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Numerical Evaluation of Phase-Space Integrals and Their Approach to Asymptotic Behavior*

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We examine the range of validity of various approximate expressions for invariant phase space and "transverse-cutoff" phase space by checking against Monte Carlo calculations. We find that there are discrepancies for the transverse-cutoff case in the transition energy region between the isotropic and one-dimensional regimes. Since the location of this transition region is multiplicity-dependent, there are observable consequences for energy dependence of multiplicity distributions.

I. INTRODUCTION

The study of models for multiparticle production in high-energy collisions, which has been of increasing interest in recent years, requires the evaluation of integrals over a phase space of large dimensionality. Such integrals in general cannot be evaluated exactly in analytic form. The most direct numerical procedure is to use Monte Carlo methods, but this can be expensive in terms of computer time. For some particularly simple cases, where the matrix element can be expressed as a product of identical factors, it is possible to reduce the multidimensional integrals to simpler quadratures. This has been done in particular for the calculation of the total invariant phase-space volume[1-3] and also for the case of "transverse-cutoff phase space" or "longitudinal phase space" when the matrix element consists of independent transverse-momentum damping factors for each particle.[4] Generalizations to somewhat more complicated cases have been made.[5] While these oversimplified models are not expected to be correct, they are extremely important in understanding which observed experimental features are simple consequences of kinematics and which give more insight into the structure of the matrix element.[6-8]

The accuracy of these so-called "statistical" methods is not uniform in energy or multiplicity, though the expressions can be evaluated with arbitrary precision. Monte Carlo methods, on the other hand, are subject to considerable statistical uncertainties, but once they converge the accuracy does not depend appreciably on energy or multiplicity.

In this paper we examine the range of validity of the approximate formulas for invariant phase-space and transverse-cutoff phase-space volumes for n identical particles by checking against Monte Carlo calculations. We confirm the fact that the best approximate formulas for invariant phase space are uniformly valid to within a few percent. For the transverse-cutoff phase space, however, there is a considerable discrepancy in the transition region between low energies, where the behavior is similar to invariant three-dimensional phase space, and high energies, where the behavior is that of one-dimensional invariant phase space. The energy range over which this transition occurs is multiplicity-dependent, and therefore there are some consequences which affect the predictions of such models.

In Sec. II we discuss invariant phase-space calculations. Section III contains the discussion of the transverse-cutoff phase-space volume, and in Sec. IV we discuss the behavior of multiplicity distributions. The results are summarized in Sec. V.

The calculations were carried out using a Monte Carlo precedure based essentially on the method of Kittel *et al.*[9,10] The details are not important for this calculation in which only total cross sections are considered. Details of the Monte Carlo calculation and studies of various model predictions for differential spectra will be presented elsewhere.

II. THREE-DIMENSIONAL ISOTROPIC PHASE SPACE

In this section we discuss invariant three-dimensional phase space and compare approximate analytical evaluation with Monte Carlo results. For the sake of simplicity we consider the phase space for n identical particles with mass m. The numerical results correspond to $m = 140$ MeV, but the general features remain the same for the case of unequal masses. The phase-space volume is given by

$$\Omega_n(W) = \int \delta^3\left(\sum_i \vec{p}_i\right) \delta\left(\sum p_{i0} - W\right) \prod_{i=1}^{n} \frac{d^3 p_i}{2p_{i0}} \qquad (2.1)$$

(in the center-of-mass frame), where n is the multiplicity, W is the total energy, and p_i is the momentum of the ith particle.

For $n=2$ this integral can be trivially evaluated to give

$$\Omega_2(W)=\tfrac{1}{2}\pi\left(\frac{W^2-4m^2}{W^2}\right)^{1/2}, \tag{2.2}$$

which rapidly increases from threshold and approaches a constant for $W \gg 2m$. For $n \geq 3$ the integral cannot be evaluated except in the nonrelativistic ($W \ll nm$) and extreme relativistic ($W \gg nm$) limits where

$$\Omega_n(W)^{\mathrm{NR}}=\frac{(2\pi)^{3(n-1)/2}\, m^{n/2}\,(W-nm)^{(3n-5)/2}}{2^n (nm)^{3/2}\,\Gamma(\frac{3}{2}(n-1))} \tag{2.3}$$

and

$$\Omega_n(W)^{\mathrm{ER}}=\frac{\pi^{n-1}\,W^{2n-4}}{2(n-1)!\,(n-2)!}, \tag{2.4}$$

respectively. These are the only cases for which the integral (2.1) can be calculated exactly.

The multiple integral (2.1) can be shown by various methods to reduce to the one-dimensional integral representation[1,11]

$$\Omega_n(W)=\frac{(2\pi)^{n-2}(2m^2)^n}{iW}\times\int_{c-i\infty}^{c+i\infty} d\alpha\,\alpha^{2-n} I_1(\alpha W)[K_1(\alpha m)]^n . \tag{2.5}$$

There have been two approaches to evaluating this. Campbell[12] and Campbell, Lepore, and Riddell[2] have used the method of steepest descents to obtain

$$\Omega_n(W)=\frac{\beta^2 I_1(\beta W)}{(2\pi)^{3/2}W}\left(\frac{4\pi m}{\beta}K_1(m\beta)\right)^n\left[\frac{4n-2}{\beta^2}+W^2+nm^2-\left(\frac{WI_0(\beta W)}{I_1(\beta W)}\right)^2-n\left(\frac{mK_0(\beta m)}{K_1(\beta m)}\right)^2\right]^{-1/2}, \tag{2.6}$$

where β is the solution of

$$\frac{1-2n}{\beta}+\frac{WI_0(\beta W)}{I_1(\beta W)}-\frac{nmK_0(\beta m)}{K_1(\beta m)}=0 . \tag{2.7}$$

A simpler procedure has been adopted by Lurcat and Mazur,[1] who first construct the Laplace transform of Ω_n,

$$\Phi_n(\beta)=\int e^{-\beta P_0}\Omega_n(P)d^4P \tag{2.8}$$

$$=\left(\frac{4\pi m^2}{\beta}K_1(m\beta)\right)^n, \tag{2.9}$$

[Eq. (2.5) being the inverse transform] and then use stochastic methods and the central-limit theorem to show that

$$\Omega_n(W)=\Phi_n(\beta)e^{\beta W}\frac{(2\pi)^{-2}}{(2m)^n}\left[\frac{\partial^2}{\partial\beta^2}\left(\frac{W^3}{\beta^3}\ln\Phi_n(\beta)\right)\right]^{1/2}+O(1/n) \tag{2.10}$$

where β is the solution of

$$\frac{2n}{\beta}+nm\frac{K_0(\beta m)}{K_1(\beta m)}=W . \tag{2.11}$$

The next-order correction to this is essentially in agreement with (2.6) and (2.7).

The stochastic method is equivalent to replacing the exact δ function in (2.1) with a sharply peaked function, in analogy with the transition from microcanonical to canonical ensemble in calculating a thermodynamic partition function. However, the values of n involved are nowhere near thermodynamic, and a certain amount of error is introduced.

We have carried out Monte Carlo calculations whose accuracy can be normalized by comparing with the nonrelativistic and extreme relativistic cases, so that they can then provide confirmation of the accuracy of the other cases. For $n=2$ the Monte Carlo calculation agrees with (2.2) to within 1%, while (2.6) differs by a few percent. For $3 \leq n \leq 8$, the agreement between (2.3) and (2.4) (when they are valid), (2.6), and the Monte Carlo result is again always within a few percent, confirming the accuracy estimate in Refs. 2 and 12 (Fig. 1).

III. TRANSVERSE-MOMENTUM CUTOFF PHASE SPACE

In the presence of a transverse-momentum cutoff, the "phase space" is given by

$$\tilde{\Omega}_n(W)=\int \delta^{(3)}(\textstyle\sum \vec{p}_i)\,\delta(\sum p_{i0}-W)\prod_{i=1}^{n} f(p_{i\perp})\frac{d^3p_i}{2p_{i0}}, \tag{3.1}$$

where $f(p_\perp)$ is the cutoff function and $p_\perp$ is the transverse momentum, usually defined to be perpendicular to the direction of the incident system in the production process. In general, the integral (3.1) cannot by analytically evaluated, not even for $n=2$ or for the nonrelativistic and extreme relativistic limits. However, since the observed transverse momentum cutoff is typically 300 to 400 MeV/c and of the same order of magnitude as the mass, such a cutoff can be neglected in the nonrelativistic limit, and the phase space is then

given by Eq. (2.3). In the extreme relativistic limit, the asymptotic behavior of Eq. (3.1) can be obtained from a simple power-counting argument and is given by

$$\tilde{\Omega}_n(W)^{\mathrm{ER}} \simeq \mathrm{const} \times \frac{(\ln W)^{n-2}}{W^2} . \tag{3.2}$$

In contrast with the isotropic case, $\tilde{\Omega}_n(W)$ is a decreasing function of W at high energies where it resembles one-dimensional phase space. The transverse-momentum cutoff thus makes the phase space behave as if it were three-dimensional at low energies and one-dimensional at high energies.[6]

An approximate formula has been derived by Kryzwicki from a stochastic method exactly analogous to that described in the last section, and is given by[4]

$$\tilde{\Omega}_n(W) = \frac{[\phi(\beta)]^n}{2\pi^2 n h(\beta)} \left[\frac{W}{\beta}\left(\frac{W}{\beta} + n h(\beta) + nm^2 - \frac{W^2}{n}\right)\right]^{-1/2} e^{\beta W} + O(1/\sqrt{n}) , \tag{3.3}$$

where

$$\phi(\beta) = \pi \int dp_\perp \, p_\perp f(p_\perp) K_0(\beta \mu_\perp) , \tag{3.4}$$

$$h(\beta) = \frac{\pi}{\phi(\beta)} \int dp_\perp \, p_\perp{}^3 f(p_\perp) K_0(\beta \mu_\perp) , \tag{3.5}$$

$$\mu_\perp = (p_\perp{}^2 + m^2)^{1/2} ,$$

β is determined from W by

$$\frac{\pi}{\phi(\beta)} \int dp_\perp \, p_\perp f(p_\perp) \mu_\perp K_1(\beta \mu_\perp) = \frac{W}{n} , \tag{3.6}$$

and $f(p)$ is normalized to

$$\int f(p_\perp) p_\perp dp_\perp = 1 . \tag{3.7}$$

The corrections to Eq. (3.3) can also be calculated.

Since the observed transverse momentum spectra are approximately Gaussian or exponential with an average value $\langle p_\perp \rangle \simeq 300$–$400$ MeV,[13] we shall take $f(p_\perp)$ for the moment to be of the form

$$f(p_\perp) = 10 e^{-5 p_\perp{}^2} \ \mathrm{GeV}^{-2} , \tag{3.8}$$

which gives $\langle p \rangle \simeq 350$ MeV for all n. Notice that due to the phase-space factor $\prod_{i=1}^{n} p_{0i}{}^{-1}$ and energy-momentum conservation, the actual transverse-momentum spectra will not be exactly Gaussian and $\langle p_\perp \rangle$ is not directly given by the width.[14]

Although we can no longer analytically evaluate the integral with the cutoff (3.8) for $n = 2$, it can be reduced to

$$\tilde{\Omega}_2(W) = \frac{\pi}{4W} \int_0^{W^2/4 - \mu^2} \frac{dp^2}{(\frac{1}{4}W^2 - p_\perp{}^2 - m^2)^{1/2}} f(p_\perp) f(-p_\perp) \tag{3.9}$$

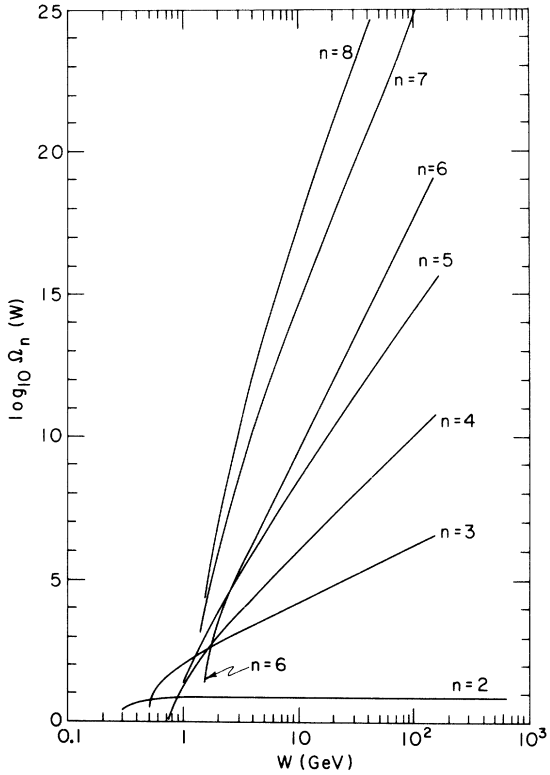

FIG. 1. The three-dimensional isotropic phase space as a function of n and W for $2 \leq n \leq 8$. Since $\Omega_n(W)$ can easily be normalized to the nonrelativistic and extreme relativistic limits, the absolute normalization on this graph is arbitrary in order to suit the spacing of the curves.

and trivially integrated numerically. Thus we can compare the results obtained from Eqs. (3.3) and (3.9) and the Monte Carlo calculation. For $n \geq 3$, we directly compare Eq. (3.3), the Monte Carlo results, and the asymptotic behavior (3.2).

For $n = 2$, the Monte Carlo results agree almost exactly with Eq. (3.9), while for $2 \leq n \leq 7$ the comparison of the Monte Carlo results with Eq. (3.3) is presented in Table I and Fig. 2. At both low and high energies these two results are almost proportional to each other, but the proportionality constants differ in these two energy ranges. In between these two limits, Eq. (3.3) gives a somewhat greater energy dependence than the Monte Carlo (and exact) results. Furthermore, for $n \geq 3$, the agreement among these two results and the asymptotic behavior is almost exact at very high energies. As seen, the statistical formula (3.3) does give a good qualitative description of the phase space, but the discrepancy should be taken into account when detailed calculations with Eq. (3.3) are attempted.[5] In Fig. 3, we present the Monte Carlo results for $2 \leq n \leq 7$ as functions of W.

TABLE I. The ratio R of the approximate formula (3.3) to the Monte Carlo (MC) results as a function of n and W, where $R = \tilde{\Omega}_n(W)_{MC}/\tilde{\Omega}_n(W)_{app}$.

W \ n	2	3	4	5	6	7
0.5	0.84					
0.6	0.84	1.26				
0.7	0.84	1.26				
0.8	0.89	1.26	1.10			
0.9	0.89	1.26	1.12			
1.0	0.89	1.26	1.12			
1.5	0.84	1.26	1.38	1.50		
2.0	0.84	1.23	1.41	1.54	1.41	
3.0	0.84	1.19	1.38	1.54	1.68	1.64
4.0	0.80	1.19	1.38	1.59	1.68	1.64
5.0	0.77	1.15	1.38	1.54	1.68	1.66
6.0	0.73	1.15	1.38	1.54	1.68	1.64
7.0	0.71	1.15	1.38	1.54	1.68	1.64
8.0	0.71	1.15	1.38	1.54	1.68	1.64
9.0	0.71	1.15	1.38	1.59	1.68	1.64
10.0	0.71	1.15	1.34	1.59	1.68	1.64
15.0	0.71	1.15	1.34	1.59	1.64	1.64
20.0	0.71	1.15	1.29	1.54	1.64	1.59
30.0	0.71	1.15	1.29	1.45	1.59	1.59
40.0	0.69	1.09	1.25	1.45	1.54	1.59
50.0	0.67	1.09	1.25	1.38	1.50	1.50
60.0	0.63	1.09	1.25	1.34	1.50	1.45
70.0	0.63	1.03	1.25	1.34	1.45	1.41
80.0	0.63	1.03	1.22	1.29	1.45	1.41
90.0	0.63	1.00	1.22	1.25	1.41	1.38
100.0	0.63	1.00	1.19	1.10	1.38	1.34
150.0		1.00	1.12	1.10	1.25	1.25
200.0		1.00	1.06	1.03	1.15	1.19
300.0			1.03		1.09	1.03
400.0			1.03		1.00	0.97

For large n, since the Monte Carlo calculation is time-consuming and the statistical formula should be more accurate than for low n, we also present the results from Eq. (3.3) in Fig. 4 for $2 \leq n \leq 20$. In both cases, the phase space $\tilde{\Omega}_n$ has been weighted by a factor $\lambda^n/n!$ with $\lambda = 1/\pi$, which is useful for the discussion in the next section.

As seen from Figs. 2–4, the phase space (3.1) does indeed behave like isotropic three-dimensional phase space at low energies, and behaves like one-dimensional phase space at high energies. The one-dimensional behavior starts to become important when the average energy per particle is about 1 GeV so that $\langle p_\perp \rangle \simeq \sqrt{2}\,\langle p_\perp \rangle \simeq 400$ MeV and, therefore, the transverse degrees of freedom are saturated. Thus, in contrast to the estimation of Ref. 6, the one-dimensional behavior is exhibited at relatively low energies.[8]

We further study the behavior of the phase space (3.1) with different $f(p_\perp)$. A typical example is shown in Fig. 5 for $n = 4$. When normalized to the same asymptotic behavior, the cutoff with smaller $\langle p_\perp \rangle$ exhibits the one-dimensional behavior earlier

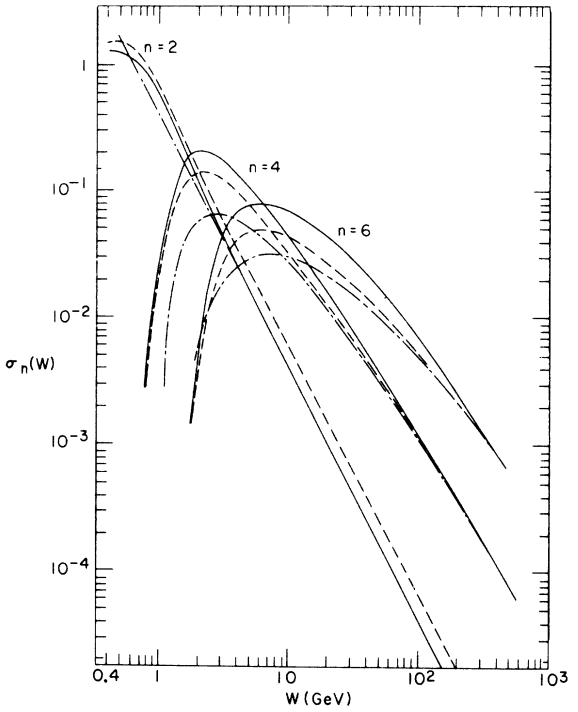

FIG. 2. Comparison of the Monte Carlo results (solid line), Eq. (3.3) (dashed line), and the extrapolated asymptotic behavior in the full energy range (dash-dotted line) for the transverse-momentum-cutoff phase space for $n = 2$, 4, and 6. The normalization is given by Eq. (4.1) with $\lambda = 1/\pi$ and const = 1.

and the phase space is larger at intermediate energies. Furthermore, for almost all reasonable $f(p_\perp)$, the phase space is always larger than expected from the asymptotic behavior in the transition region from isotropic to one-dimensional behavior. The relevance of this fact for the multiplicity distribution will be discussed in the next section.

IV. MULTIPLICITY DISTRIBUTION AND STATISTICAL MODELS

Since final states of different multiplicities can be produced in collision processes with identical initial conditions, it is important to know the multiplicity distribution at different energies. However, apart from effects of the energy-conservation condition, kinematics does not relate the relative magnitudes of the phase space for different n, so some dynamical assumption is necessary. (Of course, the transverse-momentum cutoff phase space for fixed n is already more than a pure kinematical quantity, since only dynamics can determine the precise cutoff.) In this section, we discuss the effects of phase space within the

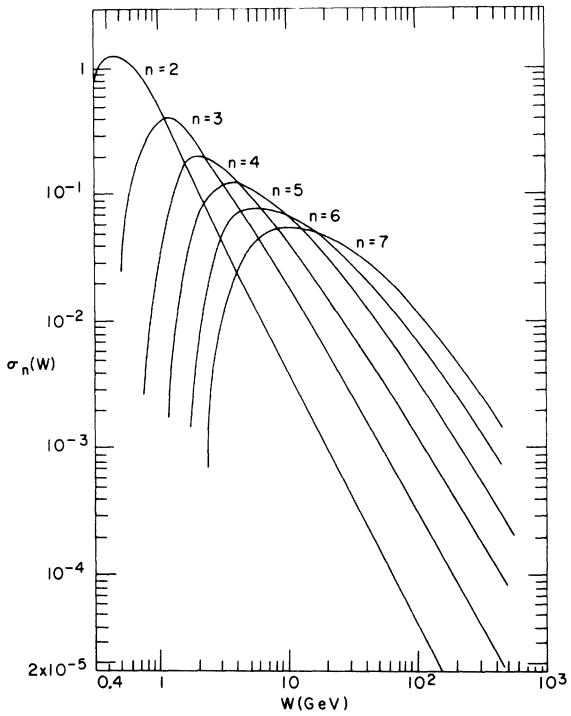

FIG. 3. The Monte Carlo results for the transverse-momentum-cutoff phase space for $2 \leq n \leq 7$. The normalization is the same as in Fig. 2.

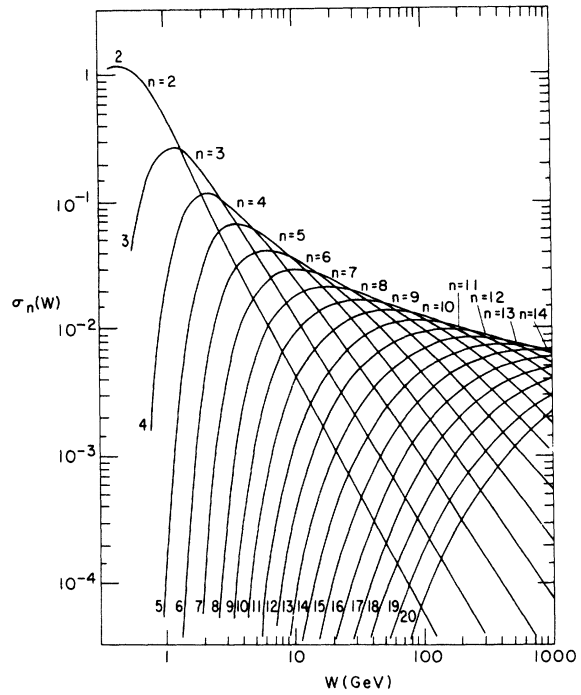

FIG. 4. The transverse-momentum-cutoff phase space given by Eq. (3.3) for $2 \leq n \leq 20$ with the same normalization as in Fig. 2.

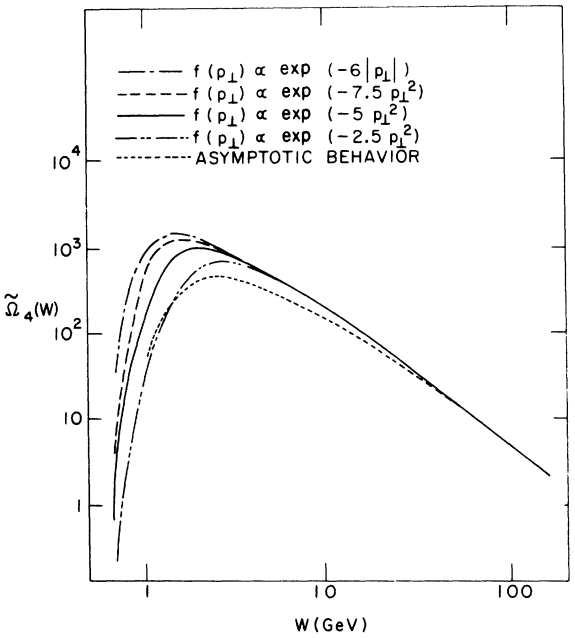

FIG. 5. Comparison of the transverse-momentum-cutoff phase space for $n = 4$ with different cutoff functions. The asymptotic behavior extrapolated to the full energy range is also given as a reference. All these curves are normalized to the same asymptotic value.

context of a statistical model, which treats all the final-state particles independently subject to the kinematical constraints and which therefore enables one to study some kinematical aspects of multiparticle production with minimal dynamical assumptions.

Within the statistical model, the partial cross section for producing n particles is given by[4]

$$\sigma_n(W) = \text{const} \times \frac{\lambda^n}{n!}\, \tilde{\Omega}_n(W)\,, \tag{4.1}$$

where λ is a certain dimensionless parameter. The multiplicity distribution is then determined by Eq (4.1), and the average multiplicity is given by

$$\bar{n} = \sum_{n=1}^{n_{\max}} n\sigma_n \Big/ \sum_{n=1}^{n_{\max}} \sigma_n\,, \tag{4.2}$$

where $n_{\max}$ is the maximum n allowed by energy conservation. $\bar{n}$ is then predicted to increase asymptotically like $\ln W$ for transverse-momentum cutoff phase space[4] and like $W^{2/3}$ for isotropic phase space.[15] In the former case, Eq. (2.2) and (4.3) imply that

$$\bar{n} = 2\pi\lambda \ln W + \text{const}\,. \tag{4.3}$$

Since the observed data [16-18] do seem to fit Eq. (4.3) at high energies with $\lambda \simeq 1/\pi$, we shall take this value to discuss the following general proper-

ties at intermediate energies.

For the Monte Carlo calculation, the phase space $\tilde{\Omega}_n(W)$ was evaluated for definite n and W. This is analogous to the microcanonical ensemble in statistical mechanics. In Ref. 4, the phase space (3.3) is evaluated at a constant "temperature" β with a certain energy dispersion, which is analogous to the canonical ensemble. On the other hand, the average multiplicity can be explicitly evaluated from a "grand canonical ensemble" model[8] to be

$$\bar{n} = 2\pi\lambda \int dp_\perp\, p_\perp f(p_\perp) K_0(\beta\mu_\perp)\,, \tag{4.4}$$

with β determined from

$$W = 2\pi\lambda \int dp_\perp\, p_\perp \mu_\perp f(p_\perp) K_1(\beta\mu_\perp)\,. \tag{4.5}$$

This has the asymptotic form

$$\bar{n} \simeq 2\pi\lambda \ln(2W/2\pi m) + b_0 + O(\ln W/W^2)\,, \tag{4.6}$$

with

$$b_0 = 2\pi\lambda\gamma - 2\pi\lambda \int dp_\perp\, p_\perp \ln(\mu_\perp/m) f(p_\perp)\,,$$

$$\gamma = 0.5772\ldots\,. \tag{4.7}$$

The correction terms to Eq. (4.6) are given in Ref. 8. However, in contrast with the situation in statistical mechanics, the canonical and grand canonical ensembles have an energy dispersion $\Delta W/W \simeq \frac{1}{2}$ and, at available energies, the multiplicity is not very high so that we are not in a thermodynamic regime, and so the results from models differ. We describe these differences and discuss their physical implications.

In Fig. 6, we give $\bar{n}$ as a function of W for the above three cases and also compare with Eq. (4.3). As expected, the Monte Carlo result and the canonical ensemble model disagree at low energies but agree with each other at higher energies with larger $\bar{n}$. The canonical ensemble and the grand canonical ensemble both approach the asymptotic limit (4.3) with the same slope but differ by a constant. This difference agrees with an analytical result of Campbell[19] that the constant terms in the asymptotic expansions for $\bar{n}$ do differ from each other and agree only in the limit of large λ.

The transition from the low-energy to the asymptotic region can be better understood from Figs. 2–5 directly. Except for $n = 2$, each multiplicity becomes important only in a region where the phase space is no longer isotropic, and the transition behavior is important at intermediate energies. At a fixed energy, the phase space for lower multiplicity is always closer to the asymptotic behavior than for higher multiplicities. Furthermore, as seen from Fig. 2, the actual phase space is always larger than the value given by the asymptotic formula. In other words, there is more phase space for larger n than that given by the asymptotic formula which predicts a purely logarithmic increase of $\bar{n}$. The average multiplicity is therefore higher than and also increases more slowly than that given by Eq. (4.3) in the transition region. For the grand canonical ensemble, the correction terms in Eq. (2.9) have been explicitly calculated in Ref. 3 and shown to be positive in the transition region for the cutoff given by (3.8). Such a positive term directly reflects the behavior of $\bar{n}$ shown in Fig. 6. In Fig. 7, we present a typical multiplicity distribution at a fixed W. As seen, there is more phase space for higher multiplicities than the pure Poisson distribution expected from the asymptotic behavior. Such a distribution is also in qualitative agreement with the observed data.[15–17]

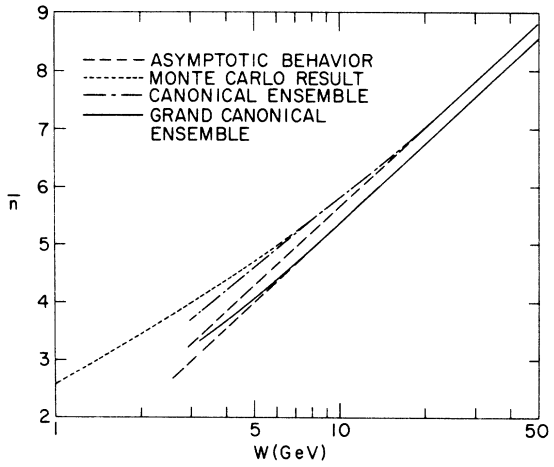

FIG. 6. Comparison of the average multiplicity as a function of W for the Monte Carlo results, the canonical ensemble, the grand canonical ensemble, and their asymptotic behaviors. The normalization is arbitrary.

Obviously, the deviations from asymptotic behavior in the transition region depend on the form of the transverse-momentum cutoff. For example, for a slow cutoff with large $\langle p_\perp \rangle$, the isotropic behavior of the phase space will dominate in a larger energy range and therefore $\bar{n}$ will indeed increase like a fractional power of W. However, as seen from Fig. 5, such a cutoff would require very large $\langle p_\perp \rangle$ and is therefore physically uninteresting.

V. CONCLUSION AND DISCUSSION

To conclude, we have studied the general properties of multiparticle phase space and numerically checked some approximate formulas. For the iso-

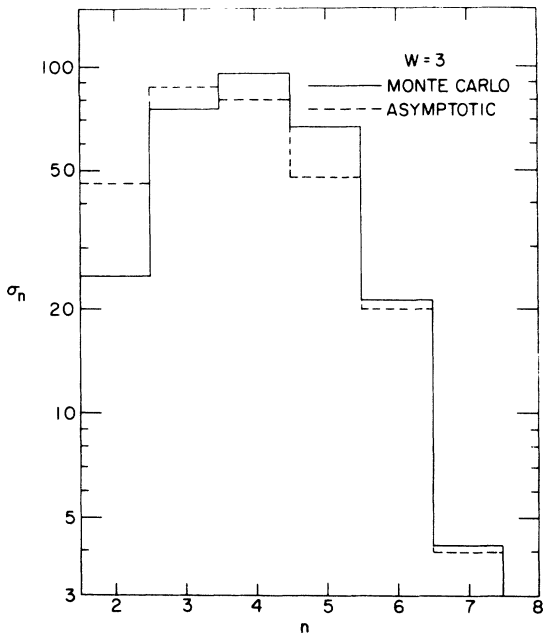

FIG. 7. Comparison of the multiplicity distribution expected from the actual phase space and from the Poisson distribution given by the asymptotic behavior. These two distributions are normalized to the same total cross section at a fixed $W = 3$.

tropic phase space, the approximate formula in Ref. 8 is in excellent agreement with the Monte Carlo result. For the transverse-momentum cutoff phase space, the lowest-order statistical formula in Ref. 1 gives a good qualitative description but the discrepancy can be as large as 50% at intermediate energies and for moderate values of n. The effects of the phase space on multiparticle processes were discussed within the statistical model and shown to be in qualitative agreement with the observed data. A comparison between the canonical and grand canonical ensemble models was made, and these two models were shown to be equivalent for the description of $\bar{n}$ even at relatively low energies.

Obviously, both kinematical and dynamical effects are important, and phase space alone will not be able to give an accurate description of the observed data. For example, in proton-proton collision processes, the leading-proton spectrum is very much different from the secondary-pion spectrum. Such effects should be definitely taken into account. For example, a leading-particle matrix element can be input to the phase-space integral (3.1).[5] At each step, more dynamical information is incorporated toward the understanding of the observed phenomena. In this work, we have limited ourselves to the study of some "kinematical" aspects which may be expected to have an important role in more detailed dynamical models.

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