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Quantum Tree Graphs and the Schwarzschild Solution

M. J. Duff*
Physics Department, Imperial College, London SW7, England
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It is verified explicitly to second order in Newton's constant, G, that the quantum-tree-graph contribution to the vacuum expectation value of the gravitational field produced by a spherically symmetric c-number source correctly reproduces the classical Schwarzschild solution. If the source is taken to be that of a point mass, then even the tree diagrams are divergent, and it is necessary to use a source of finite extension which, for convenience, is taken to be a perfect fluid sphere with uniform density. In this way both the interior and exterior solutions may be generated. A mass renormalization takes place; the total mass of the source, m, being related to its bare mass, m_0, and invariant radius, ϵ_r, by the Newtonian-like formula, $m = m_0 - 3Gm_0^2/5\epsilon_r + O(G^2)$, and the infinities in the quantum theory are seen to be a manifestation of the divergent self-energy problem encountered in classical mechanics.

I. INTRODUCTION

In an attempt to find quantum corrections to solutions of Einstein's equations, the question naturally arises as to whether the $\hbar \to 0$ limit of the quantum theory correctly reproduces the classical results. Formally, at least, the correspondence between the tree-graph approximation to quantum field theory and the classical solution of the field equations is well known,[1] i.e., the classical field produced by an external source serves as the generating functional for the connected Green's functions in the tree approximation, the closed-loop contributions vanishing in the limit $\hbar \to 0$. The purpose of this paper is to present an explicit calculation of the vacuum expectation value (VEV) of the gravitational field in the presence of a spherically symmetric source and verify, to second order in perturbation theory, that the result is in agreement with the classical Schwarzschild solution of the Einstein equations. This would appear to be the first step towards tackling the much more ambitious program of including the radiative quantum corrections.

Whereas in quantum electrodynamics it is a comparatively simple matter to obtain the Coulomb potential by means of the single-photon exchange from a stationary point charge, the analogous situation in gravidynamics, where the gauge group is non-Abelian, proves much more difficult. Firstly, as has been shown by Arnowitt, Deser, and Misner[2] (henceforth referred to as ADM) the concept of a strictly pointlike source in general relativity is untenable. There is a minimum invariant extension for a particle below which no solutions of the field equations exist, the space-time developing an intrinsic singularity at a finite point in the exterior domain of the particle for radii less than this minimum. Moreover, the total mass of the source would then become negative and eventually negatively infinite as the point-mass limit is taken. As we shall see, these difficulties manifest themselves in the quantum theory in the guise of divergent tree diagrams when a point source is used. As a model for the source, therefore, it is essential to choose a particle of finite extension.

In their work, ADM pick the simplest model for such an extended particle, a spherical "shell distribution" of pressure-free dust for which the mass density is merely proportional to $\delta(r-\epsilon)$, where r denotes the radial coordinate and ϵ the radius of the shell. From the quantum point of view, however, another dilemma arises. The quantum-field-theory calculations are most conveniently performed in a manifestly Lorentz-covariant gauge by employing, for example, the harmonic coordinate condition of de Donder,[3] $[(-g)^{1/2}g^{\mu\nu}]_{,\nu}=0$. Whereas in the canonical approach ADM are able to carry out their analysis in a frame for which the metric is continuous across the shell, in harmonic coordinates the usual regularity conditions are violated and the metric is itself discontinuous. This problem has been discussed in a previous paper.[4] One is then faced with a choice, whether to use the attractively simple δ-function source and put up with the attendant problems of discontinuity, or to abandon the shell in favor of a uniform sphere thus gaining continuity at the expense of simplicity. In this paper we shall use the latter.

Finally, there is the question of stability. A cloud of pressure-free dust for which the interactions are purely gravitational is not a static configuration. This is clear on physical grounds. In the absence of phenomenological nongravitational pressure terms the cloud of dust under its own gravitational attraction is unstable and will begin to collapse. To circumvent this problem we use a static perfect fluid with uniform mass density.

In Sec. II, the classical situation is reviewed using the most general form of the spherically symmetric time-orthogonal line element. For the pressure-free dust a coordinate-independent mass-renormalization formula is derived, valid at the moment of time symmetry, which relates the total mass of the source to its bare mass and invariant size. The generalization involved in going over to a perfect fluid is straightforward. To solve the field equations explicitly, of course, coordinate conditions must be imposed. In the third section of the paper, the particular features of de Donder coordinates are exhibited in order to facilitate a direct comparison with the quantum theory.

Section IV is devoted to the quantum calculation of the VEV of the gravitational field to fourth order in the coupling constant, κ, (i.e., second order in G). This involves a knowledge of the rather complicated 3-graviton vertex function. No attempt is made to compute the four-point graph or higher-order contributions because of the labor involved. However, the 3-point diagram in which the source acts twice will be sufficient to display the intrinsic nonlinearity of the theory, and, as we shall see, the mass-renormalization effects.

II. THE CLASSICAL SITUATION AND MASS RENORMALIZATION

In this section we shall summarize some of the classical results needed before making the comparison with the quantum theory. To begin with we confine ourselves to the pressure-free dust model and postpone discussion of the pressure effects until later. The Einstein equations

$$G_\nu{}^\mu = -\tfrac{1}{2}\kappa^2 T_\nu{}^\mu, \quad \kappa^2 = 16\pi G \tag{2.1}$$

describing the interaction of gravitation and matter, may be divided into two categories: the four initial value or constraint equations which relate the Cauchy data of the system at some initial time, and the time development equations describing the evolution of these data from their initial value. ADM have made this particularly clear using a (3 + 1)-dimensional decomposition of the dynamics. We refer the reader to Ref. 5 for a detailed discussion. The four constraint equations are just the field equations

$$G_\mu{}^0 = -\tfrac{1}{2}\kappa^2 T_\mu{}^0, \tag{2.2}$$

where for a pressure-free dust the energy-momentum tensor is given by

$$T_\nu{}^\mu = \mu u^\mu u_\nu\,. \tag{2.3}$$

μ denotes the proper rest-mass density and u^μ is a 4-velocity satisfying

$$g_{\mu\nu}u^\mu u^\nu = -1\,. \tag{2.4}$$

Since we are not interested in the time development of the system, we shall, for simplicity, concentrate on the moment of time symmetry. That is to say, we imagine the cloud of dust to be expanding at some time during the past, gradually coming to rest at $t = 0$ under its gravitational attraction, and then collapsing as we move off the hypersurface. For static initial data and for coordinate frames with time lines orthogonal to this 3-dimensional hypersurface, it follows that

$$\dot{g}_{ij} = 0, \quad i,j = 1,2,3 \tag{2.5}$$

initially, which means that the initial surface is a moment of time symmetry. In this case the initial constraints now reduce to

$$G_0{}^0 = -\tfrac{1}{2}\kappa^2 T_0{}^0, \tag{2.6a}$$

$$G_i{}^0 \equiv 0. \tag{2.6b}$$

The most general spherically symmetric time-orthogonal line element may be written in the form

$$ds^2 = F^2dr^2 + H^2d\Omega - N^2dt^2, \tag{2.7}$$

where $d\Omega$ is the conventional solid-angle element. The metric has the form

$$g_{00} = -N^2, \tag{2.8a}$$

$$g_{ij} = \frac{H^2}{r^2}\eta_{ij} + \left(F^2 - \frac{H^2}{r^2}\right)\frac{x^i x^j}{r^2} \tag{2.8b}$$

in rectangular coordinates with $x^i x^i = r^2$. F, H, and N depend only on r and t and $\dot{F} = \dot{H} = 0$ initially since $\dot{g}_{ij} = 0$ initially.

Such a metric depends on only two unknown functions and the choice of coordinates amounts to imposing a constraint on F, H, and N. For example, the choices $F = H/r$ and $H = r$ yield the isotropic and conventional Schwarzschild frames, respectively. However, for the purposes of obtaining a coordinate-independent mass renormalization formula, we shall not resort to a particular coordinate choice but merely mention that the different frames under consideration are related to each other by redefinition of the radial coordinate. Under transformations of this type H and N behave like scalars and F like a scalar density of unit weight. One may show that

$$G_0{}^0 = \tfrac{1}{2}\,{}^3\mathcal{R} = \frac{1}{H^2H'}\left[H\left(1 - \frac{H'^2}{F^2}\right)\right]', \tag{2.9}$$

where ${}^3\mathcal{R}$ is the scalar curvature of the 3-space and is independent of N. If we now consider a spherical matter distribution with uniform density ρ and radius ϵ, then

$$-T_0{}^0=\mu(r)$$
$$=\rho\theta(\epsilon-r). \qquad (2.10)$$

Both r and ϵ are coordinate-dependent quantities. ϵ is the radius of the shell as measured in a frame where r is the radial coordinate. For convenience we have put the origin at the center of the sphere. Equation (2.6a) now reads

$${}^3\mathcal{R}=\frac{2}{H^2H'}\left[H\left(1-\frac{H'^2}{F^2}\right)\right]'$$
$$=16\pi G\rho\theta(\epsilon-r). \qquad (2.11)$$

If we define the invariant quantity

$$K\equiv\frac{H'}{F} \qquad (2.12)$$

and use the subscript + (−) to denote the exterior (interior) forms, Eq. (2.11) yields[6]

$$K_-^2=1-\frac{8\pi G\rho}{H}\int_0^r dr\,H^2H'$$
$$=1-\frac{H^2(r)}{R^2}\,, \qquad (2.13a)$$
$$K_+^2=1-\frac{8\pi G\rho}{H}\int_0^\epsilon dr H^2H'$$
$$=1-\frac{2Gm}{H(r)}\,, \qquad (2.13b)$$

where $R^2=3/8\pi G\rho$ and

$$m=\tfrac{4}{3}\,\pi\rho H^3(\epsilon) \qquad (2.14)$$

is to be identified with the total mass of the system as seen by an outside observer.[7]

There are two criteria for measuring the initial size of the system. The invariant "circumference" of the sphere is $2\pi H(\epsilon)$, $H(r)$ being the coefficient of the solid-angle element $d\Omega$ in the line element. Henceforth $H(\epsilon)$ is denoted by ϵ_c. There is also the invariant "radius," ϵ_r, defined by

$$\epsilon_r=\int_0^\epsilon (g_{rr})^{1/2}\,dr \qquad (2.15)$$

For a shell distribution space is flat in the interior and $\epsilon_c=\epsilon_r$. In general, however, they are distinct, and in our case we have

$$\epsilon_r=\int_0^\epsilon F_-dr$$
$$=R\sin^{-1}\left(\frac{\epsilon_c}{R}\right) \qquad (2.16)$$

from Eq. (2.13a). The total mass, m, is clearly an invariant. We now wish to relate this total mass to the unrenormalized mass, m_0.

The standard form of the Einstein equation (2.11) is to be compared with the one obtained in the (3+1)-dimensional formalism of ADM by varying (with respect to N) the Lagrangian describing the coupling of the gravitational field to a neutral particle of bare mass m_0.[8] This bare mass is the inertial mass of the particle in the limit of zero coupling ($G=0$). There we have

$$({}^3g)^{1/2}\,{}^3\mathcal{R}=\kappa^2\rho_0(r),\quad r<\epsilon\,. \qquad (2.17)$$

${}^3g=(FH^2/r^2)^2$ is the determinant of g_{ij}. The bare-mass density $\rho_0(r)$ is related to m_0 by the equation

$$m_0=\int\rho_0\,d^3r\,, \qquad (2.18)$$

the dynamical equations determining ρ_0 uniquely and in such a way that m_0 remains constant. Comparison of (2.11) and (2.17) gives

$$\rho_0=\rho({}^3g)^{1/2} \qquad (2.19)$$

and therefore

$$m_0=\int\rho\,({}^3g)^{1/2}d^3r\,. \qquad (2.20)$$

Treating H as a radial coordinate we could now write

$$m=\int\mu\,d^3H \qquad (2.21)$$

and

$$m_0=\int\mu\frac{d^3H}{K} \qquad (2.22)$$

$$=2\pi\rho R^3\left[\sin^{-1}\left(\frac{\epsilon_c}{R}\right)-\frac{\epsilon_c}{R}\left(1-\frac{\epsilon_c^2}{R^2}\right)^{1/2}\right] \qquad (2.23)$$

$$=\tfrac{4}{3}\pi\rho\epsilon_c^3\left(1+\frac{3}{10}\frac{\epsilon_c^2}{R^2}+\cdots\right) \qquad (2.24)$$

$$=m+\frac{3}{5}\frac{Gm^2}{\epsilon_c}+\cdots\,. \qquad (2.25)$$

Turning this equation around, we obtain

$$m=m_0-\frac{3}{5}\frac{Gm_0^2}{\epsilon_c}+O(G^2)\,. \qquad (2.26)$$

This formula, valid at the moment of time symmetry, relates the total mass of the system to its bare mass and invariant extension. It resembles the Newtonian self-energy formula for a massive sphere in classical mechanics,[9] and was derived without resorting to a particular coordinate choice using the energy-momentum tensor of a dust cloud given in Eq. (2.3).

The argument is now easily extended to the case of a perfect fluid with

$$T_\nu{}^\mu=(\mu+p)u^\mu u_\nu+p\delta_\nu{}^\mu\,, \qquad (2.27)$$

p denoting the pressure. The important point is that, for constant mass density, the constraint equation (2.6a) and hence the mass renormalization formula remain unchanged even when pressure terms are introduced. Now, however, the other equations $G_j{}^i = -\frac{1}{2}\kappa^2 T_j{}^i$ may describe a truly static[10] situation and Eq. (2.26) holds not merely at the moment of time symmetry but for all time. Moreover, we may now express the remaining metric component N and the pressure p in terms of K. We omit here the algebra and state the results

$$N_- = \tfrac{3}{2}K_-(\epsilon) - \tfrac{1}{2}K_-(r), \quad N_+ = K_+, \tag{2.28}$$

$$p = \rho\left(\frac{K_-(r) - K_-(\epsilon)}{3K_-(\epsilon) - K_-(r)}\right)\theta(\epsilon - r). \tag{2.29}$$

Note that all the metric components F, H, and N and the pressure p are continuous across the boundary. As far as derivatives of the metric are concerned H' and N' are continuous; F' may or may not be, according to the choice of coordinates.

III. THE DE DONDER GAUGE

In the previous section the mass formula was derived in a coordinate independent fashion. To solve the Einstein equations explicitly, of course, coordinate conditions must be imposed. Moreover, having solved the equations in a particular frame, the invariant size ϵ_c and hence the total mass may be expressed in terms of the coordinate dependent radius ϵ. The quantum-theory calculations will be performed in the harmonic gauge of de Donder. In these coordinates the metric satisfies the four Lorentz-covariant equations

$$[(-g)^{1/2}g^{\mu\nu}]_{,\,\nu} = 0 \tag{3.1}$$

in analogy with the Lorentz gauge condition $A^\mu{}_{,\,\mu} = 0$ in electrodynamics. (Here g denotes the determinant of the four-dimensional metric $g_{\mu\nu}$.) In contrast to the isotropic and Schwarzschild conditions, the above relation is a differential constraint which involves not only g_{ij} but g_{0i} and N also. Once again choosing time lines orthogonal to the three-dimensional hypersurface we have $g^{0i} = 0$ and in the static case the de Donder condition reduces to

$$[(-g)^{1/2}g^{ij}]_{,\,j} = 0\,. \tag{3.2}$$

The relevant quantities are

$$(-g)^{1/2} = \frac{FH^2N}{r^2}\,, \tag{3.3a}$$

$$g^{ij} = \frac{r^2}{H^2}\eta^{ij} + \left(\frac{1}{F^2} - \frac{r^2}{H^2}\right)\frac{x^i x^j}{r^2}\,, \tag{3.3b}$$

and Eq. (3.2) becomes

$$\left(\frac{H^2N}{F}\right)' = 2rFN. \tag{3.4}$$

It is worthwhile noting that this implies the continuity of F' so that in de Donder coordinates all the components of the metric *and their first derivatives* are continuous across the boundary of the sphere.[11] This is in contrast to the shell situation where there are severe problems with discontinuity.[4]

The exterior solutions of the Einstein equations subject to the constraint (3.4) are

$$H_+ = r + Gm\,, \quad F_+{}^2 = \frac{r + Gm}{r - Gm}\,, \quad N_+{}^2 = \frac{r - Gm}{r + Gm}\,, \tag{3.5}$$

where r is now the harmonic radial coordinate. The invariant size of the sphere ϵ_c is merely

$$\epsilon_c = H(\epsilon) = \epsilon + Gm\,. \tag{3.6}$$

So from Eq. (2.25)

$$m = m_0 - \frac{3}{5}\frac{Gm_0{}^2}{\epsilon} + O(G^2)\,. \tag{3.7}$$

An explicit form for the interior solutions is harder to find but to lowest order we have

$$H_- = r + \left(\frac{3}{2}\frac{r}{\epsilon} - \frac{1}{2}\frac{r^3}{\epsilon^3}\right)Gm\,, \tag{3.8a}$$

$$F_- = 1 + \left(\frac{3}{2}\frac{1}{\epsilon} - \frac{1}{2}\frac{r^2}{\epsilon^3}\right)Gm\,, \tag{3.8b}$$

$$N_- = \frac{3}{2}\left(1 - \frac{Gm}{\epsilon}\right) - \frac{1}{2}\left(1 - \frac{Gmr^2}{\epsilon^3}\right)\,, \tag{3.8c}$$

with

$$p(r) = \tfrac{1}{24}\kappa^2\rho^2(\epsilon^2 - r^2)\theta(\epsilon - r)\,. \tag{3.9}$$

Finally the metric in harmonic coordinates is exhibited in the form which we will attempt to reproduce from the quantum tree graphs.

$$g^{00} = -1 - \frac{2Gm}{r} - \frac{2G^2m^2}{r^2} + O(G^3)\,, \quad r > \epsilon \tag{3.10a}$$

$$g^{ij} = \left(1 - \frac{2Gm}{r} + \frac{3G^2m^2}{r^2}\right)\eta^{ij} - \frac{G^2m^2}{r^2}\frac{x^i x^j}{r^2} + O(G^3)\,, \quad r > \epsilon \tag{3.10b}$$

and

$$g^{00} = -1 - \frac{3Gm}{\epsilon} + \frac{Gmr^2}{\epsilon^3} + O(G^2)\,, \quad r < \epsilon \tag{3.11a}$$

$$g^{ij} = \left(1 - \frac{3Gm}{\epsilon} + \frac{Gmr^2}{\epsilon^3}\right)\eta^{ij} + O(G^2)\,, \quad r < \epsilon \tag{3.11b}$$

with

$g^{0i} = 0$ everywhere.

IV. THE QUANTUM THEORY

The action integral required for the perturbation expansion

$$A = \int d^4x\, \mathcal{L}(x) \tag{4.1}$$

will be written as the sum of three terms with

$$A = A_G + A_\Phi + A_J = \int d^4x[\mathfrak{L}_G(x) + \mathfrak{L}_\Phi(x) + \mathfrak{L}_J(x)]. \tag{4.2}$$

$\mathfrak{L}_G$ is the familiar Einstein Lagrangian for the gravitational field, and $\mathfrak{L}_J$ is the additional term describing the interaction of gravity with the matter source. The part $\mathfrak{L}_\Phi$, which breaks general covariance, is added to specify the "graviton" gauge. Firstly, we examine the explicit form for $\mathfrak{L}_G$, which may be expressed in terms of the tensor densities

$$\mathfrak{g}^{\mu\nu} = (-g)^{1/2} g^{\mu\nu}, \quad \mathfrak{g}_{\mu\nu} = (-g)^{-1/2} g_{\mu\nu}, \quad \mathfrak{g}^{\mu\rho}\mathfrak{g}_{\mu\sigma} = \delta^\rho_\sigma \tag{4.3}$$

in the very convenient form given by Goldberg.[12] It differs from[13] $\kappa^{-2}(-g)^{1/2}\mathfrak{R}$ by a total divergence and contains no derivatives of $\mathfrak{g}^{\mu\nu}$ higher than the first:

$$\mathfrak{L}_G = \frac{1}{8\kappa^2}(2\mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\lambda\iota}\mathfrak{g}_{\kappa\tau} - \mathfrak{g}^{\rho\sigma}\mathfrak{g}_{\iota\kappa}\mathfrak{g}_{\lambda\tau} - 4\delta^\sigma_\kappa\delta^\rho_\lambda\mathfrak{g}_{\iota\tau})\mathfrak{g}^{\iota\kappa}{}_{,\rho}\,\mathfrak{g}^{\lambda\tau}{}_{,\sigma}\,. \tag{4.4}$$

Our quantization procedure is now to set

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + \kappa\tilde{\phi}^{\mu\nu}, \quad \tilde{\phi}^{\mu\nu} = \tilde{\phi}^{\nu\mu}, \tag{4.5}$$

and to expand $\mathfrak{L}_G$ in powers of κ. Terms independent of κ represent the free Lagrangian while the remainder gives the interaction Lagrangian. Thus, we have

$$\mathfrak{L}_G(\tilde{\phi}) = \tilde{\mathfrak{L}}_G^{(0)} + \kappa\tilde{\mathfrak{L}}_G^{(1)} + \kappa^2\tilde{\mathfrak{L}}_G^{(2)} + \cdots, \tag{4.6}$$

$$\tilde{\mathfrak{L}}_G^{(0)} = \tfrac{1}{8}(2\eta^{\rho\sigma}\eta_{\lambda\iota}\eta_{\kappa\tau} - \eta^{\rho\sigma}\eta_{\iota\kappa}\eta_{\lambda\tau} - 4\delta^\sigma_\kappa\delta^\rho_\lambda\eta_{\iota\tau})\tilde{\phi}^{\iota\kappa}{}_{,\rho}\,\tilde{\phi}^{\lambda\tau}{}_{,\sigma}, \tag{4.7}$$

$$\tilde{\mathfrak{L}}_G^{(1)} = \tfrac{1}{8}(-4\eta^{\rho\sigma}\eta_{\lambda\iota}\eta_{\kappa\alpha}\eta_{\tau\beta} + 2\eta^{\rho\sigma}\eta_{\iota\kappa}\eta_{\lambda\alpha}\eta_{\tau\beta} + 2\delta^\rho_\alpha\delta^\sigma_\beta\eta_{\kappa\tau}\eta_{\lambda\iota} - \delta^\rho_\alpha\delta^\sigma_\beta\eta_{\iota\kappa}\eta_{\lambda\tau} + 4\delta^\rho_\lambda\delta^\sigma_\kappa\eta_{\iota\alpha}\eta_{\tau\beta})\tilde{\phi}^{\alpha\beta}\tilde{\phi}^{\iota\kappa}{}_{,\rho}\,\tilde{\phi}^{\lambda\tau}{}_{,\sigma}\,. \tag{4.8}$$

To implement the de Donder gauge condition

$$\mathfrak{g}^{\mu\nu}{}_{,\nu} = 0, \tag{4.9}$$

we add the noncovariant[14] term

$$\mathfrak{L}_\Phi = \frac{1}{2\kappa^2}\eta_{\mu\nu}\mathfrak{g}^{\mu\alpha}{}_{,\alpha}\,\mathfrak{g}^{\nu\beta}{}_{,\beta} = \tfrac{1}{2}\eta_{\mu\nu}\tilde{\phi}^{\mu\alpha}{}_{,\alpha}\,\tilde{\phi}^{\nu\beta}{}_{,\beta} \tag{4.10}$$

in accordance with rules given by Fradkin and Tyutin.[15] (Since we intend to restrict ourselves to tree graphs only, the difficulties of fictitious particles occuring in closed loops may be ignored.) By choosing the density $\mathfrak{g}^{\mu\nu}$ as the interpolating field, the additional $\mathfrak{L}_\Phi$ modifies only the free Lagrangian. The free propagator of $\tilde{\phi}^{\mu\nu}$ may easily be calculated from $\tilde{\mathfrak{L}}_G^0 + \mathfrak{L}_\Phi$. It is given by the time-ordered product

$$\langle 0|\, T\{\tilde{\phi}^{\mu\nu}(x)\tilde{\phi}^{\rho\sigma}(x')\}|\, 0\rangle = G^{\mu\nu\rho\sigma}(x - x'), \tag{4.11}$$

where, in momentum space,

$$G^{\mu\nu\rho\sigma}(k^2) = d^{\mu\nu\rho\sigma}\frac{1}{k^2}, \tag{4.12}$$

$$d^{\mu\nu\rho\sigma} \equiv \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}. \tag{4.13}$$

A knowledge is also required of the bare vertex function, $\tilde{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x^1, x^2, x^3)$ describing the 3-graviton interaction. This is most easily computed using DeWitt's[16] method of repeated functional differentiation of the Einstein action A_G:

$$\tilde{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x^1, x^2, x^3) = \frac{\delta^3 A_G}{\delta\mathfrak{g}^{\alpha_1\beta_1}(x^1)\delta\mathfrak{g}^{\alpha_2\beta_2}(x^2)\delta\mathfrak{g}^{\alpha_3\beta_3}(x^3)}\bigg|_{\mathfrak{g}^{\mu\nu}=\eta^{\mu\nu}}. \tag{4.14}$$

(A_Φ has a vanishing 3rd derivative since $\mathfrak{L}_\Phi$ is bilinear in the field $\mathfrak{g}^{\mu\nu}$.) Writing

$$\tilde{A}_G^{(1)} = \int d^4x\,\tilde{\mathfrak{L}}_G^{(1)}(x), \tag{4.15}$$

Eq. (4.14) may be written

$$\tilde{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} = \frac{\delta^3 \tilde{A}_G^{(1)}}{\delta\tilde{\phi}^{\alpha_1\beta_1}\delta\tilde{\phi}^{\alpha_2\beta_2}\delta\tilde{\phi}^{\alpha_3\beta_3}} \; . \tag{4.16}$$

Using Eq. (4.8) a straightforward calculation gives, in momentum space,

$$\begin{aligned}\tilde{\Gamma}_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(k_1, k_2, k_3) = -\,\mathrm{sym}\,P_6\,\tfrac{1}{8}(&-4\eta_{\alpha_3\alpha_2}\eta_{\beta_2\alpha_1}\eta_{\beta_3\beta_1}k_2\cdot k_3 + 2\eta_{\alpha_2\beta_2}\eta_{\alpha_3\alpha_1}\eta_{\beta_3\beta_1}k_2\cdot k_3 - \eta_{\alpha_2\beta_2}\eta_{\alpha_3\beta_3}k_{2\alpha_1}k_{3\beta_1}\\ &+ 2\eta_{\alpha_3\alpha_2}\eta_{\beta_2\beta_3}k_{2\alpha_1}k_{3\beta_1} + 4\eta_{\alpha_2\alpha_1}\eta_{\beta_3\beta_1}k_{2\alpha_3}k_{3\beta_2}) \; .\end{aligned} \tag{4.17}$$

The "sym" standing in front of this expression indicates that a symmetrization is to be carried out on each index pair $\alpha_1\beta_1$, $\alpha_2\beta_2$, and $\alpha_3\beta_3$. The symbol P_6 means that a summation is to be performed over all six permutations of the momentum index triplets $\alpha_1\beta_1k_1$, $\alpha_2\beta_2k_2$, $\alpha_3\beta_3k_3$. In the above equation we have omitted an over-all δ function expressing conservation of momentum.

So far, the density $\mathfrak{g}^{\mu\nu}$ has been chosen as the interpolating field rather than $g^{\mu\nu}$ because $\mathfrak{L}_\Phi$ and $\mathfrak{L}_G$ and hence the 3-point function of (4.17) are much simpler in this form.[17] In computing the VEV of the gravitational field, however, we prefer to use the more familiar $g^{\mu\nu}$ for reasons which will become clear later. Setting

$$g^{\mu\nu} = \eta^{\mu\nu} + \kappa\phi^{\mu\nu} \; , \tag{4.18}$$

the Einstein Lagrangian may again be expanded in a fashion similar to Eq. (4.6),

$$\mathfrak{L}_G(\phi) = \mathfrak{L}_G^{(0)} + \kappa\mathfrak{L}_G^{(1)} + \kappa^2\mathfrak{L}_G^{(2)} + \cdots \; . \tag{4.19}$$

However, the explicit forms for $\mathfrak{L}_G^0$ and $\mathfrak{L}_G^{(1)}$ are rather complicated and we shall not write them down. In the de Donder gauge (though not in general), the free propagator for $\phi^{\mu\nu}$ field is the same as that for $\tilde{\phi}^{\mu\nu}$. The higher-order vertex functions, however, are different.

We now turn to the rather delicate problem of choosing the source term $\mathfrak{L}_J$. First of all we define

$$J_{\mu\nu} \equiv (-g)^{1/2}T_{\mu\nu} \; , \tag{4.20}$$

where $T_{\mu\nu}$ is the energy-momentum tensor given in Eq. (2.27). If we now insert the interior form of $g^{\mu\nu}$ known from the classical theory [Eq. (3.11)], into the above equation, then to order κ^2, $J_{\mu\nu}$ is simply

$$J_{00} = \mu(r) \, , \quad J_{ij} = p(r)\eta_{ij} \; , \tag{4.21}$$

where μ and p are given by Eqs. (2.10) and (3.9). Next, we note that if the Einstein equations

$$\frac{1}{\kappa^2}(-g)^{1/2}G_{\mu\nu} + \tfrac{1}{2}J_{\mu\nu} = 0 \tag{4.22}$$

are to be obtained by functional differentiation of the action $(A_G + A_J)$, then we must have

$$\frac{\delta A_J}{\delta g^{\mu\nu}} = \tfrac{1}{2}J_{\mu\nu} \tag{4.23a}$$

since

$$\frac{\delta A_G}{\delta g^{\mu\nu}} = \frac{1}{\kappa^2}(-g)^{1/2}G_{\mu\nu} \; . \tag{4.23b}$$

Unfortunately, in gravity theory (as in all non-Abelian gauge theories), the introduction of a purely inert external source is complicated by the fact that the source itself depends on the field. The components of the matter tensor $T^{\mu\nu}$ are not all independent but satisfy the divergence condition

$$T^{\mu\nu}{}_{;\,\nu} = 0 \; . \tag{4.24}$$

However, by adding the noncovariant piece $\mathfrak{L}_\Phi$ to the Lagrangian the gauge symmetry (general covariance) is broken and the above constraint no longer holds. If we now choose A_J to be

$$A_J = \frac{1}{2}\int d^4x\, g^{\mu\nu}(x)J_{\mu\nu}(x) \tag{4.25}$$

and regard $J_{\mu\nu}$ as being a known classical function of x [Eq. (4.21)], and no longer a functional of the metric, then functional differentiation with respect to $g^{\mu\nu}$ yields the correct term in the Einstein equation (4.23a). We may now proceed to calculate the VEV of the gravitational field in the presence of the external classical source $J_{\mu\nu}$ in the usual way.

The S matrix is given by the Feynman-Dyson expression

$$S_J = T\exp\left(i\int d^4x[\mathfrak{L}_{\mathrm{int}}(x) + \mathfrak{L}_J(x)]\right) \; , \tag{4.26}$$

where $\mathfrak{L}_{\mathrm{int}}$ describes the self-interaction of the gravitational field and subscript J reminds us of the presence of the external source. The VEV of

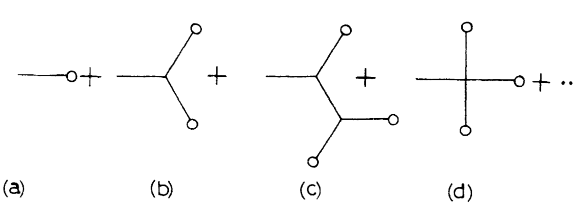

FIG. 1. Feynman diagrams for the VEV of the gravitational field in the presence of a c-number source (denoted by the circles). The closed loops have been ignored.

the gravitational field is then

$$\langle \hat{\phi}^{\mu\nu}(x)\rangle_J = \frac{\langle 0|\, T\{\phi^{\mu\nu}(x)S_J\}|\,0\rangle}{\langle 0|S_J|\,0\rangle} \ . \tag{4.27}$$

$\hat{\phi}^{\mu\nu}$ is the field operator in the Heisenberg picture and $\phi^{\mu\nu}$ the corresponding field operator in the interaction picture. $|\,0\rangle$ and $\langle 0|$ denote the unperturbed vacuum state. Ignoring closed-loop contributions, the usual Wick expansion to order κ^4 gives, in momentum space

$$\kappa\langle\hat{\phi}^{\mu_1\nu_1}(k_1)\rangle_J = \tfrac{1}{2}\kappa^2 G^{\mu_1\nu_1\alpha_1\beta_1}(k_1^{\,2})J_{\alpha_1\beta_1}(k_1) + \tfrac{1}{8}\kappa^4\int d^4k_2 d^4k_3 G^{\mu_1\nu_1\alpha_1\beta_1}(k_1^{\,2})G^{\mu_2\nu_2\alpha_2\beta_2}(k_2^{\,2})G^{\mu_3\nu_3\alpha_3\beta_3}(k_3^{\,2})$$
$$\times\delta^4(k_1+k_2+k_3)\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(k_1,k_2,k_3)J_{\mu_2\nu_2}(k_2)J_{\mu_3\nu_3}(k_3) \ . \tag{4.28}$$

These two terms may be represented by the Feynman diagrams of Figs. 1(a) and 1(b). The vertex function Γ is the 3-point function calculated using the field $\phi^{\mu\nu}$,

$$\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}(x^1,x^2,x^3) = \frac{\delta^3[A_G + A_\Phi]}{\delta g^{\alpha_1\beta_1}(x^1)\delta g^{\alpha_2\beta_2}(x^2)\delta g^{\alpha_3\beta_3}(x^3)}\bigg|_{g^{\mu\nu}=\eta^{\mu\nu}} \ . \tag{4.29}$$

It is to be distinguished from $\bar{\Gamma}$ of Eq. (4.17) which was obtained using $\hat{\phi}^{\mu\nu}$. From Eq. (4.12) we see that the above expression for the VEV requires a knowledge of

$$d^{\mu_1\nu_1\alpha_1\beta_1}d^{\mu_2\nu_2\alpha_2\beta_2}d^{\mu_3\nu_3\alpha_3\beta_3}\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3} \ . \tag{4.30}$$

Since Γ is much more complicated than $\bar{\Gamma}$, an evaluation of this quantity would involve considerable labor. Therefore, the following trick is employed. By repeated use of the relationship

$$\frac{\delta g^{\alpha\beta}(x)}{\delta \mathfrak{g}^{\mu\nu}(x')} = \tfrac{1}{2}(-g)^{-1/2}(\delta^\alpha_\mu\delta^\beta_\nu + \delta^\alpha_\nu\delta^\beta_\mu - \mathfrak{g}_{\mu\nu}\mathfrak{g}^{\alpha\beta})\delta(x,x') \ , \tag{4.31}$$

it is not difficult to show that

$$\tfrac{1}{8}d^{\mu_1\nu_1\alpha_1\beta_1}d^{\mu_2\nu_2\alpha_2\beta_2}d^{\mu_3\nu_3\alpha_3\beta_3}\Gamma_{\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3}$$
$$= \bar{\Gamma}^{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3} - \tfrac{1}{4}P_3(\delta^{\mu_1\nu_1\mu_2\nu_2}\eta^{\mu_3\nu_3} - \delta^{\mu_3\nu_3\mu_1\nu_1}\eta^{\mu_2\nu_2} - \delta^{\mu_2\nu_2\mu_3\nu_3}\eta^{\mu_1\nu_1} + \tfrac{1}{2}\eta^{\mu_1\nu_1}\eta^{\mu_2\nu_2}\eta^{\mu_3\nu_3})k_3^{\,2} \ , \tag{4.32}$$

where the symbol P_3 means sum over the three cyclic permutations of the momentum-index triplets $\mu_1\nu_1k_1$, $\mu_2\nu_2k_2$, and $\mu_3\nu_3k_3$, and also

$$\delta^{\mu\nu\rho\sigma} = \tfrac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}) \ . \tag{4.33}$$

Now the first term on the right of Eq. (4.32), $\bar{\Gamma}$, is known already and is relatively simple. So we see that by writing $\mathcal{L}_G$ and $\mathcal{L}_\Phi$ in terms of the density $\mathfrak{g}^{\mu\nu}$ it is possible to work out the quantities required in evaluating the VEV of $g^{\mu\nu}$ in a reasonably straightforward manner, and this was the reason for switching our choice of interpolating field from $\mathfrak{g}^{\mu\nu}$ to $g^{\mu\nu}$. In addition, a discussion of the source term $\mathcal{L}_J$ was facilitated by choosing $g^{\mu\nu}$.

After many diversions, we are finally in a position to work out the VEV. In addition to the invariant masses m and m_0, a third coordinate dependent mass will appear when we evaluate the integrals for $\langle\hat{\phi}^{\mu\nu}\rangle_J$. It is

$$\lambda \equiv \tfrac{4}{3}\pi\rho\epsilon^3 \tag{4.34}$$

and is related to m and m_0 by [Eqs. (2.14), (3.6), and (3.7)]

$$\lambda = m - 3\frac{Gm^2}{\epsilon} + O(G^2) = m_0 - \frac{18}{5}\frac{Gm_0^{\,2}}{\epsilon} + O(G^2) \ . \tag{4.35}$$

It will be instructive to define the quantity

$$V(x) = \tfrac{1}{4}\kappa^2\int d^4k\frac{e^{ikx}}{k^2}\mu(k) \ , \tag{4.36}$$

where

$$\mu(k) = \delta(k^0)\int d^3x e^{-i\vec{k}\cdot\vec{x}}\mu(\vec{x}) \ , \tag{4.37}$$

then V takes the form of a "potential"

$$V(r) = -G\lambda\left[\frac{1}{r}\theta(r-\epsilon) + \left(\frac{3}{2}\frac{1}{\epsilon} - \frac{1}{2}\frac{r^2}{\epsilon^3}\right)\theta(\epsilon - r)\right]. \tag{4.38}$$

If the explicit expressions for the propagators, vertex functions, and sources are substituted back into Eq. (4.28) for $\langle \phi^{\mu\nu}\rangle_J$, then Fourier-transforming back to x space, we come across integrals like

$$\tfrac{1}{16}\kappa^4 \int d\!\!{}^-{}^3k_1 d\!\!{}^-{}^3k_2 d\!\!{}^-{}^3k_3\, e^{i\vec{k}_1\cdot\vec{x}}\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{k_2^i k_2^j}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\mu(\vec{k}_2)\mu(\vec{k}_3) = \frac{1}{\nabla^2}(V\partial^i\partial^j V), \tag{4.39}$$

$$\tfrac{1}{16}\kappa^4 \int d\!\!{}^-{}^3k_1 d\!\!{}^-{}^3k_2 d\!\!{}^-{}^3k_3\, e^{i\vec{k}_1\cdot\vec{x}}\delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{k_2^i k_3^j}{\vec{k}_1^2\vec{k}_2^2\vec{k}_3^2}\mu(\vec{k}_2)\mu(\vec{k}_3) = \frac{1}{\nabla^2}(\partial^i V\partial^j V), \tag{4.40}$$

and

$$-\int d^3x' \frac{p(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|} = \frac{1}{\nabla^2}p\ , \tag{4.41}$$

where $1/\nabla^2$ denotes the inverse of the Laplacian operator. The resulting expressions for $\langle\hat{\phi}^{\mu\nu}\rangle_J$ may be simplified using momentum conservation. After much cancellation we obtain

$$\kappa\langle\hat{\phi}^{00}\rangle_J = 2V + \frac{3\kappa^2}{2}\frac{1}{\nabla^2}p - \frac{4}{\nabla^2}(\eta_{kl}\partial^k V\partial^l V) - \frac{8}{\nabla^2}(V\nabla^2 V)\ , \tag{4.42}$$

$$\kappa\langle\hat{\phi}^{ij}\rangle_J = \left(2V - \frac{\kappa^2}{2}\frac{1}{\nabla^2}p + \frac{4}{\nabla^2}(\eta_{kl}\partial^k V\partial^l V)\right)\eta^{ij} + \frac{4}{\nabla^2}(\partial^i V\partial^j V), \tag{4.43}$$

the other components vanishing.

We already have a hint of the divergence difficulties. Had a strictly pointlike source been chosen then $V\sim r^{-1}$ everywhere and quantities like

$$\frac{1}{\nabla^2}(V\nabla^2 V)$$

would be infinite.

Firstly we look at the zero-zero component of the VEV. Explicitly,

$$\frac{1}{\nabla^2}V\nabla^2 V = \left(\frac{6}{5r\epsilon}\right)G^2\lambda^2\theta(r-\epsilon) + \left(\frac{15}{8\epsilon^2} - \frac{3r^2}{4\epsilon^4} + \frac{3}{40}\frac{r^4}{\epsilon^6}\right)G^2\lambda^2\theta(\epsilon-r)\ , \tag{4.44}$$

$$\frac{1}{\nabla^2}\eta_{kl}\partial^k V\partial^l V = \left(\frac{1}{2r^2} - \frac{6}{5r\epsilon}\right)G^2\lambda^2\theta(r-\epsilon) + \left(-\frac{3}{4\epsilon^2} + \frac{r^4}{20\epsilon^6}\right)G^2\lambda^2\theta(\epsilon-r)\ , \tag{4.45}$$

$$\frac{\kappa^4}{4}\frac{1}{\nabla^2}p = \left(-\frac{1}{5r\epsilon}\right)G^2\lambda^2\theta(r-\epsilon) + \left(-\frac{3}{8\epsilon^2} + \frac{r^2}{4\epsilon^4} - \frac{3r^4}{40\epsilon^6}\right)G^2\lambda^2\theta(\epsilon-r)\ . \tag{4.46}$$

In accordance with the remarks made in the previous section one may check that all the above quantities as well as V are continuous at $r=\epsilon$ and so are their first derivatives. We now have to order G^2

$$\kappa\langle\hat{\phi}^{00}\rangle_J = -\frac{2G\lambda}{r} + \left(-\frac{2}{r^2} - \frac{6}{r\epsilon}\right)G^2\lambda^2 \quad (r>\epsilon) \tag{4.47a}$$

$$= \left(-\frac{3}{\epsilon} + \frac{r^2}{\epsilon^3}\right)G\lambda + \left(-\frac{57}{4\epsilon^2} + \frac{15r^2}{2\epsilon^4} - \frac{5r^4}{4\epsilon^6}\right)G^2\lambda^2 \quad (r<\epsilon)\ . \tag{4.47b}$$

In anticipation of the desired result we shall now replace $\langle\hat{\phi}^{\mu\nu}\rangle_J$ by the classical symbol $\phi^{\mu\nu}$. Using Eq. (4.35) to rewrite the exterior solution (4.47a) in terms of m_0 (which unlike m and λ does not depend on ϵ), we find

$$\kappa\phi^{00} = -\frac{2Gm_0}{r} + \frac{6G^2m_0^2}{5r\epsilon} - \frac{2G^2m_0^2}{r^2} + O(G^3), \quad r>\epsilon \tag{4.48}$$

and it is now clear that the limit $\epsilon\to 0$ leads to true divergences in the gravitational field *outside* the source. However, in terms of the total mass, m, the ϵ^{-1} pieces in the above equation cancel to yield

$$\kappa\phi^{00} = -\frac{2Gm}{r} - \frac{2G^2m^2}{r^2} + O(G^3), \quad r>\epsilon\ . \tag{4.49}$$

Since

$$g^{00} = -1 + \kappa\phi^{00}\ ,$$

we find, happily, that

$$g^{00} = -1 - \frac{2Gm}{r} - \frac{2G^2m^2}{r^2} + O(G^3) \quad (r>\epsilon) \tag{4.50a}$$

$$=-1-\frac{3Gm}{\epsilon}+\frac{Gmr^2}{\epsilon^3}$$
$$+\left(-\frac{21}{4\epsilon^2}+\frac{9r^2}{2\epsilon^4}-\frac{5r^4}{4\epsilon^6}\right)G^2m^2+O(G^3)\quad(r<\epsilon)\ , \tag{4.50b}$$

which are precisely the classical Schwarzschild solutions given in (3.10) and (3.11).

The computation of ϕ^{ij} proceeds in a similar fashion. To avoid too much algebra we quote only the exterior solution

$$\frac{1}{\nabla^2}(\partial^i V\partial^j V)=\left[\left(\frac{1}{4r^2}-\frac{2}{5r\epsilon}\right)\eta^{ij}-\frac{x^ix^j}{4r^4}\right]G^2\lambda^2\ ,\quad r>\epsilon \tag{4.51}$$

and

$$\kappa\phi^{ij}=\left(-\frac{2G\lambda}{r}+\frac{3G^2\lambda^2}{r^2}-\frac{6G^2\lambda^2}{r\epsilon}\right)\eta^{ij}-G^2\lambda^2\frac{x^ix^j}{r^4}+O(G^3)\ , \tag{4.52}$$

so that finally, in terms of m, the ϵ^{-1} terms again cancel to give

$$g^{ij}=\left(1-\frac{2Gm}{r}+\frac{3G^2m^2}{r^2}\right)\eta^{ij}-\frac{G^2m^2}{r^2}\frac{x^ix^j}{r^2}+O(G^3)\ , \tag{4.53}$$

which is just the classical equation (3.10b).

V. DISCUSSION

By using an extended source to avoid divergences, we have seen how, at least to order κ^4, the quantum theory correctly reproduces the classical Schwarzschild solution in the $\hbar\to 0$ limit, and how the terms ϵ^{-1}, ϵ^{-2}, etc., conspire to effect a mass renormalization consistent with that obtained from purely classical reasoning. There is every reason to believe that this correspondence holds to all orders of perturbation theory, independent of the choice of gauge or interpolating field.

Having gained an insight into the problem by the explicit evaluation of the Feynman diagrams, one is perhaps in a better position to attempt some sort of qualitative $\hbar$ corrections by including the closed loops. However, this is a formidable calculational task. Furthermore, these extra terms contain all the usual infinities of perturbation theory and would have to be renormalized. An alternative approach is possible using the superpropagator techniques of nonpolynomial Lagrangians and one might expect to achieve finite results in this way. The problem of gravitational collapse springs to mind as one arena where such $\hbar$ corrections may find applicability. As Wheeler[18] has emphasized, quantum effects must change the complexion of the problem in regions of high curvature. In this connection we may note that although we chose a static perfect fluid as the most convenient source for our calculations, it is not necessary to do so. A quantum treatment of collapsing matter, however, would necessarily be more difficult.

Finally, it is an amusing fact that there exists one coordinate frame in which each component of the exterior Schwarzschild metric, when expanded in powers of G, terminates at first order. This is the metric given by Eddington,

$$ds^2=dr^2+r^2d\Omega-dt^2+\frac{2Gm}{r}(dr+dt)^2\ .$$

If it were possible to perform the quantum calculations in Eddington (rather than de Donder) coordinates, one would no doubt find that the sum to all orders of the tree diagrams could be represented by the single-graviton-exchange graph of Fig. 1(a), with an effective source in which the bare mass is replaced by the already renormalized mass. This single graph would then describe the gravitational field everywhere outside the source and not merely its asymptotic behavior.

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*Present address: International Centre for Theoretical Physics, Trieste, Italy.

[1]See, for example, David G. Boulware and Lowell S. Brown, Phys. Rev. 172, 1628 (1968), where this is demonstrated (at least in the case where no infinite-dimensional gauge group is present in the theory).

[2]R. Arnowitt, S. Deser, and C. W. Misner, Ann. Phys. (N.Y.) 33, 88 (1965).

[3]Greek indices run over 0, 1, 2, 3 and Latin over 1, 2, 3. $\eta^{\mu\nu}$ denotes the Minkowski metric with signature (−1, 1, 1, 1). Three-dimensional quantities are sometimes marked with a prefix 3, unless the meaning is clear. Ordinary differentiation is denoted by a comma, and covariant differentiation by a semicolon. The partial derivatives with respect to r and t are symbolized by primes and dots, respectively. Natural units are used with $\hbar=c=1$ and $\kappa^2=16\pi G$, where G is the Newtonian gravitational constant.

[4]M. J. Duff, ICTP Report No. ICTP/71/19 (unpublished).

[5]R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962), Chap. 7.

[6]The function $H(r)$ must vanish at $r=0$ in order that the derivative of the metric be finite at the origin.

[7]So that the exterior solution is Schwarzschild. More rigorously this quantity must equal the total energy of the system $\int (g_{ij,j} - g_{jj,i}) dS_i$, where dS_i is a two-dimensional surface element at spatial infinity (see Ref. 5).

[8]R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. 120, 313 (1960).

[9]The analogous equation for the shell distribution is exactly the Newtonian expression, namely, $m = m_0 - Gm_0^2/2\epsilon_c$. It is curious that this mass formula was first derived by ADM in isotropic coordinates for which $\epsilon_c = H(\epsilon) = (1 + Gm/2\epsilon)^2\epsilon$, in which case $m = m_0 - Gm^2/2\epsilon$, i.e., in terms of ϵ, rather than $H(\epsilon)$, it is the total mass, m, which appears in the self-energy contribution and not the bare mass, m_0. The appearance of the total mass prompted ADM to interpret this effect as being due to the equivalence principle in general relativity (Ref. 8). However, this phenomenon is merely a fluke of the isotropic frame and does not appear to have any physical significance (Ref. 4).

[10]Provided $\epsilon_c{}^2 < 8R^2/9$, i.e., provided p remains finite.

[11]This is also true in isotropic but not Schwarzschild, coordinates. Nathan Rosen, Ann. Phys. (N.Y.) 63, 127 (1970).

[12]J. N. Goldberg, Phys. Rev. 111, 315 (1958).

[13]$\mathcal{R}$ is the usual Riemann curvature scalar. The symbol ${}^3\mathcal{R}$ is reserved for the curvature of the 3-space.

[14]By using the term "noncovariant" we mean that $\mathcal{L}_\Phi$ breaks general covariance but not Lorentz covariance.

[15]E. S. Fradkin and I. V. Tyutin, Phys. Rev. D 2, 2841 (1970).

[16]Bryce S. DeWitt, Phys. Rev. 162, 1239 (1967).

[17]Compare the five terms of Eq. (4.17) with the eleven terms of Eq. (2.6) of Ref. 15.

[18]See B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (University of Chicago Press, Chicago, Ill., 1965).

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Experimental Test of Weyl's Gauge-Invariant Geometry*

E. G. Harris, Paul G. Huray, F. E. Obenshain, and J. O. Thomson
The University of Tennessee, Knoxville, Tennessee 37916
and Oak Ridge National Laboratory, Oak Ridge, Tennessee 37830

R. A. Villecco†
University of Tennessee, Knoxville, Tennessee 37916

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Weyl's gauge-invariant geometry predicts that the periods of clocks are affected by their previous history in electromagnetic fields. We have used the Mössbauer effect to search for such an effect with negative results.

I. INTRODUCTION

The first attempt to formulate a unified field theory in which both gravitation and electromagnetism were incorporated into a single geometrical structure of a space-time manifold was made by Weyl[1,2] in 1918. In spite of some attractive features this theory was abandoned by its author and others within a few years. As will be discussed presently, at least one of the reasons for this rejection might not seem as compelling in the light of some recent astronomical observations.[3] These observations led us to reconsider Weyl's theory and to subject it to a more stringent experimental test as reported here.

Einstein[4] pointed out in 1918 that a consequence of Weyl's theory would be that the frequencies of spectral lines emitted by atoms would depend on the electromagnetic history of the atoms. Since all atoms of an element in the universe seem to emit the same frequencies the theory seemed to be contradicted by the facts.

However, observations indicate that atoms in distant galaxies have their lines shifted toward the red end of the spectrum[3] as compared with terrestrial spectra. This shift is usually attributed to the Doppler effect and accepted as evidence for expansion of the universe. Some recent astronomical observations of red shifts associated with quasistellar objects as large as $\Delta\nu/\nu \sim 1$ have made this interpretation difficult to maintain.[3] Arp[5] and Burbidge[6] have suggested that it may be necessary to revise our physical theories to account for these observations, and steps in this di-