Improved Optimal Bounds Using the Watson Theorem

M. Micu
Institute of Atomic Physics, P. O. Box 35, Bucharest, Romania
(Received 11 July 1972)

A method is presented to find an optimal lower bound on the quadratic functional integrals which appear in the derivation of bounds on the hadronic contribution to the muon magnetic moment, or on the K_{l3} form factors, when one knows that the function and its derivative take some definite values at the origin *and* that its phase is given on a part of the unitarity cut.

I. INTRODUCTION

Recently different authors established rigorous bounds for the hadronic contribution to the muon magnetic moment[1] and for some form factors of weak currents[2] in terms of certain input information taken as known. From the mathematical point of view the problem is the same for all the bounds quoted here, namely, to find the lower bound of the quadratic functional integral

$$I=\frac{1}{\pi}\int_1^\infty p(x)\,|f(x)|^2 dx \tag{1}$$

under certain conditions. The form factor $f(x)$ is an analytic function in the complex plane cut from 1 to ∞. The weight function $p(x)$ is non-negative on the cut and has the form

$$p(x)=x^{-7/2}(x-1)^{3/2}\int_0^1 \frac{u^2(1-u)du}{u^2+4(m_\pi{}^2/m_\mu{}^2)x(1-u)} \tag{2}$$

in the case of the hadronic contribution to the muon magnetic moment, and

$$p(x)=x^{-2}(x-1)^{1/2}\left[x-\left(\frac{m_K-m_\pi}{m_K+m_\pi}\right)^2\right]^{1/2} \tag{3}$$

in the case of the K_{l3} decay form factors.

The input information was represented by a set of values taken by the function $f(x)$ or (and) its derivatives at some points x_i. The purpose of this paper is to study how one can improve the lower bound of the integral I when one enlarges the input information by including the phase of $f(x)$ as known on a part of the cut. In fact, from the Watson theorem[3] the phase of the form factor $f(x)$ is equal (modulo π) to the phase of the corresponding elastic scattering from the elastic up to the inelastic threshold. This problem has already been the subject of simple consideration by Micu and Radescu[4] for a bound on the parameter $d'(0)$ (the derivative at zero momentum transfer of the form factor of the strangeness-changing vector current divergence) from K_{l3} decay and by Auberson and Li[5] for a lower bound of the hadronic contribution to the muon magnetic moment.

II. DERIVATION OF THE RESULTS

To obtain the lower bound of the integral expressed by Eq. (1), we shall first transform the complex cut plane into the interior of the unit circle $|z|=1$ through the conformal transformation

$$z=-\frac{(x-1)^{1/2}-i}{(x-1)^{1/2}+i}\ .$$

For x on the cut one has

$$z=e^{i\theta}\,,$$

$$\tan\tfrac{1}{2}\theta=(x-1)^{1/2}\,.$$

The integral I becomes

$$I=\frac{1}{2\pi}\int_{-\pi}^{\pi} q(\theta)\,|G(e^{i\theta})|^2 d\theta\,, \tag{4}$$

where

$$q(\theta)=x\,|(x-1)^{1/2}|\,p(x)\,,$$

$$G(e^{i\theta})=f(x)\,.$$

The new function $G(z)$ is analytic inside the unit circle $|z|=1$. The form of the integral of Eq. (4) can be simplified again if we take into account the function

$$D(z)=\exp\left[\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln|q(\theta)|\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\theta\right], \tag{5}$$

which is analytic for $|z|<1$, has no zeros, and satisfies almost everywhere the boundary condition

$$|D(e^{i\theta})|^2=q(\theta)\,.$$

Then

$$I=\frac{1}{2\pi}\int_{-\pi}^{\pi}|F(e^{i\theta})|^2 d\theta\,, \tag{6}$$

where the function

$$F(z)=D(z)G(z) \tag{7}$$

is also analytic inside the unit circle $|z|=1$.

Now we must obtain the optimal lower bound of the integral I given by Eq. (6) under the following conditions:

(a) The function $F(z)$ has the value $F(0)$ in the point $z=0$,

(b) The phase of the function $F(e^{i\theta})$ is $\Phi(\theta)$ for $|\theta|$ less than a certain α.

We note that the phase of the function $D(z)$ explicitly given by Eq. (5) can be immediately computed in the needed region and so, once the phase of $G(e^{i\theta})=f(x)$ is known from the Watson theorem for $|\theta|<\alpha$, the phase $\Phi(\theta)$ of the function $F(e^{i\theta})$ is known as well for $|\theta|<\alpha$.

Other optimal lower bounds can also be looked for when the condition (a) is replaced by

(a′) the function $F(z)$ and its first derivative take the values $F(0)$ and $F'(0)$, respectively, at the point $z=0$, or

(a″) the derivative of the function $F(z)$ is $F'(0)$ at the point $z=0$.

The function $F(z)$ can be developed into a series of orthogonal polynomials $p_n(z)=z^n$,

$$F(z)=F(0)+\sum_{n=1}^{\infty}F_n z^n \ , \tag{8}$$

where the first term of the series, F_0, is set equal to $F(0)$ according to the condition (a). The remaining coefficients F_n are not arbitrary, but subject to the condition (b):

$$e^{-i\Phi(\theta)}F(e^{i\theta})=e^{i\Phi(\theta)}[F(e^{i\theta})]^* , \tag{9}$$

for $|\theta|<\alpha$. It is convenient to write the condition (9) in the form

$$-i\sum_{n=1}^{\infty}F_n\rho^n e^{-i[\Phi(\theta)-n\theta]}+i\sum_{n=1}^{\infty}F_n^*\rho^n e^{i[\Phi(\theta)-n\theta]}=2F(0)\sin\Phi(\theta) . \tag{10}$$

The limit $\rho\to 1$ is introduced here in order to handle a Poisson kernel which will appear later.

Due to the orthogonality relation of the polynomials $p_n(z)=z^n$ on the unit circle, the integral (6) can be rewritten as

$$I=|F(0)|^2+\sum_{n=1}^{\infty}|F_n|^2 . \tag{11}$$

The extremum (minimum) of this expression can be obtained using the Lagrange technique which takes into account the condition (10). One obtains for the extremal coefficients the values

$$F_n=i\frac{F(0)}{\pi}\rho^n\int_{-\alpha}^{\alpha}\lambda(\theta)e^{i[\Phi(\theta)-n\theta]}d\theta , \tag{12}$$

where the Lagrange multiplier function $\lambda(\theta)$ is real and satisfies

$$\lambda(\theta)=-\lambda(-\theta) \tag{13}$$

(because $F(z)$ is a real analytic function $F(z)=[F(z^*)]^*$). The function $\lambda(\theta)$ can be obtained by introducing the parameters F_n into the condition (10). After the summation of the series one obtains

$$\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\lambda(\theta')e^{i[\Phi(\theta')-\Phi(\theta)]}\frac{\rho e^{-i(\theta'-\theta)}}{1-\rho e^{-i(\theta'-\theta)}}d\theta'+\text{c.c.}=\sin\Phi(\theta) . \tag{14}$$

Separating here the Poisson kernel, one finds in the limit $\rho=1$ an integral equation of the Fredholm type which must be satisfied by the function $\lambda(\theta)$,

$$\lambda(\theta)+\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\lambda(\theta')\frac{\sin[\Phi(\theta')-\Phi(\theta)-\frac{1}{2}(\theta'-\theta)]}{\sin\frac{1}{2}(\theta'-\theta)}d\theta'=\sin\Phi(\theta) . \tag{15}$$

The extremal value $I(F(0),\Phi)$ of the right-hand side of Eq. (11) is obtained upon introducing there the coefficients (12), summing the series, and taking into account that $\lambda(\theta)$ satisfies Eq. (14). The result is

$$I(F(0),\Phi)=|F(0)|^2\left[1+\frac{1}{\pi}\int_{-\alpha}^{\alpha}\lambda(\theta)\sin\Phi(\theta)d\theta\right] . \tag{16}$$

In a similar way, when the condition (a) is replaced by the condition (a′) or (a″), one obtains the following extremals:

$$I(F(0),F'(0),\Phi)=|F(0)|^2+|F'(0)|^2+\frac{1}{\pi}\int_{-\alpha}^{\alpha}[F(0)\lambda_1(\theta)+F'(0)\lambda_2(\theta)][F(0)\sin\Phi(\theta)+F'(0)\sin(\Phi(\theta)-\theta)]d\theta , \tag{17}$$

$$I(F'(0),\Phi)=|F'(0)|^2\left[1+\frac{1}{\pi}\int_{-\alpha}^{\alpha}\lambda_3(\theta)\sin(\Phi(\theta)-\theta)d\theta\right] , \tag{18}$$

where the functions λ_1, λ_2, and λ_3 satisfy the following integral equations:

$$\lambda_1(\theta)+\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\lambda_1(\theta')\frac{\sin[\Phi(\theta')-\Phi(\theta)-\frac{3}{2}(\theta'-\theta)]}{\sin\frac{1}{2}(\theta'-\theta)}\,d\theta'=\sin\Phi(\theta)\,, \tag{19}$$

$$\lambda_2(\theta)+\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\lambda_2(\theta')\frac{\sin[\Phi(\theta')-\Phi(\theta)-\frac{3}{2}(\theta'-\theta)]}{\sin\frac{1}{2}(\theta'-\theta)}\,d\theta'=\sin[\Phi(\theta)-\theta]\,, \tag{20}$$

$$\lambda_3(\theta)+\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\lambda_3(\theta')\left\{\frac{\sin[\Phi(\theta')-\Phi(\theta)-\frac{3}{2}(\theta'-\theta)]}{\sin\frac{1}{2}(\theta'-\theta)}+2\cos[\Phi(\theta')-\Phi(\theta)]\right\}d\theta'=\sin[\Phi(\theta)-\theta]\,. \tag{21}$$

Finally, concerning the applications of the results previously obtained to the physical problems already enunciated in the Introduction, we recall that

$$\delta a_\mu \geq \frac{1}{48\pi^2}\left(\frac{e^2}{\hbar c}\right)^2\frac{m_\mu^{\,2}}{m_\pi^{\,2}}I\,, \tag{22}$$

where δa_μ is the hadronic contribution to the muon magnetic moment, $f(x)$ from the integrand of I being the electromagnetic form factor of the π meson (the momentum transfer given in units of $4m_\pi^{\,2}$) with $f(0)=1$, and

$$\Delta(0)\geq\frac{3}{64\pi^2}(m_K^{\,2}-m_\pi^{\,2})^2 I\,, \tag{23}$$

where $\Delta(0)$ is the value at zero momentum transfer of the propagator of the divergence of the weak strangeness-changing vector current, which in the model from Ref. 6 takes on the value

$$\Delta(0)=(1.01 f_\pi m_\pi)^2\,. \tag{24}$$

Now $f(x)$ in the integral I is the K_{l3} form factor of the divergence of the weak strangeness-changing vector current, normalized in the SU_3 symmetry limit as

$$f(0)\equiv f_+(0)=1\,.$$

The momentum transfer on which the form factor depends is measured in units of $(m_K+m_\pi)^2$. In the inequalities (22) and (23) I represents one of the extremals derived before with the weight $p(x)$ given by Eqs. (2) and (3), respectively.

The elastic-scattering phase shift involved in the calculations are $\delta(l=1, T=1)$ for $\pi\pi$ scattering from the elastic threshold $t=4m_\pi^{\,2}$ up to the inelastic one $t=16m_\pi^{\,2}$ and $\delta(l=0, T=\frac{1}{2})$ for πK scattering from the elastic threshold $t=(m_K+m_\pi)^2$ up to the inelastic one $t=(m_K+3m_\pi)^2$. The angle α corresponding to these two inelastic thresholds are $\alpha=120°$ and $\alpha\approx 91°$, respectively. Practically, for computations, the angle α can be taken somewhat larger than the actual values (as done in Ref. 5). In this way the bounds can be numerically improved, the inelasticity not being expected to introduce great errors.

[1]D. Palmer, Phys. Rev. D 4, 1558 (1971); I. Raszillier, Lett. Nuovo Cimento 2, 349 (1971); P. Langacker and M. Suzuki, Phys. Rev. D 4, 2160 (1971).

[2]L.-F. Li and H. Pagels, Phys. Rev. D 4, 255 (1971); S. Okubo, *ibid.* 4, 725 (1971); M. Micu, Nucl. Phys. B44, 531 (1972).

[3]K. M. Watson, Phys. Rev. 88, 1163 (1952).

[4]M. Micu and E. E. Radescu, Phys. Rev. D 6, 1943 (1972).

[5]G. Auberson and L.-F. Li, Phys. Rev. D 5, 2269 (1972).

[6]V. S. Mathur and S. Okubo, Phys. Rev. 175, 2195 (1968).