\gamma + \tfrac{1}{2} + 2\beta - \tfrac{1}{2}(\alpha_1 + \rho_1 - \alpha_2)]\frac{Gm}{c^2r} + \delta\right\} g_s\sin(\omega - \Omega)t\,. \tag{15b}$$

Use of Appendix B yields the range perturbation

$$X(t) \simeq \left[(\gamma + 1 - 4\beta + \tfrac{3}{2}\alpha_1 + \rho_1 - \alpha_2)\frac{Gm}{c^2r} - 3\delta\right]\frac{g_s}{2\omega\Omega}\cos(\omega - \Omega)t\,. \tag{16}$$

An additional term with the same frequency and functional dependence on physical parameters as in Eq. (16) is obtained in Sec. VI. The combined range perturbation of this form then becomes

$$X(t) = \left[(\gamma + 3 - 4\beta + \tfrac{3}{2}\alpha_1 + \rho_1 - \alpha_2)\frac{Gm}{c^2r} - 3\delta\right]\frac{g_s}{2\omega\Omega}\cos(\omega - \Omega)t\,. \tag{17}$$

Evaluating this for the lunar orbit and estimating the earth's internal gravitational energy in order to give δ a value (see Appendix A) yields

$$X(t) \simeq \{6(\gamma + 3 - 4\beta + \tfrac{3}{2}\alpha_1 + \rho_1 - \alpha_2) + 10^3[(4\beta - 3 - \gamma - \alpha_1 + \alpha_2 - \rho_1) + \tfrac{1}{3}(\alpha_2 + \rho_2 - \rho_1)]\}\cos(\omega - \Omega)t \text{ cm}\,. \tag{18}$$

V. RENORMALIZATION OF G BY PROXIMITY OF MATTER

Collecting all acceleration terms of the moon proportional to $(GM/c^2R_0)(Gm\vec{r}/r^3)$, terms which incidently are also nonlinear in the source masses, the frequency-radius relationship for the moon becomes

$$\omega^3 r^3 = Gm\left[1 - (4\beta - 1 + 2\gamma - \tfrac{1}{2}\rho_2)\frac{GM}{c^2R_0}\right].$$

However, after correcting clock rates and ruler calibrations due to the solar gravitational potential in the earth's vicinity, the above relationship when quoted in earth measurements becomes

$$(\omega^2 r^3)_{\text{obs}} = Gm\left[1-(4\beta-3-\gamma-\tfrac{1}{2}\rho_2)\frac{GM}{c^2R_0}\right]. \quad (19)$$

The result of Eq. (19) agrees with previous calculations on the renormalization of Newton's gravitational constant due to proximity of matter, only here the result is quoted in the new PPN language. As the earth orbits the sun in a slightly eccentric orbit, R_0 changes with a twelve-month period, producing a perturbation on the lunar orbit. If

$$R_0(t) \simeq R_0(1-e\cos\Omega t')$$

(t' is measured from earth-orbit perigee), then e is the eccentricity of earth's orbit, and the range perturbation on the moon is

$$X(t) \simeq (4\beta-3-\gamma-\tfrac{1}{2}\rho_2)e\left(\frac{GM}{c^2R_0}\right)r\cos\Omega t' \quad (20a)$$

$$\simeq 7(4\beta-3-\gamma-\tfrac{1}{2}\rho_2)\cos\Omega t' \text{ cm} \quad (20b)$$

when evaluated for the lunar orbit.

VI. MOTIONAL EFFECT PROPORTIONAL TO vv_s

Collecting the acceleration terms of the moon proportional to $(Gm/r^2)vv_s/c^2$ leads to perturbations of frequencies $\omega-\Omega$, $2(\omega-\Omega)$, and a secular term. The $\omega-\Omega$ perturbation comes from acceleration components

$$a_{(r)} = -(2+\tfrac{1}{2}\alpha_1)\frac{Gm}{r^2}\frac{vv_s}{c^2}\cos(\omega-\Omega)t \quad (21a)$$

and

$$a_{(t)} = -\frac{Gm}{r^2}\frac{vv_s}{c^2}\sin(\omega-\Omega)t , \quad (21b)$$

which produces a range perturbation

$$X(t) = \left(\frac{\Omega}{\omega}-\tfrac{1}{4}\alpha_1\right)\frac{Gm}{r^2}\frac{vv_s}{c^2}\frac{1}{\omega\Omega}\cos(\omega-\Omega)t . \quad (22)$$

The first part of Eq. (22) combines with the results of Sec. IV to give a perturbation combination that vanishes in general relativity [Eq. (17)]. That leaves here the new effect

$$X(t) \simeq -\tfrac{1}{4}\alpha_1\frac{Gm}{r^2}\frac{vv_s}{c^2}\frac{1}{\omega\Omega}\cos(\omega-\Omega)t \quad (23a)$$

$$\simeq -40\alpha_1\cos(\omega-\Omega)t \text{ cm} \quad (23b)$$

when evaluated for lunar orbit. Additional perturbations have the components

$$a_{(r)} = -(2\gamma+1)g_s\frac{vv_s}{c^2} + g_s\frac{vv_s}{c^2}\cos 2(\omega-\Omega)t \quad (24a)$$

and

$$a_{(t)} = -g_s\frac{vv_s}{c^2}\sin 2(\omega-\Omega)t . \quad (24b)$$

The static part of Eq. (24a) alters the angular frequency of the moon's orbit:

$$\delta\omega = \tfrac{1}{2}(2\gamma+1)\left(\frac{GM}{c^2R_0}\right)\Omega . \quad (25)$$

This effect is the "geodetic precession," which results from orbiting a massive body M at distance R_0 at angular frequency Ω. Equation (25), for example, would give the precession rate of a gyroscope in orbit around the sun.[8,9] (If the moon's orbit were reversed, $\delta\omega$ would change sign). The oscillatory parts of Eqs. (24a) and (24b) give a range perturbation

$$X(t) \simeq -\tfrac{2}{3}g_s\frac{vv_s}{c^2}\frac{1}{\omega^2}\cos 2(\omega-\Omega)t ; \quad (26)$$

however, this term exactly cancels a term in Eq. (12).

VII. MACHIAN EFFECTS PROPORTIONAL TO wv

The general metric field given by Eqs. (9a)–(9c) include potentials dependent on the velocity of the sun relative to the universe $-\vec{w}$. Such potentials are called "Machian," because they produce solar-system gravitational effects dependent on the earth's relationship to the universe's mass, in line with ideas first presented by Mach in the late 19th century. In making quantitative estimates in this paper we use for $\vec{w}$ the sun's velocity in our galaxy. This velocity has magnitude of about 200 km/sec and direction given by

$$L_w \simeq -60°,$$

$$\phi_w \simeq 0°,$$

where L_w is the latitude of $\vec{w}$ with respect to the ecliptic plane, and ϕ_w is the longitude of $\vec{w}$ measured from the vernal equinox.

Perturbations proportional to wv/c^2 either multiply m/r^2 or M/R^2. The first type produces only a radial acceleration,

$$a_{(r)} = -\tfrac{1}{2}\alpha_1\frac{Gm}{r^2}\frac{vw'}{c^2}\sin(\omega t-\theta_0), \quad (27)$$

where θ_0 is the longitude angle of $\vec{w}$ ($\vec{w}'$ is the part of $\vec{w}$ in the lunar orbital plane) at some new-moon phase, and t is measured from that *particular* new-moon phase. Equation (27) produces a range oscillation,

$$X(t) = +\tfrac{1}{4}\alpha_1\frac{Gm}{r^2}\frac{vw'}{c^2}\frac{1}{\omega(\omega-\omega_0)}\sin(\omega t-\theta_0), \quad (28)$$

where ω_0 is a natural frequency for radial perturbations (in astronomical language $2\pi/\omega_0$ is the period between perigee passages for the actual lunar orbit). $\omega - \omega_0$ is the rate of precession of lunar perigee *with respect to inertial space;* this period is about 17 years. Evaluations of Eq. (28) for lunar orbit gives approximately

$$X(t) \simeq +4400\alpha_1 \sin(\omega t - \theta_0) \text{ cm}. \tag{29}$$

The perturbations proportional to $g_s wv/c^2$ produce a radial acceleration,

$$a_{(r)} = \tfrac{1}{2}\alpha_1 \frac{GM}{R_0^2}\frac{w'v}{c^2}\sin(\Omega t - \theta_0), \tag{30}$$

which gives a range oscillation

$$X(t) \simeq \tfrac{1}{2}\alpha_1 \frac{GM}{R_0^2}\frac{vw'}{c^2}\frac{1}{\omega^2}\sin(\Omega t - \theta_0), \tag{31a}$$

$$\simeq 90\alpha_1 \sin(\Omega t - \theta_0) \tag{31b}$$

when evaluated for lunar orbit.

VIII. MACHIAN EFFECTS PROPORTIONAL TO wv_s

Collecting perturbations proportional to $(Gm/r^2)wv_s/c^2$ yields the acceleration components

$$a_{(r)} = \frac{Gm}{r^2}\frac{w'v_s}{c^2}\{(\alpha_1 - \tfrac{1}{2}\alpha_2 - \alpha_3)\sin(\Omega t - \theta_0) - \tfrac{1}{2}\alpha_2 \sin[(2\omega - \Omega)t - \theta_0]\} \tag{32a}$$

and

$$a_{(t)} = \alpha_2 \frac{Gm}{r^2}\frac{w'v_s}{c^2}\cos[(2\omega - \Omega)t - \theta_0], \tag{32b}$$

which produces a range oscillation

$$X(t) = \frac{Gm}{r^2}\frac{w'v_s}{c^2}\frac{1}{\omega^2}\{(\alpha_1 - \tfrac{1}{2}\alpha_2 - \alpha_3)\sin(\Omega t - \phi_0) - \tfrac{1}{6}\alpha_2 \sin[(2\omega - \Omega)t - \theta_0]\} \tag{33a}$$

$$\simeq 2700(\alpha_1 - \tfrac{1}{2}\alpha_2 - \alpha_3)\sin(\Omega t - \theta_0) - 450\alpha_2 \sin[(2\omega - \Omega)t - \theta_0)] \text{ cm} \tag{33b}$$

when evaluated for lunar orbit.

IX. MACHIAN EFFECTS PROPORTIONAL TO w^2

The perturbations proportional to $(Gm/r^2)w^2/c^2$ give acceleration components

$$a_{(r)} = \tfrac{1}{4}\alpha_2 \frac{Gm}{r^2}\left(\frac{w'}{c}\right)^2[1 + \cos(2\omega t - 2\theta_0)], \tag{34a}$$

and

$$a_{(t)} = \tfrac{1}{2}\alpha_2 \frac{Gm}{r^2}\left(\frac{w'}{c}\right)^2 \sin(2\omega t - 2\theta_0). \tag{34b}$$

These accelerations produce a range oscillation

$$X(t) = \tfrac{1}{12}\alpha_2 \frac{Gm}{r^2}\left(\frac{w'}{c}\right)^2 \frac{1}{\omega^2}\cos(2\omega t - 2\theta_0) \tag{35a}$$

$$\simeq 1500\alpha_2 \cos(2\omega t - 2\theta_0) \tag{35b}$$

when evaluated for lunar orbit. The time-independent part of $a_{(r)}$ in Eq. (34a) rescales the magnitude of G (Newton's gravitational constant) in an unobservable manner.

X. MACHIAN TIDAL EFFECTS

The major Newtonian perturbation on the lunar orbit is the tidal force of the sun. In the PPN metric the sun has an additional gravitational potential acting on the earth and the moon of the form $GM(\vec{w}\cdot\vec{R})^2/R^3$, which yields a contribution to the acceleration of the moon *relative* to the earth:

$$\vec{a} = \alpha_2 \frac{GM}{R_0^5 c^2}\left[\tfrac{3}{2}(\vec{w}\cdot\vec{R}_0)^2\vec{r} + 3\vec{R}_0\cdot\vec{w}\,\vec{r}\cdot\vec{w}\,\vec{R}_0 - \frac{15}{2}\frac{\vec{R}_0\cdot\vec{r}(\vec{R}_0\cdot\vec{w})^2}{R_0^2}\vec{R}_0 - \vec{w}\cdot\vec{r}\,\vec{w}R_0^2 + 3\vec{R}_0\cdot\vec{w}\,\vec{R}_0\cdot\vec{r}\,\vec{w}\right], \tag{36}$$

which has acceleration components

$$a_{(r)}=\alpha_2\frac{GM}{R_0^{\,3}}\,r\left(\frac{w'}{c}\right)^2\left\{\tfrac{7}{4}+\tfrac{3}{8}\cos(2\Omega t-2\theta_0)-\tfrac{7}{8}\cos(2\omega t-2\theta_0)+\tfrac{9}{16}\cos(2\omega-2\Omega)t-\tfrac{15}{16}\cos[(2\omega+2\Omega)t-4\theta_0]\right\} \tag{37a}$$

and

$$a_{(t)}=\alpha_2\frac{GM}{R_0^{\,3}}\,r\left(\frac{w'}{c}\right)^2\left\{\tfrac{3}{8}\sin(2\omega-2\Omega)t-\tfrac{1}{16}\sin(2\omega t-2\theta_0)+\tfrac{15}{16}\sin[(2\omega-4\Omega)t+2\theta_0]\right\}; \tag{37b}$$

these acceleration components give a range oscillation of

$$X(t)=\alpha_2\frac{GM}{R_0^{\,3}}\,r\left(\frac{w'}{c}\right)^2\frac{1}{\omega^2}\left\{\tfrac{3}{8}\cos(2\Omega t-2\theta_0)+\tfrac{13}{48}\cos(2\omega t-2\theta_0)+\tfrac{5}{16}\cos[(2\omega-4\Omega)t+2\theta_0]-\tfrac{1}{16}\cos(2\omega-2\Omega)t+\tfrac{5}{16}\cos[(2\omega+2\Omega)t-4\theta_0]\right\} \tag{38a}$$

$$\simeq\alpha_2\{33\cos(2\Omega t-2\theta_0)+24\cos(2\omega t-2\theta_0)-6\cos(2\omega-2\Omega)t+28\cos[(2\omega-4\Omega)t+2\theta_0]+28\cos[(2\omega+2\Omega)t-4\theta_0]\}\text{ cm} \tag{38b}$$

when evaluated for the lunar orbit.

XI. MACHIAN SELF-ACCELERATION EFFECT

In Appendix A it is mentioned that a spinning celestial body will self-accelerate at the rate

$$\vec{a}=\tfrac{1}{3}\alpha_3\left(\frac{U_G}{Mc^2}\right)\vec{\omega}\times\vec{w}, \tag{39}$$

where U_G is the body's internal gravitational energy, M its mass, and $\vec{\omega}$ its spin angular velocity. Applying Eq. (39) to the earth-moon system, the moon acquires an acceleration relative to earth which is the negative of Eq. (39) evaluated for earth parameters. The components of $\vec{a}$ are

$$a_{(r)}=\tfrac{1}{3}\alpha_3\left(\frac{U_G}{Mc^2}\right)|\vec{\omega}\times\vec{w}'|\cos(\omega t-\theta_1), \tag{40a}$$

$$a_{(t)}=-\tfrac{1}{3}\alpha_3\left(\frac{U_G}{Mc^2}\right)|\vec{\omega}\times\vec{w}'|\sin(\omega t-\theta_1), \tag{40b}$$

where $(\vec{\omega}\times\vec{w}')$ is the component of $\vec{\omega}\times\vec{w}$ lying in the orbital plane of the moon, and θ_1 is the longitude angle of $(\vec{\omega}\times\vec{w}')$ from new-moon phase. The accelerations in Eqs. (40a) and (40b) produce a range oscillation of approximately

$$X(t)\simeq\tfrac{1}{2}\alpha_3\left(\frac{U_G}{Mc^2}\right)\frac{\omega_E w}{\omega(\omega_0-\omega)}\cos(\omega t-\theta_1) \tag{41a}$$

$$\simeq 5\times10^6\alpha_3\cos(\omega t-\theta_1) \tag{41b}$$

when evaluated for lunar orbit.

XII. RANGING TO LOW EARTH-ORBITING SATELLITES

Most of the oscillatory range effects calculated in this paper become smaller in amplitude for satellite orbits close to earth, but a few effects grow with decreasing orbital radius. This can be seen by examining the functional dependence on the physical parameters of each of the range effects calculated. Two range oscillations look particularly interesting in terms of searching for their effect on low earth-orbiting satellites. Equations (23a) and (28) expressed in terms of their dependence on orbital radius are

$$X(t)\simeq\tfrac{1}{4}\alpha_1\frac{Gm}{c^2r}w'\frac{1}{\omega-\omega_0}\sin(\omega t-\theta_0)-\tfrac{1}{4}\alpha_1\frac{Gm}{c^2r}R_0\left(\frac{\Omega}{\Omega+\omega_0-\omega}\right)\cos(\omega-\Omega)t. \tag{42}$$

For low earth orbit the second term in Eq. (42) has an approximate magnitude

$$X(t)\simeq-2000\alpha_1\cos(\omega-\Omega)t. \tag{43}$$

The first term in Eq. (42) requires special consideration. The $1/\omega-\omega_0$ factor does not affect the accuracy with which we can estimate the α_1 in that term (see Appendix C). However, Appendix C indicates that maximum estimation accuracy occurs for experiments in which $(\omega_0-\omega)T\ll1$, T being the total time of a ranging experiment. The $1/r$ factor improves the accuracy of estimating α_1 for low orbits.

APPENDIX A: SELF-GRAVITATIONAL EFFECTS ON THE EQUATIONS OF MOTION OF MASSIVE BODIES

Usually one obtains the equation of motion of a celestial body by assuming it moves in the gravitational potentials of all *other bodies of the system*. But in the typical metric theory of gravity, internal gravitational fields have been found to produce anomalous accelerations of massive bodies. Two such effects are discussed here and used in this paper to produce contributions to the lunar range.

There is a coupling of the internal gravitational fields of massive bodies to the external fields, which leads in general to a breakdown of Einstein's equivalence principle for massive bodies. The gravitational-to-inertial mass ratio of a spherical, nonspinning celestial body becomes[4,5]

$$\frac{M_G}{M_I} = 1 + [(4\beta - 3 - \gamma - \alpha_1 + \alpha_2 - \rho_1) + \tfrac{1}{3}(\alpha_2 + \rho_2 - \rho_1)] \frac{U_G}{Mc^2} \equiv 1 + \delta , \tag{A1}$$

where U_G is the internal gravitational energy of the body, and Mc^2 is its total mass energy. A $\delta \neq 0$ for earth is used in Sec. IV to obtain a relative earth-moon acceleration. $\delta = 0$ in general relativity but not in most other theories of gravity.

One of the potentials in the metric expansion of Eq. (9a) is

$$\delta g_{00} = (\alpha_1 - 2\alpha_3) \frac{G}{c^4} \sum_i \frac{m_i}{|\vec{r} - \vec{r}_i|} \vec{w} \cdot \vec{v}_i , \tag{A2}$$

where many mass sources m_i moving at velocities $\vec{v}_i$ are assumed. Considering a massive celestial body in rotation at angular velocity $\vec{\omega}$ and summing the body's interparticle forces resulting from the potential in Eq. (A2) results in a net *self-acceleration* of the rotating massive body,[6]

$$\vec{a} = \tfrac{1}{3}\alpha_3 \left(\frac{U_G}{Mc^2} \right) \vec{\omega} \times \vec{w} . \tag{A3}$$

(The α_1 part of the coefficient in Eq. (A2) has its effects canceled by another α_1 term in the total equation of motion of the body's mass elements.) The acceleration in Eq. (A3) is used in Sec. XI to produce a lunar range oscillation.

APPENDIX B: BASIC RANGE-PERTURBATION EQUATION

Consider a body in near-circular orbit around a central body. The complete secular potential is $V(r)$; $V(r)$ is not necessarily $1/r$, since a quadrupole moment of the central body or secular perturbations by other solar system bodies contribute to $V(r)$. The basic equation of motion of the body experiencing a time-dependent perturbation $\vec{a}(t)$ is then

$$\ddot{r} = -V'(r) + \frac{h^2}{r^3} + a_{(r)}(t) \tag{B1}$$

and

$$\dot{h} = r a_{(t)}(t) . \tag{B2}$$

Subscripts (r) and (t) refer to radial and tangential components; the prime denotes radial derivatives and the dot denotes time derivative. Linearizing Eqs. (B1) and (B2) about a circular orbit by assuming

$$r = r_0 + X(t) ,$$
$$h = h_0 + \delta h(t)$$

results in

$$\ddot{X} = -V''X - \frac{3h_0^2}{r_0^4} X + a_{(r)} + \frac{2h_0 \delta h}{r_0^3}$$

and

$$\delta\dot{h} = r_0 a_{(t)}(t) .$$

Defining

$$V''(r_0) + 3\omega^2 \equiv \omega_0^2 , \tag{B3}$$

where ω is the circular orbit angular frequency and ω_0 is the natural frequency for radial perturbations, the radial equation of motion becomes

$$\ddot{X} + \omega_0^2 X = a_{(r)}(t) + 2\omega \int^t a_{(t)}(t')dt' . \tag{B4}$$

If the perturbation is oscillatory,

$$a_{(r)}(t) = Ae^{i\omega' t} , \tag{B5}$$

$$a_{(t)}(t) = Be^{i\omega' t} , \tag{B6}$$

then Eq. (B4) has the inhomogeneous solution

$$X(t) = \frac{1}{\omega_0^2 - \omega'^2} \left(A - \frac{2\omega i}{\omega'} B \right) e^{i\omega' t} \tag{B7}$$

except for the case of resonance ($\omega' = \omega_0$). Equation (B7) is used throughout this paper to calculate the various range oscillations. Equation (B4) also has the homogeneous solution

$$X(t) = -r_0 e \cos(\omega_0 t - \phi_p) , \tag{B8}$$

which represents the eccentric motion of precessing elliptical orbits. Perigee occurs at time ϕ_p/ω_0 and recurs with period $2\pi/\omega_0$, the perigee precession rate being $\omega - \omega_0$.

APPENDIX C: MEASUREMENT OF NEARLY RESONANT FOURIER SIGNALS

Assume there is a time-dependent physical signal (e.g., the range data in laser ranging) of the form

$$S(t) = A \sin\omega t + B \sin\omega' t + C \cos\omega' t . \tag{C1}$$

The frequencies ω and ω' are assumed to be known and close together. The goal is to estimate A from a series of measurements S_i made at times t_i. A least-squares estimation procedure gives the algebraic equations (written here in matrix form)

$$\begin{bmatrix} \sum_i \sin^2\omega t_i & \sum_i \sin\omega t_i \sin\omega' t_i & \sum_i \sin\omega t_i \cos\omega' t_i \\ \sum_i \sin\omega t_i \sin\omega' t_i & \sum_i \cos^2\omega' t_i & \sum_i \sin\omega' t_i \cos\omega' t_i \\ \sum_i \sin\omega t_i \cos\omega' t_i & \sum_i \sin\omega' t_i \cos\omega' t_i & \sum_i \cos^2\omega' t_i \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \sum_i S_i \sin\omega t_i \\ \sum_i S_i \sin\omega' t_i \\ \sum_i S_i \cos\omega' t_i \end{bmatrix} . \tag{C2}$$

Assuming an approximately regular observational schedule, the sums in the matrix of Eq. (C2) can be well estimated by integrals:

$$\sum_i \sin^2\omega t_i = \sum_i \sin^2\omega' t_i = \sum_i \cos^2\omega' t_i \simeq \tfrac{1}{2} r T, \tag{C3}$$

$$\sum_i \sin\omega t_i \sin\omega' t_i \simeq \frac{r}{2\Omega} \sin\Omega T , \tag{C4}$$

$$\sum_i \sin\omega t_i \cos\omega' t_i \simeq \frac{r}{2\Omega} (1-\cos\Omega T), \tag{C5}$$

and

$$\sum_i \sin\omega' t_i \cos\omega' t_i \simeq 0 , \tag{C6}$$

with T the measurement interval, $\Omega \equiv \omega - \omega'$, and r the measurement rate. An error analysis of the estimation procedure then gives

$$\begin{bmatrix} T & \frac{\sin\Omega T}{\Omega} & \frac{1-\cos\Omega T}{\Omega} \\ \frac{\sin\Omega T}{\Omega} & T & 0 \\ \frac{1-\cos\Omega T}{\Omega} & 0 & T \end{bmatrix} \begin{bmatrix} \delta A \\ \delta B \\ \delta C \end{bmatrix} = \frac{2}{r} \begin{bmatrix} \sum_i \delta S_i \sin\omega t_i \\ \sum_i \delta S_i \sin\omega' t_i \\ \sum_i \delta S_i \cos\omega' t_i \end{bmatrix} , \tag{C7}$$

where δA is the error produced in estimating A due to measurement errors δS_i. We now solve Eq. (C7) for δA, square this result, and take a statistical average value. The measurement errors are assumed uncorrelated;

$$\langle \delta S_i \delta S_j \rangle_{\text{av}} = \sigma^2 \delta_{ij} , \tag{C8}$$

σ being the r.m.s. measurement error. The r.m.s. error in A is then given by

$$\Omega^2 \langle \delta A^2 \rangle_{\text{av}} = \frac{\sigma^2}{2r} \frac{1}{T^3} \frac{x^4}{x^2 - \sin^2 x} , \tag{C9}$$

where $x = \frac{1}{2}\Omega T$. (Since A in the cases of interest is proportional to $1/\Omega$, we quote $\Omega\delta A$ in Eq. (C9), as it is some PPN coefficient in A, not A itself, which we are estimating in each case.) Equation (C9) indicates that for fixed r and T, optimum estimation is when $x \to 0$, although little loss in estimation accuracy results as long as $x \equiv \frac{1}{2}\Omega T \lesssim 1$.

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