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Model Equation of State for Strongly Interacting Superdense Matter

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The equation of state for a zero-temperature system of strongly interacting baryons is calculated. Interactions, described by the exchange of a pseudoscalar meson, are treated relativistically to lowest order in the coupling constant. The finite density of the ground state is included through a fully relativistic many-body theory discussed in a previous paper. The results, which are applicable to a single-component system of baryons for densities 10^{13} g/cm$^3 < \rho$, lead to an upper limit for the speed of sound $v_s \leq c/\sqrt{3}$. Simple analytic expressions are given which fit the results of numerical computation in the density range 10^{13} g/cm$^3 \lesssim \rho \lesssim 10^{23}$ g/cm^3.

I. INTRODUCTION

A steadily growing interest in the properties of superdense matter[1] (with densities > nuclear density $\sim 2.5\times10^{14}$ g/cm^3) and matter at relativistic temperatures[2] ($T > 6\times10^9$ °K) has led to the development of a relativistic many-body theory based upon elementary particles and their interactions.[3] Systems at relativistic temperatures arise primarily in the study of the early stages of some cosmological theories, and will not concern us at this time. The interest in superdense matter arises primarily in the study of the final stages of stellar evolution, although it may be of importance in the study of stellar birth as envisioned by Ambartsumyan. The problem of superdense matter is basically a problem of relativistic many-body theory for strongly interacting systems of baryons in their ground state (at zero temperature).

In a previous paper[4] the relativistic many-body theory of strongly interacting matter in flat spacetime was developed. At this time we present numerical results based on that work. In particular, we give an analytic expression for the pressure, energy density, and resulting equation of state which closely approximates the numerical results that have been obtained. A significant feature of the result – important because of its implications for the study of relativistic interacting matter in general – is the asymptotic limit to the speed of sound

$$v_s \leq c/\sqrt{3}\ , \tag{1.1}$$

which follows from our equation of state.

Following a brief review of the material discussed earlier, we describe the calculation of the equation of state. In Sec. II the density-dependent self-energy (to lowest order in the coupling constant) is reduced to a form suitable for rapid numerical integration, and the baryon two-point function obtained by inverting (1.6) below. The details of the numerical calculation are given in Sec. III, and an analytic expression for the poles of the Green's function at the Fermi surface discussed. The latter, while extremely simple in form, is a good approximation to our results over the density range $10^{13} < \rho < 10^{23}$ g/cm^3, and leads to simple analytic equations for the pressure and energy density in Sec. IV. The pressure and energy density represent the equation of state in parametrized form, and yield a relativistic expression for the speed of sound. In Sec. V we examine the nonrelativistic limit of our analytic results.

The system which we have investigated, as described in the previous paper, consists of N baryons of number density

$$n_0 = \frac{p_F^3}{3\pi^2}, \tag{1.2}$$

interacting by the exchange of pseudoscalar mesons according to the coupling[5]

$$\mathcal{L}_{\text{int}} = ig_0\bar{\psi}(x)\gamma^5\psi(x)\phi(x). \tag{1.3}$$

There it was shown that the relativistic pressure and energy density of the interacting system could be written in terms of the fermion two-point function $G_F(p, q_F)$ as follows:

$$P(p_F) = -i\int_0^{\mu(p_F)} d\mu\,(q_F) \times \int \frac{d^4p}{(2\pi)^4} 2\cos p_0 t\,\mathrm{tr}[\gamma^0 G_F(p, q_F)], \tag{1.4}$$

$$\rho = mn_0 + 9\pi^2 n_0 \int_0^{p_F} \frac{P(q_F)dq_F}{q_F^4}. \tag{1.5}$$

The two-point function $G_F(p, q_F)$ is obtained by inverting the equation

$$[\not{p} - m - \Sigma_c^R(p, q_F)]G_F(p, q_F) = 1, \tag{1.6}$$

where $\Sigma_c^R(p, q_F)$ is the regularized self-energy obtained from the expression [see Fig. 1(a)]

$$\Sigma_c(p, q_F) = ig^2 \int \frac{d^4k}{(2\pi)^4} \gamma^5 S_F(p-k; q_F)\gamma_5 \Delta_F(k), \tag{1.7}$$

the density-dependent noninteracting two-point function[6] $S_F(p-k; q_F)$, and the usual spin-zero boson propagator $\Delta_F(k)$. Regularization was performed by renormalizing the two-point functions to the physical masses and charge. For a more complete discussion of these matters the reader is referred to the previous publication.[4]

It will be observed that the solution of (1.6) for the corrected fermion propagator is equivalent to summing an infinite number of self-energy diagrams as shown in Fig. 1(b). This procedure is crucial to obtaining a finite result since, if the expression equivalent to (1.6),

$$G = S_F + S_F \Sigma G$$

is iterated and only terms to finite order retained, the pressure obtained from (1.4) will, in the asymptotic limit $p_F/m \to \infty$, have the form of a divergent power series in (p_F/m).

In concluding this section we recall the expression for the renormalized self-energy discussed previously[4]

$$\Sigma_c^R(p, q_F) = \Sigma_c(p, q_F) - \Sigma_c(p, 0)\big|_{\not{p}=m} - (\not{p} - m)\frac{d\Sigma_c(p, 0)}{d\not{p}}\bigg|_{\not{p}=m}, \tag{1.8}$$

and note that it, as well as the Green's functions appearing in (1.4–1.6), is in fact a 4×4 matrix expression whose indices have not been explicitly exhibited. These matters are fully discussed elsewhere.[3,4]

II. SELF-ENERGY

To find the lowest-order correction to the pressure for a system of strongly interacting baryons, we solve (1.6) for $G_F(p, q_F)$ in terms of the self-energy (1.7) and perform the integrals in (1.4). The self-energy reduces, to lowest order, to a sum of two terms, the first of which contains all of the density effects. In this section the density-dependent part will be written in a form suitable for numerical integration (the details will be relegated to Appendix A). The density-independent part will be regularized by the procedure outlined in the previous paper.[4]

The second-order self-energy is obtained from (1.7), the two-point functions $S_F(p-k; q_F)$, and $\Delta_F(k)$:

$$\Sigma_c(p, q_F) = ig_0^2 \int \frac{d^4k}{(2\pi)^4}\gamma_5 \frac{\not{p} - \not{k} + m_0}{2E_{\vec{p}-\vec{k}}}\gamma^5 \times\left(\frac{1-n_F(p-k)}{p_0 - k_0 - E_{\vec{p}-\vec{k}} + i\epsilon} + \frac{n_F(p-k)}{p_0 - k_0 - E_{\vec{p}-\vec{k}} - i\epsilon} - \frac{1}{p_0 - k_0 + E_{\vec{p}-\vec{k}} - i\epsilon}\right)\frac{1}{k^2 - \mu_0^2 + i\epsilon}. \tag{2.1a}$$

where $E_{\vec{p}-\vec{k}} = (|\vec{p}-\vec{k}|^2 + m_0^2)^{1/2}$ is the free-particle energy. The distribution functions are given by

$$n(p-k) = \begin{cases} 1 \text{ if } |\vec{p}-\vec{k}| < q_F \\ 0 \text{ if } |\vec{p}-\vec{k}| > q_F \end{cases} \tag{2.1b}$$

where the variable Fermi momentum q_F is in the range $0 \le q_F \le p_F$, and p_F is fixed by the number density of particles (1.2). The unrenormalized masses and charge are denoted by m_0, μ_0, and g_0. By hypothesis[7] the ground state is devoid of antiparticles and

$$\bar{n}_F(p-k) \equiv 0.$$

Examination of (2.1) shows that the self-energy

may be written as

$$\Sigma_c(p, q_F) = \Sigma_E(p^2) + \Sigma_M(p, q_F), \tag{2.2}$$

where the first term

$$\Sigma_E(p^2) = ig_0^2 \int \frac{d^4k}{(2\pi)^4} \gamma_5 \frac{\not p - \not k + m_0}{(p-k)^2 - m_0^2 + i\epsilon} \times \gamma^5 \frac{1}{k^2 - \mu_0^2 + i\epsilon} \tag{2.3}$$

is just the zero-density self-energy arising in relativistic quantum field theory to lowest order in the interaction (2.3). The second term

$$\Sigma_M(p, q_F) = ig_0^2 \int \frac{d^4k}{(2\pi)^4} \gamma_5 \frac{\not p - \not k + m_0}{k^2 - \mu_0^2 + i\epsilon} \times \gamma^5 \frac{n_F(p-k)}{2E_{\vec p - \vec k}} 2\pi i \delta(p_0 - k_0 - E_{\vec p - \vec k}) \tag{2.4}$$

contains the entire density dependence of the system to second order. It will be observed that although (2.3) diverges as it stands, the density-dependent part is finite and well behaved. This is a consequence of the factor $n_F(p-k)$, which at zero temperature limits the range of integration to values of the virtual momentum k such that

$$|\vec p - \vec k| \leq q_F, \tag{2.5}$$

as shown in Fig. 2. By (2.2),

$$q_F \leq (3\pi^2 n_0)^{1/3}, \tag{2.6}$$

so that the upper limit is finite, and is determined naturally by the density of baryons in the system. There is no difficulty of principle involved in the evaluation of the integrals occurring in the density-dependent part of the self-energy (2.4). We shall return to it in Sec. III. The situation at finite temperature, though numerically more difficult, is basically unchanged. To show this we consider the integrand of the density-dependent term (2.4). For a given temperature and baryon density, a simple power count shows that at large virtual momenta the integrand goes as

$$\exp\{-\beta[(|\vec p - \vec k|^2 + m^2)^{1/2} - \mu]\} dk,$$

where μ is independent of k. For large k this is $\sim\exp(-\beta k)dk$, and the integral is convergent. Its value in this case is determined by the temperature as well as the density of baryons in the system.

The divergent term (2.3) may be regularized by the arguments of Sec. I. Demanding that the singularity at zero density correspond to the physical mass m, we use (1.8) which yields, to second order, the regularized self-energy

$$\Sigma_c^R(p, q_F) = \Sigma_M(p, q_F) + \Sigma^R(p), \tag{2.7}$$

where

$$\Sigma^R(p) = \Sigma_E(p) - \Sigma_E(p)\big|_{\not p = m} - (\not p - m) \frac{d\Sigma_E(p)}{d\not p}\bigg|_{\not p = m}. \tag{2.8}$$

In arriving at (2.7) use has been made of the relations $\Sigma_M(p, 0) = 0$. The notation $\not p = m$ implies that the quantity is to be evaluated on physical mass shell. It is evident that (2.8) is just the regularized elementary particle self-energy.

Examination of the γ-matrix dependence of (2.3)–(2.4) shows that they may be written in the

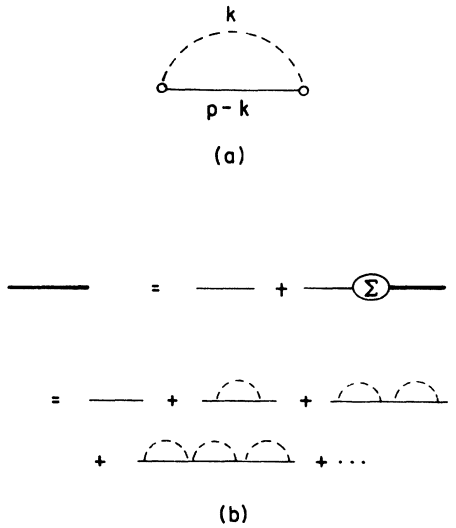

FIG. 1. Self-energy contribution to the pressure: (a) the irreducible second order self-energy; (b) the corrected baryon two-point function, which contains the self-energy (a) to infinite order.

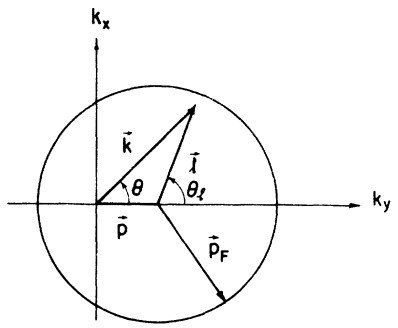

FIG. 2. The distribution function $n_F(\vec p - \vec k)$ limits the integration in (2.4) to those values of $\vec k$ lying within the Fermi sphere at $\vec p$ of radius p_F. The transformation to integration variable $l^\mu = p^\mu - k^\mu$ reduces the integral to one over the sphere centered at the origin. The $\vec l$ integral is then over $0 < l < p_F$, $0 < \theta_l < \pi$, and $0 < \phi_l < 2\pi$.

matrix form (matrix indices $\alpha, \beta = 0, 1, 2, 3$ are explicitly shown):

$$\Sigma_M(p, q_F)_{\alpha\beta} = \gamma^\mu_{\alpha\beta}\Sigma_\mu(p, q_F) + \delta_{\alpha\beta}\Sigma_I(p, q_F), \quad (2.9)$$

$$\Sigma^R(p)_{\alpha\beta} = m\delta_{\alpha\beta}S_1(p^2) + p_\mu\gamma^\mu_{\alpha\beta}S_2(p^2). \quad (2.10)$$

The functions Σ_μ and Σ_I are discussed in Appendix A, and S_1, S_2 are discussed in Appendix B. The integrals appearing in $S_1(p^2)$ and $S_2(p^2)$ are well defined and depend only on the magnitude of the 4-momentum p^μ.

Substitute (2.9)–(2.10) into (2.7), and the result into (2.6) to obtain the inverse Green's function

$$G_F(p, q_F)^{-1}{}_{\alpha\beta} = \gamma^\mu_{\alpha\beta}\{p_\mu[1 - S_2(p^2)] - \Sigma_\mu(p, q_F)\} - \{m[1 - S_1(p^2)] + \Sigma_I(p, q_F)\}\delta_{\alpha\beta}. \quad (2.11)$$

Equation (2.11) may be easily inverted by using the anticommutation relations for the matrices $\gamma^\mu_{\alpha\beta}$, and the identity $\not{a}\not{a} = 2a^\mu a_\mu$. Expressing p in polar coordinates (see Appendix A), and defining

$$\Sigma_i(p, q_F) = \Sigma_v(p, q_F)\begin{cases}\sin\theta\cos\phi, & i=1\\ \sin\theta\sin\phi, & i=2\\ \cos\theta, & i=3\end{cases}$$

leads to the following expression for the Green's function:

$$G(p, q_F)_{\alpha\beta} = \frac{N(p, q_F)_{\alpha\beta}}{D(p, q_F)}, \quad (2.12)$$

where we have set

$$N(p, q_F)_{\alpha\beta} = \gamma^\mu_{\alpha\beta}\{p_\mu[1 - S_2(p^2)] - \Sigma_\mu(p, q_F)\} + \{m[1 - S_1(p^2)] + \Sigma_I(p, q_F)\}\delta_{\alpha\beta}, \quad (2.13)$$

$$D(p, q_F) = \{p_\mu[1 - S_2(p^2)] - \Sigma_\mu(p, q_F)\}^2 - \{m[1 - S_1(p^2)] + \Sigma_I(p, q_F)\}^2$$
$$= \{p_0[1 - S_2(p^2)] - \Sigma_0(p, q_F)\}^2 - \{p[1 - S_2(p^2)] - \Sigma_v(p, q_F)\}^2 - \{m[1 - S_1(p^2)] + \Sigma_I(p, q_F)\}^2. \quad (2.14)$$

The expression (2.12) for $G_F(p, q_F)$ is correct as it stands as long as the self-energy possesses a nonzero imaginary part. For the system under consideration[7] $\Sigma_c(p, q_F)$ is real, and the net effect of the interaction is to shift the energy of the elementary excitations. In particular, the excitations remain stable (infinite lifetime). We have assumed that the particles making up the system are stable in the absence of many-body interactions; this rules out, for example, the nuclear decay $N \to \Sigma + e + \nu$ associated with the weak interactions. The stability of the elementary excitations (which result from interactions between the infinite-lifetime baryons) is a consequence of the Hartree-Fock approximation. If the calculations were carried out to higher order, by including the self-energy contributions due to the Feynman diagrams in Fig. 3 [subject to constraints (i)–(vi) in Sec. IV of the previous paper], then the elementary excitations, though still describing the interactions between stable baryons, would have finite lifetime.

In the absence of an imaginary part to $\Sigma_c(p, q_F)$, it is necessary to specify the behavior of (2.12) in the vicinity of the poles defined by the zeros of (2.14). This is done as in the free-particle case.[8] First denote the roots of $D(p_0, p, q_F)$ by $\mathcal{E}_1(p, q_F)$ and $-\mathcal{E}_2(p, q_F)$, where the first is positive and the second negative. Only two roots occur since the interaction preserves the spin degeneracy. We therefore write

$$D(p, q_F) = [p_0 - \mathcal{E}_1(p, q_F)][p_0 + \mathcal{E}_2(p, q_F)], \quad (2.15)$$

where $\mathcal{E}_1(p, q_F)$ and $\mathcal{E}_2(p, q_F)$ are defined by the requirement that $D(\mathcal{E}_i, p, q_F) = 0$ for $i = 1, 2$. To lowest order the system of N baryons plus interactions may be replaced[9] by a system of N noninteracting elementary excitations of energies $\mathcal{E}_1(p, q_F)$ and $\mathcal{E}_2(p, q_F)$. Consequently, the Green's function (2.12) will be rewritten in the form,[10] with E_p replaced by $\mathcal{E}_1(p, q_F)$ or $-\mathcal{E}_2(p, q_F)$,

$$G_F(p, q_F) = \frac{\not{p} + m}{\mathcal{E}_1 + \mathcal{E}_2}\left(\frac{1 - n_F(p)}{p_0 - \mathcal{E}_1(p, q_F) + i\epsilon} + \frac{n_F(p)}{p_0 - \mathcal{E}_1(p, q_F) - i\epsilon} - \frac{1}{p_0 + \mathcal{E}_2(p, q_F) - i\epsilon}\right). \quad (2.16)$$

In writing (2.16) we have emphasized that the excitation energies depend on the density. They will also depend on the coupling constant g_0, as will be seen in later sections. Equation (2.16) contains boundary conditions suitable to the assumed ground state of the system. The parallel between this Green's function and that describing a single particle in a noninteracting medium would be closer if $\mathcal{E}_1(p, q_F) = \mathcal{E}_2(p, q_F)$. Examination of the equation $D(p) = 0$ suggests that this is the case, as will be discussed in Sec. III. We therefore set

$$\mathcal{E}_1(p, q_F) = \mathcal{E}_2(p, q_F).$$

As a result, (2.16) is formally identical to the Green's function for a noninteracting medium. The net ef-

fect of interactions appears in the shift in the position of the poles:

$$G_F(p, q_F) = \frac{\not{p}+m}{2\mathcal{E}_1(p, q_F)}\left[\frac{1-n_F(p)}{p_0-\mathcal{E}_1(p, q_F)+i\epsilon} + \frac{n_F(p)}{p_0-\mathcal{E}_1(p, q_F)-i\epsilon} - \frac{1}{p_0+\mathcal{E}_1(p, q_F)-i\epsilon}\right].$$

The relativistic pressure is found in the following manner:

(1) Evaluate $S_1(p^2)$, $S_2(p^2)$, $\Sigma_0(p, q_F)$, $\Sigma_v(p, q_F)$, and $\Sigma_I(p, q_F)$ numerically, and find the roots $\mathcal{E}_1(p, q_F)$ and $-\mathcal{E}_1(p, q_F)$ of the equation $D(p_0, p, q_F)=0$ – Sec. III;

(2) Substitue the roots found in (1) into the expression (2.16) for $G_F(p, q_F)$, and find the chemical potential $\mu(q_F)$ – Sec. IV;

(3) Evaluate (2.4) for the pressure, (2.5) for the energy density, and obtain the equation of state – Sec. IV.

This problem will be facilitated by the introduction of an analytic expression in Sec. III, which serves as a good approximation to our numerical results for the chemical potential. The analytic expression covers the matter density range $10^{13}<\rho<10^{23}$ g/cm^3.

III. NUMERICAL COMPUTATION

To determine the equation of state, we need to evaluate the separate equations (1.4) and (1.5) for the pressure and energy density. This involves us in a contour integration where $G_F(p, q_F)$ is the only singular factor of the integrand. Therefore the first step in the calculation is to find the singularities, which are the zeros of $D(p, q_F)$ in (2.14).

The general search for the zeros of (2.14) is a difficult job, because we begin in such a case from an expression in three variables: p_0, p, and q_F. However, there is one immediate simplification. As we are interested in pressure and energy density, we need information only at the Fermi surface, i.e., at $p=q_F$. The simplified exercise of detecting the zeros of D determines p_0 as a function of q_F, which therefore gives us directly the chemical potential μ necessary to complete the evaluation of the double integral in (2.4).

Numerical computation of S_1, S_2, Σ_0, Σ_v, and Σ_I, followed by evaluation of the entire expression (2.14) for D, is the only practical way to find the poles of G_F. This is still a laborious search in (p_0, q_F) space, but the procedure is helped by the

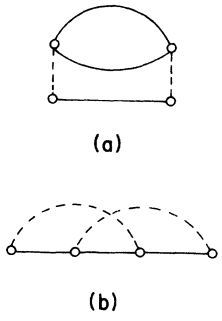

FIG. 3. Fourth order contributions to the self-energy. Inclusion of such terms would lead, among other things, to a finite lifetime for the elementary excitations in the system.

educated first guess that the two parameters may be (asympotically in q_F) linearly related, e.g.,

$$p_0 = aq_F + bm, \tag{3.1}$$

where a and b are dimensionless, and m is the nuclear mass. The same educated guess suggests that $a=1$ by analogy with the noninteracting case, but the suggestion is misleading. The linear equation (3.1) has been helpful to identify approximately and quickly the paths of $D=0$ in (p_0, q_F) space, but with $a\approx\frac{5}{3}$ for the zero corresponding to $\mathcal{E}_1$ in (2.15). If the strong-coupling constant g is varied in the computation, we find that a varies smoothly according to the relation

$$a \approx 1 + \frac{1}{22}\frac{g^2}{4\pi}. \tag{3.2}$$

In the absence of interactions, the magnitudes $\mathcal{E}_1$ and $\mathcal{E}_2$ of the roots of (2.15) are equal. Finite-density effects turn (2.15) from a simple quadratic in p_0 into a complicated transcendental equation for p_0 when $D=0$, but we find numerically that $\mathcal{E}_1 = \mathcal{E}_2$ within $1-\frac{1}{2}$ percent over the range for which (3.3) is defined below, despite rather larger changes in the sizes of individual squares in the expression for D when the sign of p_0 is changed. The differences between $\mathcal{E}_1$ and $\mathcal{E}_2$ are not systematic, except that they decrease for large q_F. Although numerical analysis alone can never give exact answers about analytic properties of such expressions we are confident that $\mathcal{E}_1$ and $\mathcal{E}_2$ may still be regarded as equal here, i.e., that every relation of the form $p_0=f(q_F)$ implies another of the form $-p_0=f(q_F)$.

We have carried out the computation for values of q_F/m between 0.01 and 100. Equation (3.1) expresses the desired condition for $D=0$ well, provided that $b=-2$ and that a is given by (3.2) (i.e., $a=\frac{5}{3}$ in practice). It fits best at the extremes of the range for q_F, but a better over-all fit involves a small exponential correction for the low end, i.e.,

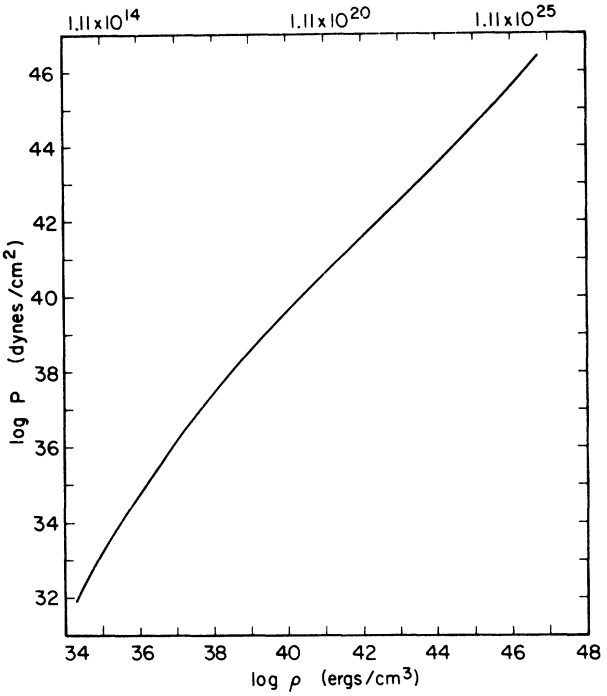

FIG. 4. Equation of state for the density range 10^{13} g/cm^3 $< \rho < 10^{26}$ g/cm^3, as calculated from the relativistic many-body Green's function (2.17). The slope of the curve $\log_{10} P$ versus $\log_{10} \rho$ approaches unity asymptotically. Numbers along the top horizontal scale correspond to the mass density in g/cm^3.

$$p_0 = \tfrac{5}{3} q_F - 2m + cm \exp(-\alpha q_F/m), \tag{3.3}$$

where $c = 3$ and $\alpha = \frac{5}{9}$. The form (3.3) still underestimates p_0 symmetrically over a small range of values of q_F near m, but we bring (3.3) into line with the results of the numerical computation by multiplying by an extra factor:

$$p_0 = \tfrac{1}{40}\left[\tfrac{5}{3} q_F - 2m + 3m \exp(-5q_F/9m)\right] \times \left[41 \exp\left(2 - \frac{q_F}{m} - \frac{m}{q_F}\right) + 39\right]. \tag{3.4}$$

Several words of explanation about (3.3) and (3.4) are in order. Firstly, it is customary[11] to write computed equations of state and their preliminaries with floating-point numerical coefficients. We could have done the same above, because (3.3) and (3.4) express fits to arrays of floating-point numbers for p_0, but we have an aesthetic preference[12] for rational approximations.

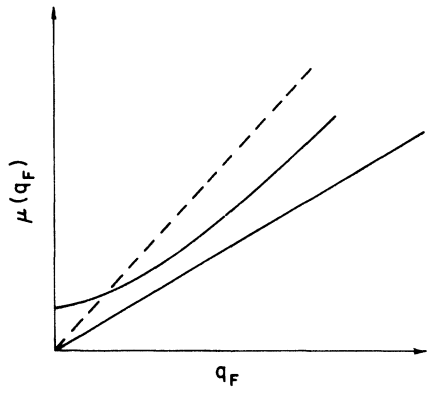

FIG. 5. Schematic plot of the chemical potential. The dashed line has slope $\frac{5}{3}$. The solid line is the free-particle asymptote.

Secondly, we provide some justification *a posteriori* on physical grounds for (3.3) and its coefficients in Sec. V below. This justification does not extend to the new term in (3.4), which is not an embarrassing fact because there is probably a limit to the applicability of simple physical reasoning to explain the entire content of (2.14). What is more embarrassing is that we have been unable to find a closed analytic form for the final integration in (2.4) when (3.4) is used. After a numerical test to ensure that the effect of (3.4) is adequately approximated by (3.3) for our purposes, we have been content to use (3.3) in the subsequent discussion. Nevertheless, (3.4) is the most accurate equation that we have been able to construct so far.

Finally, p_0 in (3.3) is actually the chemical potential μ, apart from an additive constant which is a matter of definition. In our present and earlier work,[4] we define $p_0(p, q_F)$ as derived via (2.14) from the condition $D = 0$, and evaluated at the Fermi surface, to be the chemical potential, i.e., $\mu(q_F) = p_0(q_F, q_F)$. Thus, from (3.3), we write

$$\mu(q_F) = \tfrac{5}{3} q_F - 2m + 3m \exp(-5q_F/9m). \tag{3.5}$$

[More accurately, if it is needed, (3.4) give the chemical potential.] Equation (3.5) is used in the discussion below. The range of validity of (3.5) follows from the range of q_F/m for which we have stated that it has been computed: The corresponding upper limit to the density of matter which it governs is about 7.22×10^{23} g/cm^3.

IV. EQUATION OF STATE

The relativistic pressure and energy density are determined to lowest order by the excitation energy $\mathcal{E}_1$ (Sec. III), the Green's function (2.17) and the integrals (2.4) and (2.5). The equation of state follows from

the parametric dependence of both P and ρ on p_F. The evaluation of P and ρ is easily accomplished if the analytic expression (3.3) is used for $\delta_i(q_F)$. We present the results first, and then give the derivations.[13] Finally the significance of the equation of state will be discussed.

Set $x_F = \alpha\hbar k_F/mc$; then the pressure is given by

$$P(k_F) = \frac{am^4c^5}{12\pi^2\alpha^4\hbar^3}\, x_F^4\left[1 - \frac{24}{x_F^4} + 4\frac{e^{-x_F}}{x_F^4}(x_F^3 + 3x_F^2 + 6x_F + 6)\right], \tag{4.1}$$

and the energy density by

$$\rho = \frac{m^4c^5}{3\pi^2\hbar^3\alpha^3}\left[x_F^3\left(1 - \frac{a}{\alpha}\right) + \frac{3a}{4\alpha}(x_F^4 + 8) - \frac{3a}{\alpha}e^{-x_F}(x_F^2 + 2x_F + 2)\right]. \tag{4.2}$$

The equation of state $P = P(\rho)$ is plotted in Fig. 4, for the total energy density ρ in the range[14]

$$\begin{pmatrix} 2.17\times10^{34}\ \text{ergs/cm}^3 \\ 2.43\times10^{13}\ \text{g/cm}^3 \end{pmatrix} \leq \rho \leq \begin{pmatrix} 6.45\times10^{46}\ \text{erg/cm}^3 \\ 7.22\times10^{25}\ \text{g/cm}^3 \end{pmatrix}.$$

The expressions (4.1) and (4.2) will now be derived in terms of the analytic expression (3.5) for the chemical potential shown in Fig. 5:

$$\mu(q_F) = aq_F + mb + mc\exp(-\alpha q_F/m), \tag{4.3}$$

where $a = \frac{5}{3}$, $b = -2$, $c = 3$, and $\alpha = \frac{5}{3}$. The expression for the pressure (2.4) becomes

$$P = \frac{a}{\pi^2}\int_0^{p_F}[1 - \exp(-\alpha q_F/m)]dq_F\int_0^\infty p^2dp\,\theta(q_F - p), \tag{4.4}$$

where $n(p)$ has been replaced by $\theta(q_F - p)$, the p_0 and angular integrals performed, and (4.3) used to replace the integral over chemical potential by an integral over the Fermi momentum. In arriving at (4.4) the Green's function (2.17) has been used, and the trace evaluated with the aid of the anticommutation relations[5] for the gamma matrices γ^μ and γ^5. Performing the integrals over p and q_F leads to the expression (4.1) for the relativistic pressure to lowest order in the interaction (1.3).

The expression for the total energy density (4.2) follows directly from (2.5), if the pressure is expressed in terms of the rest energy density ρ_0, defined in terms of the Fermi momentum q_F by

$$\rho_0 = m\frac{q_F^3}{3\pi^2}. \tag{4.5}$$

It is now a straightforward matter to check that the number density remains unchanged as a result of the interaction. That is, from the relativistic expression[11]

$$n = \frac{dP}{d\mu}, \tag{4.6}$$

where P and μ are given by (4.1)–(4.2), it follows [compare (1.2) for $g_0 = 0$] that

$$n = \frac{k_F^3}{3\pi^2}. \tag{4.7}$$

It is to be expected that the momentum distribution $\mu(p)$ will deviate from that of an ideal Fermi gas if higher-order corrections to the self-energy are considered (for example, those shown in Fig. 3).[15]

Examination of the expressions for the pressure and energy density reveal the following asymptotic behavior:

$$\lim_{p_F/m\to\infty} P \to \tfrac{1}{4}ap_Fcn \equiv P_\infty, \tag{4.8}$$

$$\lim_{p_F/m\to\infty} \rho \to 3P_\infty. \tag{4.9}$$

Apart from the additional factor a which is given by (3.2), (4.8)–(4.9) are just the pressure and energy density of an extreme-relativistic ideal Fermi gas. The factor a leads to an interaction-dependent enhancement in the asymptotic regime, which is independent of the density of the system.[16]

Finally, from the pressure and energy density we may calculate the speed of sound for our relativistic strongly interacting system. The speed of sound β_s is given at zero temperature by the expression

$$\beta_s^2 = \left(\frac{dP}{d\rho}\right)_{T=0},$$

which, with ρ given by (2.5), is valid for relativistic systems. A simple calculation yields

$$\beta_s^2 = \tfrac{1}{3}ax_F\frac{1 - e^{-x_F}}{\alpha - a + ax_F + ae^{-x_F}}. \tag{4.10}$$

The asymptotic limit, which also follows from (4.8)–(4.9), is

$$v_s = \frac{c}{\sqrt{3}} . \tag{4.11}$$

Further examination of (4.10) shows that (4.11) is an upper limit which is reached only at infinite density.

Early analyses of possible upper limits to v_s as a result of special relativity[17] leads to the limit (4.11). These are based upon the behavior of free fields in the asymptotic limit, and generally make no reference to interactions between the constituents of the system. In 1962, Zel'dovich presented a simple classical model of relativistic interacting matter[18] which led to the less stringent limit $v_s \leq c$. However, this result is not based on a fully relativistic interaction, nor upon a relativistic many-body theory. Our result $v_s \leq c/\sqrt{3}$ is, therefore, of particluar significance in that it is based upon relativistic interactions within a fully relativistic many-body theory. Further, it suggests that this limit may be a general consequence of relativistic interactions, at least for a broad class of couplings. In fact, a calculation of the equation of state for a system of electrons and neutrinos at zero temperature, which includes lowest order corrections due to the weak current, leads to the same asymptotic limit (4.11).

The comments above strongly suggest that results based upon models which lead to less stringent limits than (4.11) be reexamined within the context of a fully relativistic many-body theory.

V. NONRELATIVISTIC AND NONINTERACTING LIMITS

It will be instructive to examine the nonrelativistic limit of the equation of state (4.1)–(4.2), as well as the noninteracting limit of the analytic approximation discussed in the previous sections. We look first at the nonrelativistic limit $x_F \ll 1$, and then at the case $g_0 = 0$.

The nonrelativistic pressure is obtained by expanding the expression (4.1) in the limit $x_F \ll 1$, where it will be recalled that $x_F = \alpha p_F/m$. However, before doing so, we consider the chemical potential (4.3)

$$\mu = aq_F + mb + cme^{-\alpha q_F/m} . \tag{5.1}$$

We wish to show that certain constraints may be placed on the choice of constants used in fitting the chemical potential which are, to some extent, independent of the detailed numerical calculation. First, denote the nonrelativistic chemical potential at zero temperature by μ_0

$$\mu_0 = \mu - m . \tag{5.2}$$

Then expand (5.1) in powers of xq_F/m to obtain

$$\mu = m(b+c) + (a - \alpha c)x_F + \tfrac{1}{2}(\alpha^2 c x_F^2)[1 - \tfrac{1}{3}(\alpha x_F) + \tfrac{1}{12}(\alpha^2 x_F^2) + \cdots] . \tag{5.3}$$

Comparison with (5.2) yields the relation $b + c = 1$. Since the nonrelativistic limit of the pseudoscalar coupling[5] is the Yukawa potential

$$V = V_0 \frac{e^{-r/a}}{r} , \tag{5.4}$$

and insofar as nonrelativistic many-body calculations based on (5.4) lead to chemical potentials at zero temperature which do not contain terms linear in q_F/m, we expect the constraint[19] $a - \alpha c = 0$. It is quite reasonable, therefore, to demand that the constants in the analytic fit (5.1) satisfy constraints

$$b + c = 1 , \tag{5.5a}$$

$$a - \alpha c = 0 . \tag{5.5b}$$

We observe that the values $a = \frac{5}{3}$, $b = -2$, $c = 3$, and $\alpha = \frac{5}{9}$ mentioned previously, and determined independently of the above arguments (see Sec. III) satisfy conditions (5.5a)–(5.5b) exactly.

Returning to (4.1), and expanding in powers of x_F, we obtain the nonrelativistic pressure

$$P_{\text{n.r.}} = P_0^{\text{n.r.}}(1 - \tfrac{1}{12}5\alpha y + \tfrac{1}{42}5\alpha^2 y^2 - \cdots) , \tag{5.6}$$

where $y = p_F/m$, and

$$P_0^{\text{n.r.}} = \frac{p_F^5}{15\pi^2 m} \tag{5.7}$$

is the asymptotic pressure for a nonrelativistic ideal Fermi gas. Similarly, the energy density (4.2) reduces to

$$\rho = \rho_0(1 + \tfrac{3}{10}y^2 - \tfrac{1}{12}\alpha y^3 + \cdots) , \tag{5.8}$$

with ρ_0 given by (4.5). In the nonrelativistic regime the Yukawa interaction is attractive, as evidenced by the decrease in pressure and energy density over their noninteracting values. It will be noted that the pressure (5.6) has the same form (to lowest order) as that calculated from the pseudoscalar Yukawa[20] potential and nonrelativistic many-body theory.

The speed of sound in the low-density limit is readily calculated, and found to be

$$v_s^2 \simeq \frac{1}{3m^2} p_F^2 \left(1 - \frac{\alpha p_F}{m}\right) , \tag{5.9}$$

a result typical of low-lying excitations (phonons) at zero temperature.

VI. CONCLUSION

The equation of state calculated above (4.1)–(4.2) is significant for several reasons. First, it is based upon a relativistic many-body theory which includes interactions relativistically. It is to be emphasized that both relativistic and finite-density effects enter at the same elementary level. Second, since the analytic fit used in arriving at (4.1)–(4.2) reasonably describes the numerical results in the density range between 10^{13} g/cm^3 and 10^{23} g/cm^3, it is natural to consider the asymptotic limit $p_F/m \gg 1$, which gives some indication of the behavior of interacting many-body systems in the highly relativistic regime. The results (4.8)–(4.9) suggest that the principal asymptotic effect of interactions is to modify the free-particle pressure and energy density by a factor proportional to the coupling constant, but independent of the density to lowest order. Since the constant of proportionality is the same for both P and ρ, we obtain the asymptotic speed of sound

$$v_s \leq c/\sqrt{3}\ ,$$

a result which is of significance in its own right.[21]

Furthermore, our approach serves to emphasize that any distinction between self-interactions (zero density) and interparicle interactions is unnecessary. In fact, for densities $\rho_0 \gtrsim 10^{15}$ g/cm^3 the interparticle separation is less than the range of interactions, and it is difficult to talk about particles as such. Under these conditions the separation of interactions into self- and interparticle interactions is meaningless. This is most easily seen by examining (2.1a), which includes all interactions to lowest order. The decomposition $\Sigma_c(p, q_F)$ given by (2.2)–(2.4) is purely arbitrary. The form chosen here is dicated by questions of calculational convenience, and should imply nothing more.

Our calculation has demonstrated, at least to lowest order in the coupling constant, that all divergences associated with a renormalizable interaction in vacuum may be eliminated in discussing the corresponding many-body problem. Further, the procedure used in relativistic quantum field theory is sufficient for regularization. The infinite contributions due to negative-energy states are removed when we work in terms of normal-ordered operators (this has been discussed in greater detail in Ref. 4).

The simple ansatz represented by (2.16), and the equality $\mathcal{E}_1(p, q_F) = \mathcal{E}_2(p, q_F)$ which is independent of the reasoning leading to (2.16), makes the subsequent determination of the pressure and energy density straightforward. However, we stress that this state of affairs is a consequence of our approximating the self-energy to lowest order (Fig. 1). The inclusion of higher-order terms in the coupling constant, such as those shown in Fig. 3, complicates the problem considerably. It then becomes necessary to use an equation for the Green's function such as (2.12)–(2.14), and to solve for the zeros of (2.15) as functions of $|\vec{p}|$ and q_F. It then seems likely that the integrations over p and $\mu(q_F)$ in (1.4) for the pressure and the resulting integration in (1.5) for the energy density must all be done numerically.

The additional complexity resulting from higher-order corrections in g_0 may not be of as much interest as those obtained by generalizing the present approach to include more than one type of intermediate boson in describing the interaction. The latter would lead to an equation of state more nearly descriptive of superdense matter, particularly that which can be expected to occupy the interiors of neutron stars and pulsars. For this reason we are currently solving the generalized problem which includes the eight particles of the first baryon octet, with interactions included through the SU(3)-invariant exchange of members of the first pseudoscalar meson octet. The resulting equation of state will then be used as the basis of a stellar structure calculation for matter at relativistic densities $\rho > 10^{15}$ g/cm^3.

We conclude by observing, as suggested by the last comment above, that the approach used in this calculation is of a quite general nature. As described, it is applicable to renormalizable (in the sense of regularization) quantum field theories possessing normal ground states. We are currently examining systems with non-normal ground states, such as those leading to superfluidity, superconductivity, and Bose-Einstein condensation.

APPENDIX A

The functions Σ_μ and Σ_I appearing in (2.10) are given by the integrals

$$\Sigma^\mu(p, q_F) = -\pi g_0^2 \int \frac{d^4k}{(2\pi)^4} \frac{n_F(p-k)}{E_{\vec{p}-\vec{k}}} \delta(p_0 - k_0 - E_{\vec{p}-\vec{k}}) \frac{(k-p)^\mu}{k^2 - \mu^2}\ , \tag{A1}$$

$$\Sigma_I(p, q_F) = -\pi g_0^2 \int \frac{d^4k}{(2\pi)^4} \frac{n_F(p-k)}{E_{\vec{p}-\vec{k}}} \delta(p_0 - k_0 - E_{\vec{p}-\vec{k}}) \frac{m}{k^2 - \mu^2}\ . \tag{A2}$$

Since the system is assumed to be homogeneous and isotropic, the self-energy terms Σ_μ and Σ_I will depend only on the magnitude of the three-vector momentum $p \equiv |\vec{p}|$:

$$\Sigma_\mu(p, q_F) = \Sigma_\mu(|\vec{p}|, p_0, q_F),$$
$$\Sigma_I(p, q_F) = \Sigma_I(|\vec{p}|, p_0, q_F). \tag{A3}$$

Furthermore, the Fermi sea is spherically symmetric as tacitly assumed by (2.16) and (2.5). Expressing $\vec{p}$ in spherical coordinates (p, θ, ϕ) and noting that the γ matrices γ^μ are defined with respect to the Cartesian components of $\vec{p}$, it can be shown that (A1)–(A2) reduce to two distinct integrals:

$$\gamma^\mu \Sigma_\mu = \gamma^0 \Sigma_0 - (\gamma^1 \sin\theta\cos\phi + \gamma^2 \sin\theta \sin\phi + \gamma^3 \cos\theta)\Sigma_v, \tag{A4}$$

where $\Sigma_v = \Sigma_v(p, q_F)$. It is convenient to perform a change in integration variable $k^\mu - p^\mu = l^\mu$, where the polar coordinates of l are defined by (l, θ_l, ϕ_l). Then

$$\Sigma_0(p, q_F) = \frac{g_0^2}{8\pi^2} \int_{-1}^{1} d(\cos\theta_l) \int_0^{q_F} \frac{l^2 dl}{\Phi(p, l, \theta_l)}, \tag{A5}$$

$$\Sigma_v(p, q_F) = -\frac{g_0^2}{8\pi^2} \int_{-1}^{1} d(\cos\theta_l) \int_0^{q_F} \frac{l^3 dl}{E_l} \frac{\cos\theta}{\Phi(p, l, \theta_l)}. \tag{A6}$$

The last term of (2.10) similarly reduces to

$$\Sigma_I(p, q_F) = -\frac{g_0^2 m}{8\pi^2} \int_{-1}^{1} d(\cos\theta_l) \int_0^{q_F} \frac{l^2 dl}{E_l} \frac{1}{\Phi(p, l, \theta_l)}. \tag{A7}$$

The function $\Phi(p, l, \theta_l)$ is defined by

$$\Phi(p, l, \theta_l) = p_0^2 - p^2 + m^2 - \mu^2 - 2p_0 E_l - 2pl\cos\theta_l, \tag{A8}$$

where $E_l = (l^2 + m^2)^{1/2}$, and $|\vec{p}| \equiv p$, $|\vec{l}| \equiv l$. The integration over l_0 is trivially performed, due to the δ functions in (A1)–(A2). Furthermore in the integration over ϕ_l, all terms in $\sin\phi_l$ or $\cos\phi_l$ vanish.

The evaluation of (A5)–(A7) is straightforward but tedious. For this reason only the results will be given. It was found that

$$\Sigma_0(p, p_0, q_F) = I_3, \tag{A9}$$

$$\Sigma_I(p, p_0, q_F) = I_2, \tag{A10}$$

where

$$I_n \equiv \frac{g^2 m^2}{16\pi^2 p} \frac{(-1)^{n+1}}{n-1} \left\{ \left[\frac{(q_F^2 + m^2)^{1/2}}{m} \right]^{n-1} \ln \left| \frac{\alpha - 2p_0(q_F^2 + m^2)^{1/2} - 2pq_F}{\alpha - 2p_0(q_F^2 + m^2)^{1/2} + 2pq_F} \right| + 2f(n, 2mp) - 2f(n, -2mp) \right\}, \tag{A11}$$

$$f(n, c) = \int_{-T}^{0} dt \left(\frac{1+t^2}{1-t^2} \right)^n \frac{4p_0 mt + (1+t^2)c}{(1-t^4)\alpha - 2(1+t^2)^2 p_0 m - 2ct(1-t^2)}, \tag{A12}$$

$$q_F T = m - (q_F^2 + m^2)^{1/2}, \tag{A13}$$

$$\alpha = p^\mu p_\mu - \mu^2 + m^2. \tag{A14}$$

Finally we find for Σ_v

$$\Sigma_v(p, p_0, q_F) = \frac{g^2}{32\pi^2 p} \Bigg\{ 2m^2 \ln\left(\tan\tfrac{1}{4}\pi + \tfrac{1}{2} \tan^{-1} \frac{q_F}{m} \right) - 2q_F(q_F^2 + m^2)^{1/2} + \frac{(q_F^2 + m^2)^{1/2}}{p} \left(\alpha - p_0(q_F^2 + m^2)^{1/2} \right) \ln \left| \frac{\alpha - 2p_0(q_F^2 + m^2)^{1/2} + 2pq_F}{\alpha - 2p_0(q_F^2 + m^2)^{1/2} - 2pq_F} \right| + \frac{m^2}{p} p_0 [f(2, 2mp) - f(2, -2mp)] - \frac{m\alpha}{p} [f(1, 2mp) - f(1, -2mp)] \Bigg\}. \tag{A15}$$

It will be noted that in (A9)–(A15) the energy p_0 and magnitude of the vector momentum p are written separately. The 4-momentum when it occurs [as in (A14)] is denoted by p^μ.

Further evaluation of the integrals was done numerically as described in Sec. III.

APPENDIX B

The elementary particle self-energy has been discussed extensively in the literature.[22] In renormalized form it has been shown that

$$\Sigma^R(p) = -\frac{3g^2}{16\pi^2}\int_0^1 dx[-(\not{p}-m)(1-x)+mx]\ln\left|\frac{m^2x+\mu^2(1-x)-p^2x(1-x)}{m^2x^2+\mu^2(1-x)}\right|$$

$$-\frac{3g^2}{16\pi^2}\int_0^1 dx\frac{2m^2x^2(1-x)}{m^2x^2+\mu^2(1-x)}(\not{p}-m). \tag{B1}$$

Comparison of (B1) with (2.10) gives the expressions for $S_1(p^2)$ and $S_2(p^2)$. The integral above is finite, and vanishes on mass shell. The two terms $S_1(p^2)$ and $S_2(p^2)$ were evaluated and included in the numerical computation for the pressure. (See Sec. III.)

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[5]The notation is that of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[6]These are discussed in detail in Sec. IV of Ref. 4.

[7]The boundary conditions imposed on the ground state, and other assumptions leading to (2.1a) are fully discussed in Sec. IV of Ref. 4.

[8]Introduction of finite-density boundary conditions through the homogeneous part of the Green's function is discussed in detail in Ref. 3, Sec. III.

[9]It is to be emphasized that this replacement is purely formal, and that the elementary excitations serve only to describe in a quantum-mechanical manner the collective nature of the interacting system. They are not to be thought of as localizable particles with energies $\mathcal{E}_i(p, q_F)$. For a discussion of this point see any text on many-body theory, for example, A. A. Abrikosov, L. P. Gor'kov, and I.-E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics*, translated and edited by R. Silverman (Prentice-Hall, Englewood Cliffs, N. J., 1963).

[10]Note the similarity between (2.17) and the noninteracting Green's function (4.2) of Ref. 4.

[11]B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (Univ. of Chicago, Chicago, 1965).

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[13]Since the Green's function is expressed in the form (2.17), the derivation of the pressure parallels the general procedure outlined earlier in Sec. III of Ref. 4 for noninteracting baryons, with the exception that we here treat a system devoid of antiparticles.

[14]The upper limit on the density represents the largest value of q_F/m actually used in the numerical program. We certainly do not claim that the quoted results give a detailed description of matter at all of these densities since other interactions enter, and the system will contain more than a single species of baryons. On principle, we advise caution about using the results above densities of immediate current interest for which there is no other relativistic equation of state, e.g., above approximately 10^{17} g/cm^3.

[15]It is well known that $n(p)$ for a nonrelativistic Fermi gas is no longer the step function $\theta(q_F-p)$, but leads to nonzero occupancy for $q_F<p$; see, for example, V. A. Belyakov, Zh. Eksp. Teor. Fiz. 40, 1210 (1961) [Sov. Phys. JETP 13, 850 (1961)]; P. Nozières, *Theory of Interacting Fermi Systems* (Benjamin, New York, 1964).

[16]See Sec. III.

[17]Nonrelativistic arguments generally lead to limits which are finite, but exceed the speed of light. For example, T. H. R. Skyrme [Nucl. Phys. 9, 665 (1959)] finds $v_s \lesssim 4.3c$; A. G. W. Cameron [Astrophys. J. 130, 884 (1959)] finds $v_s \lesssim 1.28c$. Such arguments are certainly suspect in the relativistic regime.

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[19]This condition further guarantees that the chemical potential is a monotonically increasing function of the density – a condition which is necessary if the system is to be stable against microscopic collapse; see Ref. 11.

[20]The pseudoscalar nature of the nonrelativistic potential may be taken into account by (5.4) and using only the direct term corresponding to Fig. 2(b) of Ref. 4.

[21]R. L. Bowers, Phys. Rev. Letters 29, 509 (1972).

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