Decay $\eta' \to \eta\pi\pi$ as a Test of Chiral Symmetry Breaking

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The width Γ of the decay $\eta' \to \eta\pi\pi$ is estimated in the (8, 8) model of chiral symmetry breaking. Under plausible additional assumptions we find $\Gamma > 0.27$ MeV together with an experimentally uninteresting upper limit. In particular, the (8, 8) model is – at least under present extrapolation errors in the Dalitz plot of the decay – consistent with any Γ above our lower limit and below the experimental upper limit $\Gamma < 2.7$ MeV. The actual width may be expected to be around 1 MeV in perfect agreement with the (8, 8) model. Our assumptions include that the dimension l_u of the chiral-symmetry-breaking Hamiltonian density has the particularly attractive value 2. For this value, the conventional $(3, \bar{3}) \oplus (\bar{3}, 3)$ model predicts $\Gamma < 0.065$ MeV. Despite the large extrapolation errors the decay $\eta' \to \eta\pi\pi$ therefore clearly distinguishes between the two models and might be decisive. For any given extrapolation the estimated Γ of the (8, 8) model is at least 13 times bigger than the prediction of the $(3, \bar{3}) \oplus (\bar{3}, 3)$ model for $l_u = 2$. For $l_u = 3$ a prediction can also be obtained in the $(3, \bar{3}) \oplus (\bar{3}, 3)$ model with a c-number $\dot{\sigma}$. It reads $\Gamma < 0.25$ MeV and is roughly compatible with the expected width. The $\eta'(958)$ is taken to be a possible ninth pseudoscalar meson throughout. It is however emphasized that nonexistence of an SU(3) scalar meson might be a plausible alternative in the (8, 8) model.

Recently Riazuddin and Oneda[1] and Weisz, Riazuddin, and Oneda[2] have estimated the width Γ of the decay $\eta' \to \eta\pi\pi$ in the $(3, \bar{3}) \oplus (\bar{3}, 3)$ model of chiral symmetry breaking. The present paper is devoted to an estimate of Γ in the competing chiral-symmetry-breaking scheme,[3-9] the (8, 8) model. For reasons to become clear later, we shall restrict ourselves to $l_u = 2$ [with l_u the dimension of the chiral-symmetry-breaking Hamiltonian density $u(x)$]. For this value, the prediction of the $(3, \bar{3}) \oplus (\bar{3}, 3)$ model is $\Gamma < 0.065$ MeV, which is considered to be unacceptably small.[1,2,10,11] In the (8, 8) model for $l_u = 2$, we predict a value of Γ around the expected 1 MeV. Details of our results are given at the end of the paper.

We mention already at this point that the (8, 8) model contains no pseudoscalar SU(3)-singlet operator.[12] This might suggest that the (8, 8) predicts that there should also not be a pseudoscalar SU(3)-singlet particle. Whereas this attitude is certainly not excluded experimentally, we attempt here a conventional treatment of the mixing problem. In doing so, we find that two possible attractive assumptions are contradictory in the (8, 8). These are (I) the SU(3) formula

$$\langle \eta_0 | S_0 | \eta_8 \rangle = 0$$

[with S_0 the scalar SU(3)-scalar density of the model] and (II) validity of the soft-pion limit for the matrix elements

$$\langle \eta_0 | S_S^{\alpha\beta} | \pi(q) \rangle$$

(with $S_S^{\alpha\beta}$ the positive-parity operators of the model).

The incompatibility of assumptions I and II will be derived in the paper [Eq. (21)]. The basic reason for this incompatibility is the nonexistence of a pseudoscalar SU(3)-scalar field for the η_0. Namely, the pseudoscalar operators $S_A^{\alpha\beta}$ of the model transform according to $\underline{8} + \underline{10} + \overline{\underline{10}}$ under SU(3). This implies that $f_{8\rho\sigma} S_A^{\rho\sigma}$ (i.e., the eight component of the 8) is the *only* $I = Y = 0$ pseudoscalar operator of the (8, 8). Thus

$$\langle \eta_0 | S_A^{\alpha\beta} | \Omega \rangle \equiv 0$$

if and only if

$$\langle \eta_0 | f_{8\rho\sigma} S_A^{\rho\sigma} | \Omega \rangle = 0 .$$

The soft-pion limit applied to $\langle \eta_0 | S_S^{\alpha\beta} | \pi(q) \rangle$ evidently leads to a linear combination of the form

$$c^{\alpha\beta}_{\rho\sigma}\langle\eta_0|S^{\rho\sigma}_A|\Omega\rangle = c^{\alpha\beta} f_{8\rho\sigma}\langle\eta_0|S^{\rho\sigma}_A|\Omega\rangle$$

for any α and β. In particular, the soft-pion limit applied to $\langle\eta_0|\hat{S}_\Theta|\pi(q)\rangle$ [with $\hat{S}_\Theta$, $\Theta = 1-27$, the positive-parity $\underline{27}$ component of the (8, 8)] yields a nonvanishing proportionality constant c^Θ. According to a reliable SU(3) formula,

$$\langle\eta_0|\hat{S}_\Theta|\pi(q)\rangle = 0,$$

and thus, assuming the soft-pion limit to be valid, we have

$$\langle\eta_0|f_{8\rho\sigma}S^{\rho\sigma}_A|\Omega\rangle = 0,$$

and in turn

$$\langle\eta_0|S^{\alpha\beta}_A|\Omega\rangle = 0.$$

On the other hand, we shall see in the paper that $\langle\eta_0|S_0|\eta_8\rangle = 0$ implies $\langle\eta_0|S_\alpha|\pi\rangle \neq 0$ (with S_α, $\alpha = 1-8$, the symmetric 8), which, since $\langle\eta_0|S_\alpha|\pi\rangle$ is proportional to $c^\alpha\langle\eta_0|f_{8\rho\sigma}S^{\rho\sigma}_A|\Omega\rangle$ in the soft-pion limit, would then imply that $\langle\eta_0|S^{\alpha\beta}_A|\Omega\rangle \neq 0$. Therefore, assumptions I and II are contradictory. The only doubtful assumption that went into this derivation is the formula $\langle\eta_0|S_0|\eta_8\rangle = 0$ [the further SU(3) assumptions are found to be valid later on in the paper] and thus we have investigated assumptions I and II separately.

The prediction for Γ in each of the two cases is different, and thus the decay $\eta' \to \eta\pi\pi$ might furthermore be useful to distinguish between two different possible schemes of (8, 8) chiral symmetry breaking. The present experimental information[13] consists in the upper limit $\Gamma < 2.7$ MeV. This agrees with both the $(3, \overline{3}) \oplus (\overline{3}, 3)$ and the (8, 8) for $l_u = 2$. Within the (8, 8) it only implies weak restrictions. A prediction closer to the expected value and to the experimental upper limit $\Gamma < 2.7$ MeV has been obtained[2] in the $(3, \overline{3}) \oplus (\overline{3}, 3)$ model with a c-number δ for $l_u = 3$. It is $\Gamma < 0.25$ MeV. We shall emphasize below that to us $l_u = 2$ still appears to be a particularly attractive choice. The arguments are essentially those of Ref. 14. We would like to make two additional introductory remarks.

First, we never consider any soft-meson limit except the soft-pion limit. Instead, our additional theoretical inputs are dimensional arguments and $l_u = 2$. This is partly as in Ref. 2. Our assumptions would be perfectly unambiguous in the $(3, \overline{3}) \oplus (\overline{3}, 3)$. In the (8, 8) they lead to the one ambiguity connected with $\eta\eta'$ mixing discussed above. This ambiguity has not prevented us from obtaining a result in clear distinction from the $(3, \overline{3}) \oplus (\overline{3}, 3)$ model. Present information from meson-baryon scattering incidentally does not imply a definite distinction.[15-18]

Second, we adopt the linear Dalitz-plot extrapolation of Refs. 2 and 10. The errors of this linear extrapolation are always taken into account. We have no convincing way to discuss the additional error coming from the assumed linearity. Reference 2 expects about 50% in the final result. If the error is at most 50%, none of our conclusions is affected in any essential way.

It was shown in Refs. 1, 2, and 10 by standard current-algebra techniques that Γ can be written as

$$\Gamma = \xi \left| \frac{2}{f_\pi^{\,2}} \sigma_{\eta\eta'} \right|^2 . \tag{1}$$

In the above, $f_\pi = 0.95\, m_\pi$ and

$$\sigma_{\eta\eta'} = \tfrac{1}{3} i \left\langle \eta \left| \sum_{\alpha=1}^{3} [Q^\alpha_A, \partial^\mu A^\alpha_\mu(0)] \right| \eta' \right\rangle . \tag{2}$$

The numerical value of ξ depends on the slope in the Dalitz plot of the decay. Including the errors quoted in Refs. 2 and 10 we have

$$9 \text{ keV} < \xi = 15 \text{ keV} < 28 \text{ keV}. \tag{3}$$

We start by considering $\eta\eta'$ mixing under our assumptions and write $(p^2 + q^2 = 1)$

$$\begin{aligned} |\eta\rangle &= p|\eta_8\rangle + q|\eta_0\rangle , \\ |\eta'\rangle &= q|\eta_8\rangle - p|\eta_0\rangle \end{aligned} \tag{4}$$

for the SU(3)-singlet and -octet components $|\eta_0\rangle$ and $|\eta_8\rangle$ of the $|\eta\rangle$ and $|\eta'\rangle$. The physical states $|\eta\rangle$ and $|\eta'\rangle$ are eigenstates of the Hamiltonian such that $\langle\eta|H(0)|\eta'\rangle = 0$. We shall also use

$$\langle\eta|T^\mu_\mu(0)|\eta'\rangle = 0 \tag{5}$$

with T^μ_μ the trace of the new and improved energy-momentum tensor. This plausible relation can also be *derived* from $\langle\eta|H(0)|\eta'\rangle = 0$ together with $\langle a|H(0)|a\rangle = \langle a|T^\mu_\mu|a\rangle = 2m_a^{\,2}$ for $a = \eta, \eta', \eta_8$, and η_0. As usual, $m_{\eta 8}^{\,2}$ is given by the Gell-Mann–Okubo mass formula for the pseudoscalar octet. Considering

$$\langle\eta_0|T^\mu_\mu|\eta_0\rangle \pm \langle\eta_8|T^\mu_\mu|\eta_8\rangle ,$$

it is then easy to compute $|pq|$. We find (Refs. 1 and 2 have $|pq| = 0.21$)

$$|pq| = 0.17. \tag{6}$$

Furthermore,

$$\langle\eta_0|T^\mu_\mu|\eta_8\rangle = 2pq(m_\pi^{\,2} - m_{\eta'}^{\,2}). \tag{7}$$

The 64 scalar and pseudoscalar operators $S^{\alpha\beta}$ (definition: $S^{\alpha\beta}_S = S^{\alpha\beta} + S^{\alpha\beta}$ for $\alpha, \beta = 1-8$) of the (8, 8) model obey SU(3)$\otimes$SU(3) transformation properties as noted in several places. The positive-parity SU(3) singlet, octet, and $\underline{27}$ components of the S are

$$S_0 = \sum_{\alpha=1}^{8} S_S^{\alpha\alpha}\,, \tag{8a}$$

$$S_\alpha = \sum_{\rho,\sigma=1}^{8} d_{\alpha\rho\sigma} S_S^{\rho\sigma}\,, \tag{8b}$$

and $(\Theta = 1-27)$

$$\hat{S}_\Theta = -\frac{20}{\sqrt{30}} \sum_{\rho,\sigma=1}^{8} \xi_{\Theta\rho\sigma} S_S^{\rho\sigma}\,. \tag{8c}$$

We need in particular the $I=0$, $Y=0$ component $\hat{S}_{27}$ to be computed from

$$-\frac{20}{\sqrt{30}}\,\xi_{27,\alpha\beta} = \delta_{\alpha\beta} - \tfrac{4}{3}\sum_{p=1}^{3} \delta_{\alpha p}\delta_{\beta p} - 4\delta_{\alpha 8}\delta_{\beta 8}\,. \tag{9}$$

We also define $S_A^{\alpha\beta}$ by $S_A^{\alpha\beta} = S^{\alpha\beta} - S^{\beta\alpha}$.

The chiral-symmetry-breaking Hamiltonian density of the (8, 8) model can be written as

$$u(x) = \frac{B}{2\sqrt{5}}\left[zS_0(x) + \sqrt{3}\,S_8(x)\right]. \tag{10}$$

The constant z has been estimated by several methods.[3,5,6] The result is consistently[19]

$$z = -\frac{1}{4}\,\frac{2m_K{}^2 + m_\pi{}^2}{m_K{}^2 - m_\pi{}^2} \approx -0.565\,. \tag{11}$$

Computing the σ term from the above we find for $\sigma_{\eta\eta'}$ the following expression:

$$\sigma_{\eta\eta'} = \left\langle \eta \left| \frac{3}{4\sqrt{5}}(z+\tfrac{1}{2})BS_0 + \frac{3\sqrt{3}}{5\sqrt{5}}(\tfrac{3}{2}+z)BS_8 + \frac{1}{20\sqrt{5}}(z-\tfrac{7}{2})BS_{27} \right| \eta' \right\rangle. \tag{12}$$

We shall assume in what follows that $u(x)$ has dimension 2, at least between the pseudoscalar-meson states. To motivate this value we note that only then does the soft-pion limit for $\langle \pi | T^\mu_\mu | \pi \rangle$ yield the correct result $2m_\pi{}^2$. It is furthermore known that only this value[20] allows SU(3) for the meson matrix elements of S_0, S_α, and $\sum_i (4-l_i)\delta_i = \delta$. We may also remark that in the attractive although more restrictive (8, 8) model proposed in Ref. 3, $u(x)$ has dimension 2. Note, however, that we in no way make the additional assumptions made in that reference. We know from our analysis of meson-baryon scattering data[7] that for $l_u = 2$ we need a q-number δ in the (8, 8). Thus we allow for a q-number δ and write $\delta = \sum_i \delta_i$ where the δ_i have dimensions l_i. It then follows that

$$T^\mu_\mu = \sum_i (4-l_i)\delta_i + 2u\,. \tag{13}$$

In order to compute $\sigma_{\eta\eta'}$ we may choose to eliminate $\langle \eta | S_0 | \eta' \rangle$ from Eq. (12) by use of Eq. (13). Due to Eq. (5) the result is then an expression for $\sigma_{\eta\eta'}$ as a linear function of the following matrix elements:

$$\langle \eta_0 | \sum_i (4-l_i)\delta_i | \eta_0 \rangle = 2\mu^2\,, \tag{14a}$$

with

$$0 \le \mu^2/m_\pi{}^2 \le m_{\eta_0}{}^2/m_\pi{}^2 \approx 46\,,$$

$$\langle \eta_0 | BS_8 | \eta_0 \rangle = 0\,, \tag{14b}$$

$$\langle \eta_0 | B\hat{S}_{27} | \eta_0 \rangle = 0\,, \tag{14c}$$

$$\langle \eta_8 | \sum_i (4-l_i)\delta_i | \eta_8 \rangle = 0\,, \tag{14d}$$

$$\langle \eta_8 | BS_8 | \eta_8 \rangle = \frac{4\sqrt{5}}{3\sqrt{3}}(m_K{}^2 - m_\pi{}^2)\,, \tag{14e}$$

$$\langle \eta_8 | B\hat{S}_{27} | \eta_8 \rangle = \frac{2\sqrt{5}\,m_\pi{}^2}{z+\frac{1}{2}}\,, \tag{14f}$$

$$\langle \eta_0 | \sum_i (4-l_i)\delta_i | \eta_8 \rangle = 0\,, \tag{14g}$$

$$\langle \eta_0 | BS_8 | \eta_8 \rangle = \frac{2\sqrt{5}}{\sqrt{3}}\,pq\,(m_\eta{}^2 - m_{\eta'}{}^2)\,, \tag{14h}$$

and

$$\langle \eta_0 | B\hat{S}_{27} | \eta_8 \rangle = 0\,. \tag{14i}$$

We shall critically discuss in particular Eq. (14h) and also obtain as a possible alternative

$$\langle \eta_0 | BS_8 | \eta_8 \rangle = 0\,. \tag{15}$$

In order to derive Eqs. (14) and (15) we start with the uncontroversial results.

In (14a) the contribution of δ to the mass of the ninth pseudoscalar meson has been parametrized by μ. Since μ^2 is approximately the mass squared of the η_0 in the limit $u \to 0$, and since in this limit the η_0 should certainly not be a tachyon, we must have $\mu^2 > 0$. The value of μ^2 could be anything between $\mu^2 \approx m_{\eta'}{}^2$ and $\mu^2 \approx 0$ (as, for example, in a theory with a c-number δ).

In proposing the further estimates made in Eqs. (14) we use the value $l_u = 2$ in an essential way. It is only for this value that our SU(3) and soft-pion assumptions can possibly be correct. This has been explained above. We shall throughout assume SU(3) for the matrix elements of $(4-l_i)\delta_i$, S_8, and $\hat{S}_{27}$. As explained below there is no reason to doubt this approximation. It yields directly Eqs. (14b), (14c), (14g), and (14i). If SU(3) is combined with the soft-*pion* limit[21] for the matrix elements

$$\langle \pi^3 | \sum_i (4-l_i)\delta_i | \pi^3(q_\mu) \rangle\,,$$

$$\langle \pi^3 | BS_8 | \pi^3(q) \rangle\,, \tag{16}$$

and

$\langle \pi^3 | B\hat{S}_\Theta | \pi^3(q)\rangle$,

we arrive at (14d)–(14f). For (14e) a perhaps more compelling derivation has been given in Ref. 2. It consists in differentiating

$$\langle K|V_\mu^K|\pi\rangle = if_{KK\pi}(P_K + P_\pi)_\mu \tag{17}$$

(as given by the Ademollo-Gatto theorem) with respect to x_μ. Incidentally, it follows from the above by use of the relation $2m_a{}^2 = \langle a|T_\mu^\mu|a\rangle$ for $a=\pi$ and K that

$$\langle \pi|BS_0|\pi\rangle = \langle K|BS_0|K\rangle = \frac{2\sqrt{5}}{z+\frac{1}{2}} m_\pi{}^2 \tag{18}$$

in agreement with SU(3) for S_0. Validity of the Gell-Mann–Okubo mass formula is then equivalent to having also the SU(3) relation,

$$\langle \pi|BS_0|\pi\rangle = \langle \eta_8|BS_0|\eta_8\rangle \tag{19}$$

(use $\langle \pi|T_\mu^\mu|\pi\rangle - \langle \eta_8|T_\mu^\mu|\eta_8\rangle$). One might furthermore assume, as done in Ref. 2, that differentiating $\langle K|V_\mu^K|\eta\rangle$ with respect to x_μ amounts to multiplication of the matrix element with $i(P^K - P^{\eta_8})_\mu$. (We are not entirely convinced that this procedure is correct.) This yields the Gell-Mann–Okubo mass formula as shown in Ref. 2 and implies Eq. (19). Equations (14d)–(14f) now agree with the result obtained by formally applying the soft-η_8 limit. Conversely, if this limit is valid it also implies the Gell-Mann–Okubo mass formula. In summary, we feel safe in using $m_{\eta_8}{}^2 = \frac{1}{3}(4m_K{}^2 - m_\pi{}^2)$.

It only remains to derive Eq. (14h). For this purpose we use the SU(3) formula (also assumed in the form $\langle \eta_0|u_0|\eta_8\rangle = 0$ in Ref. 2)

$$\langle \eta_0|S_0|\eta_8\rangle = 0. \tag{20}$$

If one accepts the Gell-Mann–Okubo mass formula, this is the only place in which we have to make use of SU(3) for S_0. Due to the established validity of (18) and (19) one may feel confident in using (20). Then Eq. (14h) follows using (7), (13), and (14g). Before turning to a critical discussion of (20) and a possible alternative, we devote the next paragraph to demonstrating that SU(3) should in fact be valid for matrix elements of S_8, $\hat{S}_{27}$, and $\sum_i (4-l_i)\delta_i$.

According to present understanding, the origin of a violation of SU(3) for matrix elements of scalar operators is the possible contribution of ϵ-meson poles. Thus, if the ϵ does not couple to a scalar operator, SU(3) is satisfied for its matrix elements. Since

$$\langle \hat{S}_{27}\rangle_0 = \sum_{\alpha=1-8,\,\Theta=1-27} \langle [Q_V^\alpha, \hat{S}_\Theta]\rangle_0 c_{\alpha\Theta},$$

and since SU(3) should be valid for the vacuum, we have $\langle \hat{S}_{27}\rangle_0 = 0$. Similarly, it may also be seen that $\langle S_8\rangle_0 = 0$. However, the preceding argument does not hold for S_0 since it is an SU(3) scalar and we expect $\langle S_0\rangle_0 \neq 0$. This is also implied by saturation of the vacuum expectation values of the pseudoscalar σ terms.[6,8] Furthermore, we shall assume on the grounds of Ref. 8 that $\langle [Q_D, \delta]\rangle_0$ vanishes (Q_D is the dilatation charge). Namely, we know[7] that for $l_u = 2$ we need a q-number δ and that most likely it cannot have a dimension[8] $l_\delta \geq 1$. The most simple assumption is then that δ has a q-number part δ_q with a dimension and a c-number part. The analysis of Ref. 8 then shows (even for a reasonable amount of $\eta\eta'$ mixing) that $\langle \delta_q\rangle_0 = 0$ for possible values of the width and the mass of the ϵ particle, such as, e.g., $m_\epsilon = 670$ MeV and $\Gamma_\epsilon = 600$ MeV (or, e.g., $m_\epsilon = 600$ MeV and $\Gamma_\epsilon = 340$ MeV). We take this as motivation for the assumption $\langle \delta_q\rangle_0 = 0$. These remarks imply that out of the operators S_0, S_8, $\hat{S}_{27}$, and $\sum_i (4-l_i)\delta_i$, at most S_0 couples to the ϵ. Namely, we have seen that $\langle S_0\rangle_0 \neq 0$, whereas

$$\langle S_8\rangle_0 = \langle \hat{S}_{27}\rangle_0 = \langle \delta_q\rangle_0 = 0 .$$

We continue to assume ϵ saturation of $\langle [Q_D, X(0)]\rangle_0$ with $X(0) = S_8$, $\hat{S}_{27}$, S_0, and δ. It then follows from

$$\langle [iQ_D, S_0]\rangle_0 = 2\langle S_0\rangle \neq 0$$

that $\langle \Omega|T_\mu^\mu|\epsilon(\vec{k})\rangle \neq 0$ and $\langle \Omega|S_0|\epsilon(\vec{k})\rangle \neq 0$, upon using

$$\langle \Omega|Q_D|\epsilon(\vec{k})\rangle = -(i/m_\epsilon)(2\pi)^3\delta^{(3)}(\vec{k})\langle \Omega|T_\mu^\mu|\epsilon(\vec{k})\rangle .$$

Furthermore, we have, for example,

$$i\langle [Q_D, S_8]\rangle_0 = 2\langle S_8\rangle_0 .$$

The assumed ϵ saturation now implies that

$$\langle \Omega|Q_D|\epsilon(k)\rangle\langle \epsilon(k)|S_8|\Omega\rangle = (2\pi)^3\left(\frac{-i}{m_\epsilon}\right)\delta^{(3)}(\vec{k})\langle \Omega|T_\mu^\mu|\epsilon(0)\rangle\langle \epsilon(\vec{0})|S_8|\Omega\rangle = 0 .$$

Since, as we have seen, $\langle \Omega|T_\mu^\mu|\epsilon(0)\rangle \neq 0$, the ϵ decouples from S_8, i.e., $\langle \Omega|S_8|\epsilon\rangle = 0$. Therefore, S_8 is *not* an interpolating field for the ϵ and the matrix elements of S_8 have no ϵ poles. Thus SU(3) is valid for the matrix elements of S_8. The same argument applies to $\hat{S}_{27}$ and δ_q. Of course, for S_8 and $\hat{S}_{27}$ the assumption that the ϵ is an SU(3) scalar would lead to the same result. This is not so for δ_q. We have given in the above a complete description of the argument (which is basically due to Ref. 22) since we found that it is not generally known.

We next turn to a critical discussion of Eq. (20). It implies that the soft-pion limit does not hold for the matrix elements $\langle\eta_0|\hat{S}_\Theta|\pi^3(q)\rangle$ and $\langle\eta_0|S_\alpha|\pi^3(q)\rangle$. First note that since SU(3) is valid for $\hat{S}_{27}$ and S_8, it follows that $\langle\eta_0|\hat{S}_\Theta|\pi^3(q)\rangle$ vanishes and $\langle\eta_0|S_\alpha|P_\beta\rangle$ is proportional to $\delta_{\alpha\beta}$, where according to Eq. (14h) the proportionality constant is nonvanishing. However, noticing that

$$\sum_{\alpha=1}^{8}[Q_A^3,(d_{44\alpha}-d_{66\alpha})S_\alpha]$$

$$=-\tfrac{1}{8}\sqrt{30}\sum_{\Theta=1}^{27}[Q_A^3,(\xi_{\Theta 44}-\xi_{\Theta 55})\hat{S}_\Theta]$$

$$=i(S_A^{45}+S_A^{67}), \tag{21}$$

the soft-pion limit is seen to yield (up to a constant nonvanishing factor) the same result for both of the above matrix elements, namely, essentially $\langle\eta_0|S_A^{45}+S_A^{67}|\Omega\rangle$. Thus from

$$\langle\eta_0|\hat{S}_\Theta|\pi^3(q)\rangle=0$$

and the soft-pion limit it would then follow that

$$\langle\eta_0|S_A^{45}+S_A^{67}|\Omega\rangle=0, \tag{22}$$

which would then imply the vanishing of the expression in Eq. (14h) upon making use of the soft-pion limit for $\langle\eta_0|S_\alpha|\pi^3(q)\rangle$. It could in fact be reasonable to have SU(3) for S_0 and the validity of the soft-pion limit for matrix elements of the type $\langle\pi^3(p)|\cdots|\pi^3(q)\rangle$ but not for $\langle\eta_0(p)|\cdots|\pi^3(q)\rangle$. One reason for this could for example be that in the variable $p\cdot q$ the extrapolation distance from the soft-pion limit to the physical value is much larger for $\langle\eta_8(p)|\cdots|\pi^3(q)\rangle$ than for $\langle\pi^3(p)|\cdots|\pi^3(q)\rangle$.

We have seen that Eq. (14h) disagrees with the soft-pion limit in matrix elements of the type $\langle\eta_0|\cdots|\pi\rangle$. The only doubtful assumption going into the derivation of (14h) is SU(3) for S_0 as assumed in (20). Giving up that formula and using the soft-pion limit instead, we arrive at Eq. (15) and $\langle\eta_0|S_A^{\alpha\beta}|\Omega\rangle=0$. This latter result is in agreement with SU(3). We do not see at present any real reason to prefer either set of assumptions. An experimental decision might come from the decay under consideration here. We repeat that it might also be reasonable to neglect mixing altogether in the (8, 8).

We conclude by presenting our numerical estimates for Γ taking into account the errors given in Eq. (3). We have from Eqs. (1) and (3) the width formula

$$\Gamma=\begin{cases}0.044(\sigma_{\eta\eta'}/m_\pi^2)^2\text{ MeV} & (\text{for }\xi=9\text{ keV}),\\ 0.074(\sigma_{\eta\eta'}/m_\pi^2)^2\text{ MeV} & (\text{for }\xi=15\text{ keV}),\\ 0.137(\sigma_{\eta\eta'}/m_\pi^2)^2\text{ MeV} & (\text{for }\xi=28\text{ keV}).\end{cases} \tag{23}$$

In the $(3,\bar{3})\oplus(\bar{3},3)$ for $l_u=2$, the estimate[23]

$$\frac{|\sigma_{\eta\eta'}|}{m^2}=0.68-0.015\mu^2 \tag{24}$$

has been obtained in Refs. 1 and 2. Thus we may conclude

$$\Gamma<0.065\text{ MeV} \tag{25}$$

in the $(3,\bar{3})\oplus(\bar{3},3)$ with $l_u=2$. The limit is assumed for both $\mu=0$ and $(\mu/m_\pi)^2=46$. The prediction obtained in the $(3,\bar{3})\oplus(\bar{3},3)$ with a c-number δ is smaller for $l_u=2$, and for $l_u=3$ it is $\Gamma<0.25$ MeV.

We next turn to our estimates in the (8, 8). The assumption in Eq. (20) [leading to (14h)] yields

$$\frac{|\sigma_{\eta\eta'}|}{m_\pi^2}\approx 7+0.03\left(\frac{\mu}{m_\pi}\right)^2=\begin{cases}7 & \text{for }\mu=0,\\ 8.4 & \text{for }(\mu/m_\pi)^2=46.\end{cases} \tag{26}$$

In this case, the lower limit

$$\Gamma\gtrsim 2.2\text{ MeV} \tag{27}$$

is most interesting in view of the experimental value

$$\Gamma<2.7\text{ MeV}.$$

Unrestricted validity of the soft-pion limit has yielded Eq. (15) and in turn

$$\frac{|\sigma_{\eta\eta'}|}{m_\pi^2}\approx 2.5+0.03\left(\frac{\mu}{m_\pi}\right)^2=\begin{cases}2.5 & \text{for }\mu=0,\\ 3.6 & \text{for }(\mu/m_\pi)^2=46.\end{cases} \tag{28}$$

In this case we explicitly note both lower and upper limits,

$$0.27<\Gamma<1.8\text{ MeV}, \tag{29}$$

obtained for various values of ξ and μ. Thus even taking these errors into account we get a clear distinction between the $(3,\bar{3})\oplus(\bar{3},3)$ and the (8, 8) for $l_u=2$. Since the errors in the Dalitz-plot extrapolation (i.e., the errors in ξ) affect all predictions in the same way and since one might hope that these errors can be reduced by future work it is important to notice that the distinction between the two models is even larger for any given value of ξ. This may be read off Eqs. (24), (26), and (28). For the widths the results are

$$\Gamma^{(8,8)}>100\Gamma^{(3,\bar{3})\oplus(\bar{3},3)}$$

and

$$\Gamma^{(8,8)} > 13\,\Gamma^{(3,\bar{3})\oplus(\bar{3},3)}$$

under the two alternative assumptions we have made. Except for an experimental Γ around 2 MeV already under present errors on ξ, a distinction between these alternatives is possible [Eqs. (27) and (29)] using the decay $\eta' \to \eta\pi\pi$.

We note in conclusion that the decay $\eta' \to \eta\pi\pi$ clearly distinguishes between the $(3,\bar{3}) \oplus (\bar{3},3)$ and the (8, 8) model for $l_u = 2$ and may be decisive. The same decay can also distinguish between two different forms of the (8, 8) model.

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[12]According to $\underline{8} \otimes \underline{8} = \underline{1} + \underline{8}_s + \underline{8}_a + \underline{10} + \underline{\overline{10}} + \underline{27}$, only one SU(3)-scalar field is contained in the (8, 8). This must have positive parity or else u has no SU(3)-scalar part.

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[15]If (as it appears to be the case) the πN σ term has actually the value given in Ref. 16, present measurements of the $KN\,\sigma$ terms are certainly not sufficiently accurate to exclude the $(3,\bar{3}) \oplus (\bar{3},3)$ model. In the (8,8), in addition to the πN σ term, another σ term can be fitted. It turns out that (given the πN σ term) the independent estimates of Refs. 17 and 18 for the $(KN)_{I=1}$ σ term and the $(KN)_{I=0}$ σ term, respectively, are roughly equivalent in the (8,8). [They also roughly agree with the prediction of the $(3,\bar{3}) \oplus (\bar{3},3)$.] If one of them is accepted, a big difference exists in the predictions of the two models for the $(\pi\Sigma)$ σ terms of isospin 1 and 0, respectively. In the $(3,\bar{3}) \oplus (\bar{3},3)$, these σ terms are $(\pi\Sigma)_{I=1} = (\pi\Sigma)_{I=0} \approx 20$ MeV, whereas in the (8,8) the prediction is $(\pi\Sigma)_{I=1} \approx -(\pi\Sigma)_{I=0} \approx 1000$ MeV. There may be hope to detect such a large discrepancy; see Ref. 7 for details.

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[19]In order to derive Eq. (11) one may for example (Ref. 6) compute $\langle\Omega|\partial^\mu A_\mu^\pi|\pi\rangle = f_\pi$ and $\langle\Omega|\partial^\mu A_\mu^K|K\rangle = f_K$ in terms of the $\langle\Omega|BS_A^{\alpha\beta}|\pi\rangle$ and $\langle\Omega|BS_A^{\alpha\beta}|K\rangle$. Writing (at least for $|\pi\rangle$ and $|K\rangle$) SU(3) as $\langle\Omega|S_A^{\alpha\beta}|P_\gamma\rangle = rf^{\alpha\beta\gamma}$ (P_γ: pseudoscalar octet states) and as $f\pi/m_\pi^2 = f_K/m_K^2$, the ratio of $\langle\Omega|\partial^\mu A_\mu^\pi|\pi\rangle$ to $\langle\Omega|\partial^\mu A_\mu^K|K\rangle$ can be computed to obtain Eq. (11) together with $rB = \frac{1}{3}\sqrt{5}\,f_\pi/(z+\frac{1}{2})$.

[20]To see this, it is sufficient to consider the difference $\langle\pi|T_\mu^\mu|\pi\rangle - \langle K|T_\mu^\mu|K\rangle = 2(m_\mu^2 - m_K^2)$. Using Eq. (13), the Ademollo-Gatto theorem, and SU(3), this expression is also equal to $(4-l_u)(m_\pi^2 - m_K^2)$. The result $l_u = 2$ therefore follows. Notice however that the analogous reasoning for baryon matrix elements yields $l_u = 3$.

[21]The soft-pion limit directly yields expressions proportional to $\langle\Omega|S_A^{\alpha\beta}|\pi\rangle$. These may easily be computed from the value of r given in footnote 19.

[22]J. Ellis, P. Weisz, and B. Zumino, Phys. Letters 34B, 91 (1971).

[23]Our method can also immediately be extended to the $(3,\bar{3}) \oplus (\bar{3},3)$ upon assuming SU(3) for the meson matrix elements of $\sum_i (4-l_i)\delta_i$, u_0, and u_8. For $l_u = 2$ there is no real reason to doubt this approximation. It should certainly be valid for u_8 and can be supported in case of $\sum_i (4-l_i)\delta_i$ by the sum rule of Ref. 24. The Γ of Refs. 1 and 2 for $l_u = 2$ may then be easily obtained.

[24]H. Kleinert and P. Weisz, Nuovo Cimento 3A, 479 (1971).