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Coulomb-Nuclear Interference*

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The phase difference between strong and electromagnetic contributions to the scattering amplitude is derived from diffraction theory. The results are consistent with relativistic calculations. Bethe's phase is shown also to be consistent with relativistic calculations. New expressions for the amplitude are obtained to all orders in $Z_1Z_2e^2/\hbar v$ for the point-charge and extended-charge cases. Cross sections calculated from the original expression and from the point-charge solution are compared with those of the extended-charge solution.

When charged strongly interacting systems collide, both strong and electromagnetic interactions contribute to the scattering amplitude. Bethe[1] showed that with certain assumptions the modulus of the amplitude $F(q)$ for elastic scattering of a point unit charge by a nucleus of charge Z may be approximated by

$$|F(q)| = |f_s(q) + f_e(q)e^{in\phi}| \tag{1}$$

where f_s is the strong-interaction amplitude, f_e is the electromagnetic amplitude apart from its phase, $n = Ze^2/\hbar v$, $\hbar\vec{q}$ is the momentum transfer, and the phase ϕ is given by

$$\phi \approx -2\ln(aq/1.06)\,. \tag{2}$$

Both f_s and the form factor for the charge distribution were taken to be proportional to $\exp(-\frac{1}{4}a^2q^2)$.

Subsequently, West and Yennie,[2,3] using relativistic methods, derived for the phase ϕ the expression

$$\phi \approx -2\ln(a'q) - \gamma\,, \tag{3}$$

where $\gamma = 0.577\ldots$ is Euler's constant and

$$a' = \tfrac{1}{2}(a^2 + c^2 + d^2)^{1/2}\,. \tag{4}$$

It was assumed that the charge form factors of the target and incident particles are $\exp(-\frac{1}{4}c^2q^2)$ and $\exp(-\frac{1}{4}d^2q^2)$, respectively, and that f_s is proportional to $\exp(-\frac{1}{4}a^2q^2)$.

We point out that no reinterpretation of a in Eq. (2) is needed in order that Eq. (2) be consistent with Eq. (3). Bethe assumed $d = 0$ (point charge for the incident particle) *and* $c = a$. Therefore from Eq. (4) $a' = a/\sqrt{2}$ and from Eq. (3)

$$\phi \approx -2\ln(aq/\sqrt{2}) - \gamma$$
$$\approx -2\ln(aq/1.06)\,,$$

which is Bethe's formula Eq. (2).[4]

Using relativistic methods West and Yennie also showed[2] that for point charges and arbitrary f_s, ϕ is given by

$$\phi = -2\ln\sin\tfrac{1}{2}\theta - 2\int_0^{2k} \frac{[1 - f_s(q')/f_s(q)]q'}{|q^2 - q'^2|}\,dq'\,. \tag{5}$$

Equation (5) can be derived from diffraction theory. We use techniques similar to those of the Glauber approximation.[5] The scattering amplitude is given by[5]

$$F(\vec{q}) = \frac{ik}{2\pi}\int e^{i\vec{q}\cdot\vec{b}}(1 - e^{i\chi(\vec{b})})d^2b\,, \tag{6}$$

where $\hbar\vec{k}$ is the incident momentum and $\vec{b}$ is an impact-parameter vector. To exhibit the effects of the Coulomb interaction we write[6]

$$\chi(\vec{b}) = \chi_c(b) + \chi_s(b)\,, \tag{7}$$

where $\chi_c(b)$ is due to the Coulomb interaction alone and $\chi_s(b)$ is due to the strong interaction alone. The purely strong and purely Coulombic amplitudes f_s and f_c are given by expressions similar to Eq. (6),

$$f_j(q) = \frac{ik}{2\pi}\int e^{i\vec{q}\cdot\vec{b}}(1 - e^{i\chi_j(b)})d^2b\,,\quad j = s, c\,. \tag{8}$$

Using Eqs. (6)–(8) together with the identity

$$1 - e^{i(\chi_c+\chi_s)} = (1 - e^{i\chi_c}) + (1 - e^{i\chi_s}) - (1 - e^{i\chi_c})(1 - e^{i\chi_s})\,, \tag{9}$$

we obtain

$$F(q) = f_c(q) + f_s(q) - \frac{ik}{2\pi}\int e^{i\vec{q}\cdot\vec{b}}(1 - e^{i\chi_c(b)})(1 - e^{i\chi_s(b)})d^2b\,. \tag{10}$$

If we invert Eq. (8) with $j = s$, we obtain

$$1 - e^{i\chi_s(b)} = (2\pi i k)^{-1} \int e^{-i\vec{q}\cdot\vec{b}} f_s(q) d^2q\,, \tag{11}$$

and the amplitude F becomes

$$F(q) = f_c(q) + f_s(q) + (2\pi)^{-1} \int f_s(q') J_0(b|\vec{q} - \vec{q}'|)(e^{i\chi_c(b)} - 1) b\,db\, d^2q'\,. \tag{12}$$

For a screened Coulomb field with screening radius R and for $q \gg R^{-1}$, we may write[5]

$$\begin{aligned} \chi_c(b) &= 2n \ln(b/2R)\,, \quad b < R \\ &= 0\,, \qquad\qquad\quad\; b > R\,. \end{aligned} \tag{13}$$

Since R is of atomic dimensions, Eq. (13) will be valid for $\hbar^2 q^2 \gg 10^{-11}$ $(\text{GeV}/c)^2$. Furthermore,[5]

$$f_c(q) = -2nkq^{-2} \exp\{-2i[n \ln(qR) - \arg\Gamma(1+in)]\}\,. \tag{14}$$

For $n \ll 1$, Eq. (12) with Eq. (13) becomes[7]

$$F(q) \approx \lim_{R\to\infty} \left[f_c(q) + f_s(q) + in\pi^{-1} \int f_s(q') d^2q' \int_0^R J_0(|\vec{q} - \vec{q}'|b) \ln(b/2R) b\,db \right] \tag{15}$$

$$\approx \lim_{R\to\infty} \left[f_c(q) + f_s(q) \exp\left(-in\pi^{-1} \int \frac{d^2q'}{|\vec{q} - \vec{q}'|^2} \frac{f_s(q')}{f_s(q)} [1 - J_0(R|\vec{q} - \vec{q}'|)] - 2in \ln 2 \right) \right]\,. \tag{16}$$

Therefore the phase difference ϕ is

$$\phi = -2 \ln \sin \tfrac{1}{2}\theta - 2\gamma - \lim_{R\to\infty} \left\{ 2 \ln kR + \frac{1}{\pi} \int_{q'=0}^{q'=2k} \frac{d^2q'}{|\vec{q} - \vec{q}'|^2} \left[\left(1 - \frac{f_s(q')}{f_s(q)} \right) + \left(J_0(R|\vec{q} - \vec{q}'|) \frac{f_s(q')}{f_s(q)} - 1 \right) \right] \right\}. \tag{17}$$

We have assumed, as is usual in high-energy diffraction theory,[5] that $f_s(q)$ is strongly peaked in the forward direction and that the q' integration is taken over the physical region $q' \leqslant 2k$. For $(q/2k)^2 \ll 1$, we obtain[7]

$$\lim_{R\to\infty} \pi^{-1} \int_0^{2k} \frac{d^2q'}{|\vec{q} - \vec{q}'|^2} \left[1 - J_0(R|\vec{q} - \vec{q}'|) \frac{f_s(q')}{f_s(q)} \right] \approx \lim_{R\to\infty} \pi^{-1} \int_0^{2kR} \frac{d^2q'}{q'^2} [1 - J_0(q')] \tag{18}$$

$$= \lim_{R\to\infty} [2 \ln(kR) + 2\gamma]\,. \tag{19}$$

If we perform the angular integration on the remaining integral in Eq. (17) we obtain Eq. (5).

We have thus demonstrated that the results obtained by relativistic methods [which are calculated to $O(n)$] and the results of diffraction theory (which are calculated to all orders in n) agree to $O(n)$. For scattering of hadrons by heavy nuclei, or for high-energy heavy-ion experiments being performed,[8] the parameter $n = Z_1 Z_2 e^2/\hbar v$ is not small (even at high energies) and calculations good to first order in n will be inaccurate. Let us therefore use diffraction theory to calculate the complete amplitude in such cases. We shall assume that the charge form factors of the target and incident nuclei are $\exp(-\frac{1}{4}c^2q^2)$ and $\exp(-\frac{1}{4}d^2q^2)$, respectively, and that the incident-nucleus-target strong-interaction amplitude is[9]

$$f_s(q) = f_0 \exp(-\tfrac{1}{4}a^2q^2)\,.$$

We find that

$$\chi_c(b) = \chi_c^{\text{pt}}(b) + nE_1(b^2/r^2)\,, \tag{20}$$

where $r^2 = c^2 + d^2$, $\chi_c^{\text{pt}}(b)$ is the Coulomb phase-shift function for a point charge, and $E_1(x) = -\text{Ei}(-x)$, where Ei is the exponential integral. There are many ways of separating the Coulomb and nuclear effects and of separating the point-charge and charge-distribution effects. The final result may be written, for example, as

$$\begin{aligned} |F(q)| = \Bigg| & F^{\text{pt}}(q) + ikr^2 e^{2in\ln(r/a)} \int_0^\infty J_0(rqx) x^{2in+1} [1 - e^{inE_1(x^2)}] dx \\ & + (2r^2 f_0/a^2) e^{2in\ln(r/a)} \int_0^\infty J_0(rqx) [e^{inE_1(x^2)} - 1] x^{2in+1} e^{-r^2x^2/a^2} dx \Bigg|\,, \end{aligned} \tag{21}$$

where $F^{\text{pt}}(q)$ is the point-charge solution[10]

$$F^{\text{pt}}(q) = -2nkq^{-2} \exp\{-2i[n \ln(\tfrac{1}{2}aq) - \arg\Gamma(1+in)]\} + f_s(q)\Gamma(1+in)\,{}_1F_1(-in; 1; \tfrac{1}{4}a^2q^2) \tag{22}$$

in which ${}_1F_1$ is the confluent hypergeometric function.

As an illustration, we consider the case $d=0$, $c=a$ [so that Eqs. (2) and (3) are identical]. We compare in Fig. 1 cross sections near the Coulomb interference region obtained from (i) Eq. (1) with ϕ given by Eqs. (2) or (3), and (ii) $F(q)=F^{\text{pt}}(q)$, with those obtained from the "exact" solution Eq. (21). The parameters a and f_0 were chosen to correspond approximately to appropriate strong-interaction data. Their precise values are unimportant since we are comparing theoretical expressions.

We see from Fig. 1 that with Eq. (2) or (3) errors in the cross sections of ~20% occur for p-Pb scattering, ~10% for C-C scattering, ~1% for p-C and π^--C scattering, ~0.2% for α-α scattering, and ~0.03% for p-p scattering.

It is interesting to note that the point-charge solution Eq. (22) is generally no more accurate (and often less accurate) than Eq. (2) or (3), even at very small angles.

In obtaining his result, Bethe made the approximations that the average Coulomb phase shift of the nuclear scattering be evaluated at $q=0$ and that the calculation neglect terms of $O(n^2)$. The remarkable accuracy of Bethe's result for p-p, α-α, π-C, and p-C scattering stems in part from the compensating effects of these two approximations. Figure 1 shows that neglecting the q dependence of the phase generally induces an error opposite in sign to that induced by neglecting terms which are of higher order in n. (The dotted and solid curves generally lie on opposite sides of the $-t$ axis.) We also see that for p-C, π-C, α-α, and p-p scattering, retention of the q dependence of the phase (dotted curve) generally increases the accuracy of the results significantly. It is straightforward to modify Eq. (2) to take the q dependence into account by means of a simple rapidly converging power series in q^2. However, for $n \gtrsim 0.1$, the q dependence modification still yields inaccurate results. It is clear from our calculations that for collisions in which $n \gtrsim 0.1$, Eq. (21) should be used in analyses of data in the Coulomb-nuclear interference region.

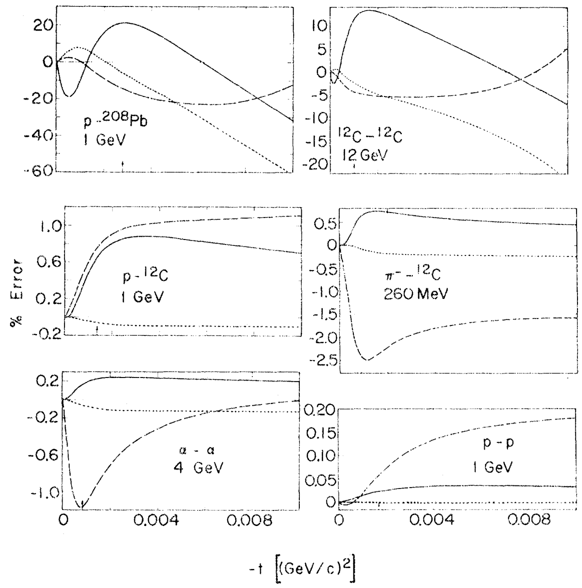

FIG. 1. Percentage error in calculated cross sections when compared with the "exact" solution Eq. (21), as a function of the squared momentum transfer. The solid curves compare cross sections calculated using the phase given by Bethe and by West and Yennie, Eqs. (2) and (3), with cross sections obtained from Eq. (21). The dashed curves compare the point-charge solution, Eq. (22), with Eq. (21). The dotted curves are obtained by modifying Bethe's calculation to take into account the q dependence of the average Coulomb phase shift of the nuclear scattering. The arrows indicate the t values for which Coulomb and strong-interaction contributions to the cross section are approximately equal.

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