The problem of "covariantizing" an amplitude has been discussed by a number of authors; see for example L. S. Brown, Phys. Rev. 150, 1338 (1966); L. S. Brown and D. Boulware, *ibid.* 156, 1724 (1967); R. F. Dashen and S. Y. Lee, *ibid.* 187, 2017 (1969); D. J. Gross and R. Jackiw, Nucl. Phys. B14, 269 (1969).

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[13]For a general discussion of fixed poles, Zee's review article (Ref. 6) is recommended.

[14]The data analysis in Ref. 3 assumed that

$$\lim_{q^2\to\infty} \beta_i(q^2, \alpha) \to (1/q^2)^{\alpha_i - n},$$

n being so adjusted that the combinations $\mathrm{Im}T_1^*$ and $\nu\mathrm{Im}T_2^*$ scale in the region $\nu\to\infty$, $q^2\to\infty$, ν/q^2 fixed. This kind of assumption – some sort of "asymptotic smoothness" [see R. Brandt, Phys. Rev. 187, 2192 (1969)], of being able to go from the Regge limit ($\nu\to\infty$, q^2 fixed) to the scaling region ($\nu\to\infty$, q^2, ν/q^2 fixed) – has been made by a number of authors [see for instance H. D. I. Abarbanel, M. L. Goldberger, and S. B. Treiman, Phys. Rev. Letters 22, 500 (1969); H. Harari, Phys. Rev. Letters 22, 1078 (1969)]. Our assumption is in this spirit. [See also G. Altarelli and H. R. Rubenstein, Phys. Rev. 187, 2111 (1969).]

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Energy Eigenvalues for Charged Particles in a Homogeneous Magnetic Field – An Application of the Foldy-Wouthuysen Transformation*

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We find that the equation of motion for spin-$\frac{1}{2}$ and spin-1 particles with an anomalous magnetic moment in a homogeneous magnetic field can be diagonalized by applying the Foldy-Wouthuysen transformation. The energy eigenvalues are then easily obtained by observation.

The traditional method for obtaining the energy eigenvalues of a system is to solve the eigenvalue equations.[1-3] This method becomes increasingly complicated when anomalous-magnetic-moment couplings are introduced and higher-spin particles are involved.[2,3] It was only recently that the energy eigenvalues for the motion of a Dirac particle and a spin-1 particle with an anomalous magnetic moment in a homogeneous magnetic field were calculated.[2-5] A simpler method for obtaining the eigenvalues of a system without solving the equation of motion was proposed by Tsai and Yildiz.[4,5] They observed that, even though the second-order form of the eigenvalue equation is not diagonal, it can be diagonalized by going to the fourth-order form.

The purpose of this paper is to present an even simpler method, by applying the Foldy-Wouthuysen transformation,[6-8] to obtain the energy eigenvalues of the spin-$\frac{1}{2}$ and spin-1 systems with anomalous-magnetic-moment couplings in a homogeneous magnetic field. The transformation method of Foldy and Wouthuysen is well known in its application to the reduction of a relativistic equation to the nonrelativistic form. One of its virtues is that the Dirac equation for a free particle and for a particle moving in a homogeneous magnetic field can be diagonalized by this transformation.[6,7,9] The extension of this feature to the cases when anomalous-magnetic-moment couplings are introduced and higher-spin particles are involved enables us to obtain the energy eigenvalues easily.

For the spin-$\frac{1}{2}$ system, the eigenvalue equation is

$$p^0\psi=\gamma^0\left(m+\vec{\gamma}\cdot\vec{\pi}-\frac{eq\kappa}{2m}\vec{\sigma}\cdot\vec{H}\right)\psi\equiv\mathcal{H}_{1/2}\psi\,. \tag{1}$$

The application of the canonical transformation $U_{1/2}=e^{-\vec{\gamma}\cdot\vec{\pi}_\perp\theta/2}$ implies

$$\begin{aligned}\mathcal{H}_{1/2}\to\mathcal{H}'_{1/2}&=e^{\vec{\gamma}\cdot\vec{\pi}_\perp\theta/2}\mathcal{H}_{1/2}e^{-\vec{\gamma}\cdot\vec{\pi}_\perp\theta/2}\\&=\gamma^0(m+\vec{\gamma}\cdot\vec{\pi}_\perp)e^{-\vec{\gamma}\cdot\vec{\pi}_\perp\theta}+\gamma^0\gamma_3p_3-\frac{eq\kappa}{2m}\gamma^0\vec{\sigma}\cdot\vec{H}\\&=\gamma^0\left[\left(m\cos\xi\theta+\xi\sin\xi\theta+\gamma_3p_3-\frac{eq\kappa}{2m}\vec{\sigma}\cdot\vec{H}\right)+\vec{\gamma}\cdot\vec{\pi}_\perp\left(\cos\xi\theta-\frac{m}{\xi}\sin\xi\theta\right)\right],\end{aligned} \tag{2}$$

where we have used the relations

$$\{\gamma^0,\vec{\gamma}\}=0,\quad[\gamma^0\vec{\sigma}\cdot\vec{H},\vec{\gamma}\cdot\vec{\pi}_\perp]=0,\quad\vec{\pi}_\perp^{\,2}=\pi_1^{\,2}+\pi_2^{\,2},\quad\xi^2\equiv-(\vec{\gamma}\cdot\vec{\pi}_\perp)^2=\vec{\pi}_\perp^{\,2}-eq\vec{\sigma}\cdot\vec{H},\quad e^{-\vec{\gamma}\cdot\vec{\pi}_\perp\theta}=\cos\xi\theta-\frac{\vec{\gamma}\cdot\vec{\pi}_\perp}{\xi}\sin\xi\theta\,. \tag{3}$$

The choice of θ such that $\tan\xi\theta=\xi/m$ enables us to obtain a diagonal Hamiltonian

$$\mathcal{H}'_{1/2}=\gamma^0\left[(m^2+\vec{\pi}_\perp^{\,2}-eq\vec{\sigma}\cdot\vec{H})^{1/2}-\frac{eq\kappa}{2m}\vec{\sigma}\cdot\vec{H}+\gamma_3p_3\right]. \tag{4}$$

The corresponding eigenvalues are then

$$p^0=\pm\left[p_3^{\,2}+\left([m^2+(2n+1-q\sigma_3)eH]^{1/2}-\frac{eq\kappa}{2m}\vec{\sigma}\cdot\vec{H}\right)^2\right]^{1/2}, \tag{5}$$

where we have used the quantization condition

$$\vec{\pi}_\perp^{\,2}=(2n+1)eH\,,\quad n=0,1,2,\ldots\,.$$

This is precisely the same result obtained in Refs. 2 and 4.

For the spin-1 system, we start from the eigenvalue equation[10]

$$(m^2+\pi^2)\phi_\mu-\pi_\mu(\pi\phi)+ieq(1+\kappa)F_{\mu\nu}\phi^\nu=0\,, \tag{6}$$

which implies the subsidiary condition

$$m^2\pi^\mu\phi_\mu-ieq(1-\kappa)\pi^\mu F_{\mu\nu}\phi^\nu=0\,. \tag{7}$$

Substituting Eq. (7) into Eq. (6) and picking up the spatial components, we obtain

$$(p^0)^2\phi_i=(m^2+\vec{\pi}^{\,2})\phi_i+ieq(1+\kappa)F_{ij}\phi^j-\frac{ieq}{m^2}(1-\kappa)\pi_i\pi_jF^{jk}\phi_k\,, \tag{8}$$

or in the matrix form (in the following, without loss of generality, we consider the case when $\vec{H}$ is along the z axis and $p_3=0$)

$$(p^0)^2\Phi=\left[m^2+\vec{\pi}_\perp^{\,2}-eq(1+\kappa)\vec{S}\cdot\vec{H}+\frac{eq(1-\kappa)}{m^2}(\vec{\pi}_\perp^{\,2}-eq\vec{S}\cdot\vec{H})\vec{S}\cdot\vec{H}-\frac{eq(1-\kappa)}{m^2}(\vec{S}\cdot\vec{\pi}_\perp)^2\vec{S}\cdot\vec{H}\right]\Phi\,, \tag{9}$$

where we have used the relations

$$\pi_i\pi_j=\vec{\pi}^{\,2}\delta_{ij}-eq(\vec{S}\cdot\vec{H})_{ij}-[(\vec{S}\cdot\vec{\pi})^2]_{ij}\,,\qquad F_{ij}=i(\vec{S}\cdot\vec{H})_{ij},\quad F_{12}=H,\quad(S_j)_{ik}=i\epsilon_{ijk}\,. \tag{10}$$

Further simplification can be accomplished by introducing the definitions

$$S_\pm=S_1\pm iS_2,\quad\pi_\pm=\pi_1\pm i\pi_2\,,\quad\lambda=\frac{eq(1-\kappa)H}{4m^2}\,,\quad N_\pm=S_-^{\,2}\pi_+^{\,2}\pm S_+^{\,2}\pi_-^{\,2}\,, \tag{11}$$

and the identities

$$S_iS_jS_k+S_kS_jS_i=\delta_{ij}S_k+\delta_{kj}S_i,\quad N_\pm=-S_3N_\mp\,,\qquad(\vec{S}\cdot\vec{\pi}_\perp)^2\vec{S}\cdot\vec{H}=\tfrac{1}{2}(\vec{\pi}_\perp^{\,2}-eq\vec{S}\cdot\vec{H})\vec{S}\cdot\vec{H}+\tfrac{1}{4}HN_-\,. \tag{12}$$

The simplified form of Eq. (9) is

$$(p^0)^2\Phi = \left[m^2 + \vec{\pi}_\perp{}^2 - eq(1+\kappa)\vec{S}\cdot\vec{H} + \frac{eq(1-\kappa)}{2m^2}(\vec{\pi}_\perp{}^2 - eq\vec{S}\cdot\vec{H})\vec{S}\cdot\vec{H} + \lambda S_3 N_+\right]\Phi \equiv \mathcal{H}_1\Phi\ . \tag{13}$$

This is the eigenvalue equation we want to study. Before we go on, we note that $\zeta = \vec{\pi}_\perp{}^2 - 2eq\vec{S}\cdot\vec{H}$ commutes with all other scalar quantities, and it is more convenient to rewrite $\mathcal{H}_1$ as

$$\mathcal{H}_1 = \left[m^2 + \zeta + \frac{(1-\kappa)e^2H^2}{2m^2}S_3{}^2 + (1-\kappa)\left(1 + \frac{\zeta}{2m^2}\right)eq\vec{S}\cdot\vec{H} + \lambda S_3 N_+\right]. \tag{14}$$

The application of the canonical transformation $U_1 = e^{-\lambda N_+\Theta/2}$ implies

$$\begin{aligned}\mathcal{H}_1 \to \mathcal{H}_1' &= e^{\lambda N_+\Theta/2}\mathcal{H}_1 e^{-\lambda N_+\Theta/2}\\ &= \left\{\left[m^2 + \zeta + \frac{(1-\kappa)e^2H^2}{2m^2}S_3{}^2\right] + (1-\kappa)eq\vec{S}\cdot\vec{H}\left[\left(1 + \frac{\zeta}{2m^2}\right) + \frac{N_+}{4m^2}\right]e^{-\lambda N_+\Theta}\right\}\\ &= \left\{m^2 + \zeta + \frac{(1-\kappa)e^2H^2}{2m^2}S_3{}^2 + eq(1-\kappa)\vec{S}\cdot\vec{H}\left[\left(1 + \frac{\zeta}{2m^2}\right)\cosh\xi\Theta - \frac{\lambda N_+{}^2}{4m^2\xi}\sinh\xi\Theta\right]\right.\\ &\qquad \left. + \frac{eq(1-\kappa)}{4m^2}\vec{S}\cdot\vec{H}N_+\left[\cosh\xi\Theta - \left(1 + \frac{\zeta}{2m^2}\right)\frac{eq(1-\kappa)H}{\xi}\sinh\xi\Theta\right]\right\},\end{aligned} \tag{15}$$

where we have used the relations

$$\begin{aligned}&N_+{}^2 = -N_-{}^2 = 4S_3{}^2(\zeta^2 - e^2H^2)\,, \quad [S_3{}^2, N_\pm] = 0\,,\\ &\{S_3, N_\pm\} = 0, \quad \xi^2 \equiv (\lambda N_+)^2 = \left[\frac{eq\vec{S}\cdot\vec{H}(1-\kappa)}{4m^2}\right]^2(\zeta^2 - e^2H^2)\,,\\ &e^{-\lambda N_+\Theta} = \cosh\xi\Theta - \frac{\lambda N_+}{\xi}\sinh\xi\Theta\ .\end{aligned} \tag{16}$$

To obtain a diagonal form for $\mathcal{H}_1$, we choose

$$\tanh\xi\Theta = \xi\left[eq(1-\kappa)H\left(1 + \frac{\zeta}{2m^2}\right)\right]^{-1}, \tag{17}$$

which then yields

$$\mathcal{H}_1' = m^2 + \zeta + \frac{(1-\kappa)e^2H^2}{2m^2}S_3{}^2 + eq(1-\kappa)\vec{S}\cdot\vec{H}\left[\left(1 + \frac{\zeta}{2m^2}\right)^2 - \frac{S_3{}^2}{4m^4}(\zeta^2 - e^2H^2)\right]^{1/2}. \tag{18}$$

The corresponding eigenvalues are[11]

$$(p^0)^2 = m^2 + (2n+1)eH, \quad \text{for } S_3 = 0 \tag{19}$$

$$(p^0)^2 = m^2 + (2n+1-2qS_3)eH + \frac{(1-\kappa)e^2H^2}{2m^2} + eq(1-\kappa)\vec{S}\cdot\vec{H}\left[1 + (2n+1-2qS_3)\frac{eH}{m^2} + \frac{e^2H^2}{4m^4}\right]^{1/2}, \quad \text{for } S_3{}^2 = 1\ . \tag{20}$$

These results are the same as those of Refs. 4 and 5.

This investigation was stimulated by a discussion with Professor Richard E. Norton on the possible ways to transform the relativistic equation to the corresponding classical form. We thank him for the interesting conversation. We also thank Professor Julian Schwinger and Professor Thomas Erber for arousing the author's interest in this area, and Dr. Kimball A. Milton for reading the manuscript.

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[9]In Ref. 7, Case pointed out that the Dirac equation (without an anomalous-magnetic-moment coupling term) for the homogeneous-magnetic-field case could be diagonalized. However, he was unable to find an appropriate transformation in the spin-1 case.

[10]The method presented in this paper can be applied to a more general eigenvalue equation, such as that discussed in Ref. 5.

[11]For the discussion of these results and their connection with the consistency of the spin-1 theory, see Ref. 5.

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Errata

Short-Distance Behavior of Quantum Electrodynamics and an Eigenvalue Condition for α, Stephen L. Adler [Phys. Rev. D 5, 3021 (1972)]. 1. Page 3025, Eq. (20): $\pi[y]$ should read $\pi[w]$. 2. Page 3025, second column, line 10: $\alpha_w d_c[x, y, \alpha_w]$ should read $\alpha_w d_c[x, w, \alpha_w]$. 3. Page 3026, first column, line following Eq. (31a): Eq. (21) should read Eq. (28). 4. Page 3030, first column, fourth line following Eq. (61): "the case" should read "this case." 5. Page 3031, first column, line 2: Eq. (30) should read Eq. (31a). 6. Page 3031, first column, third line following Eq. (67a): Eq. (17) should read Eq. (66). 7. Page 3036, first column, second line from bottom: Ref. 17 should read Ref. 18. 8. Page 3043, first column, Eq. (B3): $\tilde{S}_F'[p, \mu, m, \alpha, \eta]$ should read $\tilde{S}_F'^{-1}[p, \mu, m, \alpha, \eta]$. 9. Page 3043, second column, second line following Eq. (B5): Eq. (B4) should read Eq. (B3). 10. Page 3044, second column, paragraph headed *Type I*: Ref. 17 should read Ref. 18 throughout. 11. Page 3045, second column, fourth line following Eq. (B19): Ref. 17 should read Ref. 18.

Mellin-Transform Analysis of Light-Cone Structure and Scaling in Inelastic Electron Scattering, D. Bhaumik, O. W. Greenberg, and R. N. Mohapatra [Phys. Rev. D 6, 2989 (1972)]. The bound given in (A14) is incorrect because $\partial^2 F_s^{(+)}/\partial s^2$ is not in $\mathfrak{L}^1$ under the assumptions made. The statements on p. 2996, first column, concerning R. Jaffe's results are incorrect. We now agree with Jaffe's bound under corresponding assumptions.

Massive Particles and the Spontaneous Breakdown of Dilation Invariance, S. K. Bose and W. D. McGlinn [Phys. Rev. D 6, 2304 (1972)]. Replace Eq. (7) with $\tilde\sigma^i_{10}(\mu^2=0,\vec{0})=0$. Replace Eq. (11) with

$$\int_0^\infty d\mu^2[\bar\beta_{10}(\mu^2)+(2\pi)^3\tilde\sigma^i_{1i}(\mu^2,0)]\cos(x_0\mu)=0$$

and Eq. (12) with

$$\bar\beta_{10}(\mu^2)+(2\pi)^3\tilde\sigma^i_{1i}(\mu^2,0)=0\,.$$

Add the terms

$$i\int d^3x\int_0^\infty d\mu^2\,\bar\beta_{10}(\mu^2)\frac{\partial}{\partial x_\mu}\Delta^+(x,x_0,\mu^2)$$

and

$$i\int_0^\infty d\mu^2\,k_\mu\bar\beta_{10}(\mu^2)e^{ix_0k}\theta(k_0)\delta(k_0^2-|\vec{k}|^2-\mu^2)$$

to the right-hand sides of Eqs. (15) and (16), respectively.

The statement immediately following Eq. (16) should read: It is straightforward to see that the term multiplying x_0 as well as the last term in square brackets in Eq. (16) vanish. The remaining terms also vanish due to Eqs. (7) and (12) whenever μ takes up values 1, 2, or 3.

The right-hand side of Eq. (17) should contain an additional term,

$$\tfrac{1}{2}\delta_{\mu0}\,i\int_0^\infty d\mu^2[\bar\beta_{10}(\mu^2)+(2\pi)^3\tilde\sigma^i_{1i}(\mu^2,0)]e^{ix_0\mu}\,,$$

which, however, is zero due to Eq. (12).