Eq. (32) follows directly. One way of deriving Eq. (32) based upon a set of assumptions which are not more economical than the ones we make in Sec. III is the following. Define

$$j^{\mu}_{H+L+W,\alpha} \cong j^{\mu}_{H+L,\alpha} + \frac{(m_W^2)_{\alpha\beta}}{g} W^{\mu}_{\beta},$$

where $(m_W^2)_{\alpha\beta}$ is the W-meson (mass)^2 matrix. Assume

$$\lim_{g\to 0} (m_W^2)_{\alpha\beta}/g \to 0,$$

$$\langle \chi_\alpha | j^{\nu}_{H+L+W,\beta}|0\rangle = -iq^{\nu}\langle X_\alpha | \tilde{j}^{\nu}_{H+L,\beta}|0\rangle + O(g),$$

$$\langle W^{\mu}_{\alpha} | j^{\nu}_{H+L+W,\beta}|0\rangle = \frac{-i}{g}[(m_W^2)_{\alpha\beta}g^{\mu\nu} - k^{\mu}k^{\nu}].$$

The desired result follows from

$$\langle 0| T(j^{\mu}_{H+L+W,\alpha}(k) j^{\nu}_{H+L+W,\beta}(-k))|0\rangle$$
$$= i q^{\mu} q^{\nu}\left(\sum_{\delta} \frac{\langle 0|\tilde{j}_{H+L,\alpha}|X_\delta\rangle\langle X_\delta|\tilde{j}_{H+L,\beta}|0\rangle}{q^2}\right)$$
$$+ \frac{1}{g^2}[(m_W^2)_{\alpha\sigma}g^{\mu\lambda} - k^{\mu}k^{\lambda}]\frac{i}{k^2-(m_W^2)_{\sigma\rho}}$$
$$\times [(m_W^2)_{\rho\beta}g^{\lambda\nu} - k^{\lambda}k^{\nu}] + O(g).$$

N.b. the first two terms for the right-hand side of this are of order unity if $m_W^2 \approx g^2$. Taking the divergence of both sides and *assuming* $\partial_\mu j^{\mu}_{H+L+W}$ is of order "g" we obtain

$$\theta^{\mu}_1(g) = i q^{\mu}\left[\sum_{\delta}\langle 0|\tilde{j}_{H+L+W,\alpha}|X_\delta\rangle\langle X_\delta|\tilde{j}_{H+L+W,\beta}|0\rangle - (1/g^2)(m_W^2)_{\alpha\beta}\right] + \theta^{\mu}_2(g).$$

Canceling terms of order unity gives the desired relation.

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Finite Mass Splitting of the Nucleons*

Anthony Duncan and Peter Schattner
Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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We calculate the neutron-proton mass difference in second order in a simple semirealistic gauge model of the weak and electromagnetic interactions. The mass splitting is finite. It is found that in the limit of infinitely heavy gauge mesons, the mass difference becomes the divergent one of conventional quantum electrodynamics. In the course of the calculation, certain general features of such models, which might be relevant for analogous calculations with more complicated groups, will be exhibited.

I. INTRODUCTION

Since the renaissance of interest in the possibility of renormalizable unified theories of the weak and electromagnetic interactions, there has been much activity in applying examples of these theories to various observable physical processes.[1] One intriguing possibility has been that of computing mass differences as finite higher-order corrections in the Feynman-Dyson expansion.[2,3] Weinberg[2] has pointed out that mass relations valid in zeroth order for arbitrary values of the parameters appearing in the Lagrangian must acquire only finite corrections in higher order. More specifically, he has shown[2] just how one must set up the Lagrangian so that, once one has specified the representation content of the scalar fields, any counterterms corresponding to mass differences will destroy the exact invariance of the renormalized Lagrangian and hence cannot appear.

Weinberg also suggests[2] a "semirealistic" model of the neutron-proton mass difference along these lines. In this paper we work out Weinberg's model in considerable detail[4] in order to determine what explicit mechanisms occur in this model to produce the general results shown in Ref. 2. This should be useful in constructing new, more realistic models of the weak and electromagnetic interactions of the hadrons.

The structure of the paper is as follows. In Sec. II, we show how to extract finite mass differences from the divergent self-energy integrals for any model of the class described by Weinberg,[2] provided only that $m \ll \lambda_i$, where m is the zeroth-order fermion mass and λ_i is the zeroth-order mass of the ith massive gauge meson. Our main result in this section will be an expression for the fermion mass difference as a function of the generators of the gauge group and the resulting vector-meson mass matrix. In Sec. III, we present the model itself in detail and verify that it is indeed a model of the type suggested by Weinberg.[2] It then follows that the mass difference will be finite and given by the formula derived in Sec. II. Section IV examines the constraints placed on the parameters appearing in the model by experiment. It is found that these constraints seem to force the proton-neutron mass difference to have the wrong sign. Indeed, in the limit of infinite gauge meson masses, the nucleon mass difference becomes that of quantum electrodynamics. Section V is devoted to a discussion of our results and observations on the type of modifications necessary to improve the model.

II. SECOND-ORDER CORRECTIONS TO THE MASS MATRIX

In this section we calculate the finite second-order corrections to the fermion mass matrix in the general class of theories described by a Lagrangian[2,5]

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - \gamma^\mu t_\alpha W_{\alpha\mu})\psi + \tfrac{1}{2}(\partial_\mu \phi + i\theta_\alpha \phi W_{\alpha\mu})(\partial^\mu \phi + i\theta_\alpha \phi W^\mu_\alpha) - \tfrac{1}{4} F_{\alpha\mu\nu} F^{\mu\nu}_\alpha - F(\phi) - i\bar{\psi}\Gamma_i \phi_i \psi . \quad (2.1)$$

Here ψ, W_μ, and ϕ refer to fermion, vector-boson, and scalar fields, respectively. $\mathcal{L}$ is constructed to be invariant under a gauge group G whose generators acting on ψ (ϕ) are represented by t_α (θ_α). The coupling constants are contained implicitly in the t_α, θ_α, Γ_i. $F(\phi)$ is a G-invariant quartic polynomial in the scalar fields. With reference to choices of kinematics and gauge, the following should be noted:

(1) G invariance of the Yukawa coupling requires[2]

$$[t_\alpha, \gamma_0 \Gamma_i] = (\theta_\alpha)_{ji} \gamma_0 \Gamma_j .$$

(2) The zeroth-order vector-boson mass matrix is given by

$$\mu^2{}_{\alpha\beta} = \mu^2{}_{\beta\alpha} = (\mu^2{}_{\alpha\beta})^* = -(\theta_\alpha \langle \phi \rangle_0)_i (\theta_\beta \langle \phi \rangle_0)_i , \quad (2.2)$$

provided we choose a totally antisymmetric set of structure constants.

(3) The zeroth-order fermion mass matrix is

$$\gamma_0 m = i\gamma_0 \Gamma_i \langle \phi_i \rangle_0 ,$$

where $\gamma_0 \Gamma_i$ is skew-Hermitian.

We have assumed in writing (2.1) that $D_F^* \otimes D_F$ (where D_F denotes the reducible representation formed by the fermions) does not contain the trivial representation, so that no bare-mass term is possible.

(4) Calculations are performed in the unitarity gauge in which all Goldstone bosons are absent, so the scalar-boson propagator is

$$iD(k) = \frac{i\Pi}{k^2 - M^2} \quad (2.3a)$$

with

$$\Pi_{ij} = \delta_{ij} + (\theta_\beta \langle \phi \rangle_0)_i \, \mu^{-2}{}_{\beta\alpha} (\theta_\alpha \langle \phi \rangle_0)_j . \quad (2.4)$$

Actually, since the second term in Π projects onto the subspace of massless Goldstone bosons, the scalar propagator has the explicit form

$$D(k) = \left(\frac{1}{k^2 - M^2}\right)_{ij} + \frac{(\theta_\beta \langle \phi \rangle_0)_i \, \mu^{-2}{}_{\beta\alpha} (\theta_\alpha \langle \phi \rangle_0)_j}{k^2} . \quad (2.3b)$$

Here M is the scalar mass matrix in zeroth order.

In general the zeroth-order fermion mass matrix is a linear combination of matrices belonging to some given subset of representations occurring in $D_F^* \otimes D_F$. A higher-order correction to the fermion mass matrix is said to be representation-conserving if it is a linear combination of the same matrices. It will then satisfy the same mass relations as the zeroth-order mass matrix and will not contribute to corrections to these relations. As Weinberg has shown,[2] all divergent contributions to the mass matrix in second order (and, presumably, in all higher orders) are representation conserving. Here we are interested in the finite, non-representation-conserving contribution to the mass matrix in second order. To second order, the fermion propagator is

$$S(p) = \frac{1}{\not{p} - m} + \frac{1}{\not{p} - m} \Sigma(p) \frac{1}{\not{p} - m} , \quad (2.5)$$

$$\Sigma(p) \equiv \Sigma_1(p) + \Sigma_2(p) , \quad (2.6a)$$

$$\Sigma_1(p) \equiv \Sigma_1'(p) + \Sigma_1''(p) = -i \int \frac{dk}{(2\pi)^4} \gamma_\mu t_\beta \frac{1}{\not{p} - \not{k} - m} \gamma_\nu t_\alpha \times \left(\frac{g^{\mu\nu} - k^\mu k^\nu \mu^{-2}}{k^2 - \mu^2}\right)_{\beta\alpha} , \quad (2.6b)$$

$$\Sigma_2(p) \equiv \Sigma_2'(p) + \Sigma_2''(p) = -i \int \frac{dk}{(2\pi)^4} \Gamma_i \frac{1}{\not{p} - \not{k} - m} \Gamma_j \left(\frac{\Pi}{k^2 - M^2}\right)_{ij} , \quad (2.6c)$$

$$\Sigma_1'(p) \equiv -i \int \frac{dk}{(2\pi)^4} \gamma_\mu t_\beta \frac{1}{\not{p} - \not{k} - m} \gamma^\mu t_\alpha \left(\frac{1}{k^2 - \mu^2}\right)_{\beta\alpha} , \quad (2.7a)$$

$$\Sigma_1''(p) \equiv i \int \frac{dk}{(2\pi)^4} \not{k} t_\beta \frac{1}{\not{p} - \not{k} - m} \not{k} t_\alpha \left(\frac{\mu^{-2}}{k^2 - \mu^2}\right)_{\beta\alpha} , \quad (2.7b)$$

$$\Sigma_2'(p) \equiv -i \int \frac{dk}{(2\pi)^4} \Gamma_i \frac{1}{\not{p} - \not{k} - m} \Gamma_j \left(\frac{1}{k^2 - M^2}\right)_{ij} , \quad (2.7c)$$

$$\Sigma_2''(p) \equiv -i \int \frac{dk}{(2\pi)^4} \Gamma_i \frac{1}{\not{p} - \not{k} - m} \Gamma_j \frac{X_{ij}}{k^2} , \quad (2.7d)$$

$$X_{ij} \equiv (\theta_\beta \langle \phi \rangle_0)_i \mu^{-2}{}_{\beta\alpha} (\theta_\alpha \langle \phi \rangle_0)_j . \qquad (2.8)$$

One can verify that

$$\Sigma_1'' + \Sigma_2'' = Q + F , \qquad (2.9)$$

where Q is divergent but representation-conserving.[2] F is finite (and complicated). One also finds the logarithmic divergences in Σ_1' and Σ_2' to be representation-conserving.

The contributions from Σ_2' and F are expected to be small compared to the finite part of Σ_1' in the limit where the vector-boson masses far exceed the fermion masses. We make this specialization at the outset, and henceforth consider only the finite, non-representation-conserving part of $\Sigma_1'(p)$.

Finally, since we are particularly interested in models in which there is (at least) a completely unbroken U(1) subgroup of G, such as electromagnetic gauge invariance, we specialize to the case in which one of the vector bosons is massless (see, however, Appendix A). It is convenient to introduce the eigenvectors $\{\epsilon_\alpha^{(n)}\}$ and eigenvalues $\{\lambda_n\}$ of the vector mass matrix

$$\mu^2{}_{\beta\alpha}\epsilon_\alpha^{(n)} = \lambda_n{}^2 \epsilon_\beta^{(n)}, \quad \epsilon_\alpha^{(n)}\epsilon_\alpha^{(m)} = \delta_{nm} \qquad (2.10)$$

and the projections of the generators in the directions of well-defined mass

$$t^{(n)} \equiv \epsilon_\beta^{(n)} t_\beta , \quad t_\alpha = \epsilon_\alpha^{(n)} t^{(n)} . \qquad (2.11)$$

For any matrix χ in the internal space of the mesons we then have

$$t_\beta \chi t_\alpha \left(\frac{1}{k^2 - \mu^2}\right)_{\beta\alpha} = \sum_n t^{(n)} \chi t^{(n)} \frac{1}{k^2 - \lambda_n{}^2} = t^\gamma \chi t^\gamma \frac{1}{k^2} + \sum_{\tilde{n}} t^{(\tilde{n})} \chi t^{(\tilde{n})} \frac{1}{k^2 - \lambda_{\tilde{n}}{}^2} , \qquad (2.12)$$

where we have explicitly separated the photon contribution and introduced tildes to denote a sum over the subspace of massive particles. We can now write

$$\gamma_0 \Sigma_1' = \gamma_0 (C_1 + C_2) , \qquad (2.13a)$$

$$C_1 \equiv -i \int \frac{dk}{(2\pi)^4} \gamma_\mu t^{(n)} \frac{1}{\not{p} - \not{k} - m} \gamma^\mu t^{(n)} \frac{1}{k^2} , \qquad (2.13b)$$

$$C_2 \equiv -i \int \frac{dk}{(2\pi)^4} \gamma_\mu t^{(\tilde{n})} \frac{1}{\not{p} - \not{k} - m} \gamma^\mu t^{(\tilde{n})} \frac{\lambda_{\tilde{n}}{}^2}{k^2 (k^2 - \lambda_{\tilde{n}}{}^2)} \qquad (2.13c)$$

(sums over n and $\tilde{n}$ implied). Defining $\alpha \equiv p^2 - m^2$ and $f \equiv k^2 - 2p \cdot k$, we find

$$\gamma_0 C_1 = -2i \int \frac{dk}{(2\pi)^4} \left[-\gamma_0 m t^{(n)} \left(1 - \frac{\alpha}{\alpha + f}\right) t^{(n)} + t^{(n)} \gamma_0 \not{k} \left(1 - \frac{\alpha}{\alpha + f}\right) t^{(n)} + 2 t^{(n)} \gamma_0 m \left(1 - \frac{\alpha}{\alpha + f}\right) t^{(n)} \right] \frac{1}{f k^2} , \qquad (2.14a)$$

$$\gamma_0 C_2 = -2i \int \frac{dk}{(2\pi)^4} \left(-\gamma_0 m t^{(\tilde{n})} \frac{1}{\alpha + f} t^{(\tilde{n})} + t^{(\tilde{n})} \gamma_0 \not{k} \frac{1}{\alpha + f} t^{(\tilde{n})} + 2 t^{(\tilde{n})} \gamma_0 m \frac{1}{\alpha + f} t^{(\tilde{n})} \right) \frac{\lambda_{\tilde{n}}{}^2}{k^2 (k^2 - \lambda_{\tilde{n}}{}^2)} . \qquad (2.14b)$$

It should be noted that all terms in $\gamma_0 C_1$ and $\gamma_0 C_2$ beginning in $\gamma_0(\not{p} - m)$ or ending in $(\not{p} - m)$ are neglected since they only lead to an external wave-function renormalization, and do not affect the mass matrix. The non-representation-conserving part of $\gamma_0 C_1$ involves the integrals

$$i \int \frac{dk}{(2\pi)^4} \frac{\alpha}{f k^2 (\alpha + f)} = \frac{1}{16\pi^2} \left[\frac{\alpha}{p^2} \ln\left(1 - \frac{p^2}{\alpha}\right) - \ln\left(1 - \frac{\alpha}{p^2}\right) \right] , \qquad (2.15a)$$

$$i \int \frac{dk}{(2\pi)^4} \frac{(\gamma_0 \not{k})\alpha}{f k^2 (\alpha + f)} = \frac{\gamma_0 \not{p}}{32\pi^2} \left[\frac{\alpha}{p^2} + \ln\left(\frac{p^2}{-\alpha}\right) + \left(\frac{\alpha^2}{p^4} - 1\right) \ln\left(1 - \frac{p^2}{\alpha}\right) \right] . \qquad (2.15b)$$

These formulas are valid for any Hermitian matrix α (we need only regard them as applying separately in the various eigensubspaces of α) and both integrals vanish in any subspace in which p^2 is on the mass shell. It simplifies matters considerably to assume α to be a multiple of the identity, in which case $\gamma_0 C_1$ may be neglected altogether in second order.

$\gamma_0 C_2$ is finite and involves the integrals

$$\left.\begin{matrix} I_1 \\ I_2 \end{matrix}\right\} \equiv i \int \frac{dk}{(2\pi)^4} \frac{\lambda^2}{k^2 (\alpha + f)(k^2 - \lambda^2)} \times \left\{\begin{matrix} 1 \\ \gamma_0 \not{k} . \end{matrix}\right. \qquad (2.16a)$$

These are found to be

$$I_1 = \frac{1}{16\pi^2} \left[\frac{\alpha + \lambda^2}{2p^2} \ln\left(\frac{\lambda^2}{p^2 - \alpha}\right) + \frac{\alpha}{p^2} \ln\left(1 - \frac{p^2}{\alpha}\right) + \left(\frac{(\alpha + \lambda^2)^2}{4p^4} - \frac{\lambda^2}{p^2}\right)^{1/2} \ln\left(\frac{\lambda^2 - \alpha - [(\lambda^2 + \alpha)^2 - 4\lambda^2 p^2]^{1/2}}{\lambda^2 - \alpha + [(\lambda^2 + \alpha)^2 - 4\lambda^2 p^2]^{1/2}}\right) \right] , \qquad (2.16b)$$

$$I_2 = \frac{\gamma_0 \not{p}}{32\pi^2}\left[-\frac{\lambda^2}{p^2} + \left(\frac{(\lambda^2+\alpha)^2}{2p^4} - \frac{\lambda^2}{p^2}\right)\ln\left(\frac{\lambda^2}{p^2-\alpha}\right) + \frac{\alpha^2}{p^4}\ln\left(1-\frac{p^2}{\alpha}\right) + \frac{\lambda^2+\alpha}{p^2}\left(\frac{(\lambda^2+\alpha)^2}{4p^4} - \frac{\lambda^2}{p^2}\right)^{1/2}\ln\left(\frac{\lambda^2-\alpha-[(\lambda^2+\alpha)^2-4\lambda^2p^2]^{1/2}}{\lambda^2-\alpha+[(\lambda^2+\alpha)^2-4\lambda^2p^2]^{1/2}}\right)\right]. \tag{2.16c}$$

Equations (2.16b) and (2.16c) are valid provided $p^2 >$ largest eigenvalue of α. We will need the on-mass-shell limit of these formulas in the case where $x \equiv \lambda^2/p^2 \gg 1$. Namely,

$$I_1(\alpha=0) = \frac{1}{16\pi^2}\left[1 + \ln x + O\left(\frac{1}{x}\ln x\right)\right], \tag{2.17a}$$

$$I_2(\alpha=0) = \frac{\gamma_0 \not{p}}{32\pi^2}\left[-\tfrac{1}{2} + \ln x + O\left(\frac{1}{x}\ln x\right)\right]. \tag{2.17b}$$

Substituting (2.17a) and (2.17b) into (2.14a) and (2.14b), we obtain finally the correction to the zeroth-order mass matrix

$$\Delta m \simeq C_2 = \frac{1}{16\pi^2}[\gamma_0 t^{(\vec{n})}(\tfrac{5}{2} + \ln x_{\vec{n}})t^{(\vec{n})}\gamma_0 m - 4\gamma_0 t^{(\vec{n})}(1+\ln x_{\vec{n}})\gamma_0 m t^{(\vec{n})}], \tag{2.18}$$

where $x_{\vec{n}} \equiv \lambda_{\vec{n}}^2/m^2$.

One should note at this point that since the generators may contain γ_5's, it is in principle possible for C_2 to have the form $\alpha(p) + \beta(p)\gamma_5$, where α and β are proportional to the identity in Dirac space. The term linear in γ_5 does not, however, result in a mass shift in second order, only a wave-function renormalization. This is because

$$S_F = (\not{p}-m)^{-1} + (\not{p}-m)^{-1}[\alpha(p)+\beta(p)\gamma_5](\not{p}-m)^{-1}$$
$$= (\not{p}-m)^{-1} + (p^2-m^2)^{-2}(\not{p}+m)\alpha(p)(\not{p}+m) + \beta(p)(m^2-p^2)^{-1}.$$

III. GENERAL COMMENTS

The program we are following aims at establishing zeroth-order mass relations by restricting the representation content of the scalar fields which could couple G invariantly to the fermions. We must also ensure that after all the neutral scalar fields are given nonzero vacuum expectation values $\langle\phi\rangle_0$, the zeroth-order fermion mass matrix still preserves isospin (or more generally, whatever "slightly broken" symmetry we are interested in). This imposes stringent restrictions on the remaining Yukawa-coupled scalar fields: In general, it will imply that isospin must be contained in the little group of an arbitrary $\langle\phi\rangle_0$. There are, however, cases in which the extremization of an at most quartic scalar self-interaction seems to restrict automatically the possible $\langle\phi\rangle_0$ and in such cases the resulting enlargement of the little group may be sufficient to ensure preservation of isospin in zeroth order.[3,6] It should be noted that the model described below is *not* of this latter sort.

From the above discussion, it is clear that, if D_F denotes the representation of the fermions, $D_F^* \otimes D_F$ must be nontrivially reducible. Furthermore, we are especially interested in chiral groups of the form $G_L \times G_R$. In such models, if D_F is the representation $(\alpha, 0) \oplus (0, \alpha)$, then

$$D_F^* \otimes D_F = (\alpha^*, \alpha) \oplus (\alpha, \alpha^*).$$

Although this appears to be explicitly reducible, it must be noted that the presence in the Lagrangian of $\phi \in (\alpha^*, \alpha)$ implies the presence of $\phi^* \in (\alpha, \alpha^*)$. Our method therefore requires that (α^*, α) [or (α, α^*)] be itself reducible. In Appendix B, we show that if α (β respectively) are pseudoreal[7] irreducible representations of G_L (G_R respectively), where pseudoreal refers to a representation equivalent to its conjugate but not transformable into a real form, then the representation (α, β) of $G_L \times G_R$ is reducible into exactly two real irreducible component representations. Reducibility is of course essential if we are to obtain zeroth-order mass relations by a nontrivial restriction on the representation content of the scalar fields coupled to the nucleons. Furthermore, in the simplest case which comes to mind, namely, chiral SU(2)×SU(2), we show below that isospin is contained in the little group of an arbitrary $\langle\phi\rangle_0$ belonging to one of the two possible irreducible components.

The Model in Detail

We consider the most general renormalizable, gauge-invariant Lagrangian on the fields $\{\psi_N, \psi_l, W^\mu_{\alpha L}, W^\mu_{\alpha R}, Y_\mu, \phi, \chi_L, \chi_R\}$, the gauge group being $SU(2)_L \times SU(2)_R \times U(1)$. The fields are, respectively, the nucleons, leptons, "left-handed," "right-handed," and "hypercharge" vector mesons, Yukawa-coupled scalar fields, and two types of non-Yukawa-coupled scalars whose vacuum expectation values will eventually break isospin. These fields belong to the following representations of $SU(2)_L \times SU(2)_R \times U(1)$:

$$\begin{aligned}
&\psi_N^L \equiv \frac{1-\gamma_5}{2}\psi_N \in (\tfrac{1}{2}, 0), \quad Y=1,\\
&\psi_N^R \equiv \frac{1+\gamma_5}{2}\psi_N \in (0, \tfrac{1}{2}), \quad Y=1,\\
&\psi_l^L \in (\tfrac{1}{2}, 0)_e \oplus (\tfrac{1}{2}, 0)_\mu, \quad Y=-1,\\
&\psi_l^R \in (0,0)_e \oplus (0,0)_\mu, \quad Y=-2,\\
&W^\mu_{\alpha L} \in (1,0), \quad Y=0,\\
&W^\mu_{\alpha R} \in (0,1), \quad Y=0,\\
&Y^\mu \in (0,0), \quad Y=1,\\
&\phi \in (\tfrac{1}{2}, \tfrac{1}{2}), \quad Y=0,\\
&\chi_L \in (\tfrac{1}{2}, 0), \quad Y=1,\\
&\chi_R \in (0, \tfrac{1}{2}), \quad Y=1.
\end{aligned} \tag{3.1}$$

The hypercharges have been chosen so that $Q \equiv T_{3L} + T_{3R} + \frac{1}{2}Y$ is the conventional charge operator.

Restricting our attention for the moment to the nucleons, the most general Yukawa coupling involves the complex field $\phi \in (\frac{1}{2}, \frac{1}{2})$, which we represent as a complex 2×2 matrix. Under an arbitrary transformation of $SU(2)_L \times SU(2)_R$, the fields ψ_N and ϕ transform as follows:

$$\psi_N^L \to U_L \psi_N^L, \quad \psi_N^R \to U_R \psi_N^R, \quad \phi \to U_L \phi U_R^\dagger. \tag{3.2}$$

The scalar-nucleon coupling has the form

$$\mathcal{L}_{\text{Yukawa}} = -G_N(\bar{\psi}_N^L \phi \psi_N^R + \text{H.c.}). \tag{3.3}$$

The theorem mentioned above (cf. Appendix B) asserts the invariance under all $SU(2)_L \times SU(2)_R$ transformations of the subspaces

$$V_\pm \equiv \{\phi' \,|\, \phi' = \tfrac{1}{2}(\phi \pm S\phi^* S^{-1}),\ \phi \in (\tfrac{1}{2}, \tfrac{1}{2})\}, \tag{3.4}$$

where S is the similarity transformation defined by

$$S^{-1} U S = U^*, \quad \forall U \in SU(2), \tag{3.5}$$

and may be taken to be

$$i\tau_2 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In order to obtain a zeroth-order mass relation we now restrict the representation content of $D_F^* \otimes D_F$ by imposing the $SU(2)_L \times SU(2)_R$-invariant constraint

$$\phi = S\phi^* S^{-1}, \tag{3.6}$$

which eliminates the V_- representation and implies that ϕ is of the form

$$\phi = \begin{pmatrix} \eta & \zeta \\ -\zeta^* & \eta^* \end{pmatrix}$$

$$(\eta, \zeta \text{ arbitrary complex fields}). \tag{3.7}$$

The neutral scalar fields with hypercharge zero are those satisfying

$$\tau_3 \phi_0 = \phi_0 \tau_3 \Rightarrow \phi_0 = \begin{pmatrix} \eta & 0 \\ 0 & \eta^* \end{pmatrix}, \tag{3.8}$$

so the vacuum expectation value is necessarily of the form

$$\langle \phi \rangle_0 = \sigma \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}, \tag{3.9}$$

which is seen to have (in addition to hypercharge) the little group SU(2) of transformations

$$\phi \to U\phi \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} U^\dagger \begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix}, \tag{3.10}$$

where U is an arbitrary unitary unimodular 2×2 matrix. On redefining the right-handed nucleon doublet by a rotation of φ about the right-handed three axis, the zeroth-order nucleon mass matrix becomes simply

$$\sigma\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

exhibiting the desired zeroth-order mass relation.

In order to apply the results of Sec. II, we work in a representation in which all fields are Hermitian, the structure constants totally antisymmetric, and the (Hermitian) generators purely imaginary. We define the generators acting on the various scalar fields in the usual way (where, for convenience, we suppose the χ_R field to be absent for the time being):

$$U_{L,R}\begin{pmatrix} \Phi \\ \chi_L \end{pmatrix} = (1 + i\epsilon_\alpha^{L,R}\theta_{L,R}^\alpha)\begin{pmatrix} \Phi \\ \chi_L \end{pmatrix}, \tag{3.11a}$$

$$U_Y\begin{pmatrix} \Phi \\ \chi_L \end{pmatrix} = (1 + i\epsilon^Y \theta_Y)\begin{pmatrix} \Phi \\ \chi_L \end{pmatrix}, \tag{3.11b}$$

where

$$(\Phi, \chi_L) \equiv (\text{Re}(\zeta), \text{Im}(\zeta), \text{Re}(\eta), \text{Im}(\eta), \text{Re}(\chi_L^1), \text{Im}(\chi_L^1), \text{Re}(\chi_L^2), \text{Im}(\chi_L^2)).$$

Here U_L, for example, is represented in the fundamental representation of $SU(2)_L$ by $(1 + ig_L \epsilon_\alpha^L \tau^\alpha)$, where $\tau^\alpha \equiv \frac{1}{2}\sigma^\alpha$, and the σ^α's are the usual Pauli matrices.

The matrices representing the generators on the eight-dimensional reducible representation of the scalars are found to be

$$i\theta_L^1 = g_L\begin{pmatrix} 0 & \tau_1 & 0 & 0 \\ -\tau_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\tau_2 \\ 0 & 0 & -i\tau_2 & 0 \end{pmatrix},\quad i\theta_L^2 = g_L\begin{pmatrix} 0 & \tau_3 & 0 & 0 \\ -\tau_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}I \\ 0 & 0 & -\frac{1}{2}I & 0 \end{pmatrix},\quad i\theta_L^3 = g_L\begin{pmatrix} -i\tau_2 & 0 & 0 & 0 \\ 0 & -i\tau_2 & 0 & 0 \\ 0 & 0 & -i\tau_2 & 0 \\ 0 & 0 & 0 & i\tau_2 \end{pmatrix},$$

$$i\theta_R^1 = g_R\begin{pmatrix} 0 & i\tau_2 & 0 & 0 \\ i\tau_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\quad i\theta_R^2 = g_R\begin{pmatrix} 0 & -\frac{1}{2}I & 0 & 0 \\ \frac{1}{2}I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.12}$$

$$i\theta_R^3 = g_R\begin{pmatrix} -i\tau_2 & 0 & 0 & 0 \\ 0 & i\tau_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\quad \frac{i\theta^Y}{2} = g'\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i\tau_2 & 0 \\ 0 & 0 & 0 & -i\tau_2 \end{pmatrix}.$$

Inclusion of another scalar field $\chi_R \in (0, \frac{1}{2})$ is straightforward in principle, although we can no longer rotate away phases by a transformation $e^{i\lambda\theta^Y}$, so that this generalization introduces two additional real parameters.

The vacuum expectation values of the fields will be taken to be

$$\langle(\Phi, \chi_L)\rangle_0 = (0, 0, \sigma, 0, 0, 0, \chi, 0). \tag{3.13}$$

The zeroth-order vector-meson mass matrix is now obtained directly from (2.2), the result being

$$\mu^2 = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & B \end{pmatrix}, \tag{3.14a}$$

where $A(B)$ are the mass matrices for the charged (neutral) mesons and are given explicitly by

$$A = \tfrac{1}{4}\begin{pmatrix} g_L^2(\sigma^2+\chi^2) & -g_L g_R\sigma^2 \\ -g_L g_R \sigma^2 & g_R^2\sigma^2 \end{pmatrix}, \tag{3.14b}$$

$$B = \tfrac{1}{4}\begin{pmatrix} g_L^2(\sigma^2+\chi^2) & -g_L g_R\sigma^2 & -2g'g_L\chi^2 \\ -g_L g_R\sigma^2 & g_R^2\sigma^2 & 0 \\ -2g'g_L\chi^2 & 0 & 4g'^2\chi^2 \end{pmatrix}. \tag{3.14c}$$

Finally, we note that as far as the fermion masses are concerned, if G_N, G_e, and G_μ are the Yukawa coupling strengths of the ϕ, χ_L, and χ_L fields to the nucleon, electron, and muon doublets, respectively, we have in zeroth order

$$m_N = G_N\sigma, \qquad m_{e,\mu} = G_{e,\mu}\chi. \tag{3.15}$$

Of course, the leptons are treated here in a manner identical to original broken symmetry model of leptons advanced by Weinberg.[8] The χ_L field is necessary to give the leptons a mass; we find below that implementation of weak universality leads in addition to the presence of a field χ_R.

IV. EXPERIMENTAL CONSTRAINTS AND THE NUCLEON MASS SPLITTING

In view of the many qualitative features of hadronic interactions omitted, the model described above is really only useful for illustrative purposes. It is nevertheless amusing to see what constraints experiment places on the parameters of the model, and hence, on the mass relations obtained. For example, we will require that the electric charge and the β and muon decay constants should have their experimental values.

We first look at the mass matrix of the neutral vector bosons to determine the electric charge. The massless vector boson (namely, the photon) is

$$A^\mu = [g_L^{-2} + g_R^{-2} + (2g')^{-2}]^{-1/2} \times\left(\frac{1}{g_L}W_{3L}^\mu + \frac{1}{g_R}W_{3R}^\mu + \frac{1}{2g'}Y^\mu\right).$$

To find the electric charge e,[9] one must substitute A^μ back into the original Lagrangian and determine its coupling constant. At first this appears to require knowledge of C^{-1}, where C is the matrix defined by

$$\begin{pmatrix} A^\mu \\ H_1^\mu \\ H_2^\mu \end{pmatrix} = C\begin{pmatrix} W_{3L}^\mu \\ W_{3R}^\mu \\ Y^\mu \end{pmatrix},$$

with $H_{1,2}^\mu$ the heavy neutral fields of definite mass. However, to find the photon coupling we only need the first column of C^{-1} and, since C is orthogonal, this equals the first row of C which is just

$$[g_L^{-2} + g_R^{-2} + (2g')^{-2}]^{-1/2}\left(\frac{1}{g_L}, \frac{1}{g_R}, \frac{1}{2g'}\right).$$

Hence, one can easily substitute back into the Lagrangian and find

$$e^{-2} = g_L^{-2} + g_R^{-2} + (2g')^{-2}. \tag{4.1}$$

Similarly, let $M_\pm$ be the masses of the two positively charged vector mesons of definite mass, with $M_+ > M_-$. Let α, β determine the eigenvectors by

$$\alpha W^{\mu}_{1+i2,L} + \beta W^{\mu}_{1+i2,R} = W^{\mu}_{+}, \qquad -\beta W^{\mu}_{1+i2,L} + \alpha W^{\mu}_{1+i2,R} = W^{\mu}_{-}, \tag{4.2}$$

with $\alpha^2 + \beta^2 = 1$. Then, inverting this equation and substituting into the parts of the Lagrangian relevant to β and muon decay, one obtains

$$\mathcal{L}_\beta = G_\beta[\bar{p}\gamma^\mu(1 + r\gamma_5)n]\left[\bar{e}\gamma_\mu\left(\frac{1-\gamma_5}{2}\right)\nu_e\right], \tag{4.3}$$

$$\mathcal{L}_\mu = G_\mu\left[\bar{e}\gamma^\mu\left(\frac{1-\gamma_5}{2}\right)\nu_e\right]\left[\bar{\nu}_\mu\gamma_\mu\left(\frac{1-\gamma_5}{2}\right)\mu\right], \tag{4.4}$$

$$G_\beta \equiv \frac{g_L}{2}\left[\frac{g_L}{2}\left(\frac{\alpha^2}{M_+^2} + \frac{\beta^2}{M_-^2}\right) + \frac{g_R}{2}\alpha\beta\left(\frac{1}{M_+^2} - \frac{1}{M_-^2}\right)\right], \tag{4.5}$$

$$r \equiv \frac{\alpha\beta g_R(M_-^2 - M_+^2) - g_L(M_-^2\alpha^2 + M_+^2\beta^2)}{\alpha\beta g_R(M_-^2 - M_+^2) + g_L(M_-^2\alpha^2 + M_+^2\beta^2)}, \tag{4.6}$$

$$G_\mu \equiv \frac{g_L^2}{4}\left(\frac{\alpha^2}{M_+^2} + \frac{\beta^2}{M_-^2}\right). \tag{4.7}$$

Weak universality indicates that $G_\beta \simeq G_\mu$. This would also give $r \simeq -1$, a constraint which is consistent with the fact that we have ignored the strong interactions throughout. In order to implement this constraint, one needs

$$\left|\frac{g_R(\alpha/\beta)(M_+^2 - M_-^2)}{g_L[(\alpha/\beta)^2 M_-^2 + M_+^2]}\right| \ll 1. \tag{4.8}$$

It would be convenient, if this inequality could be used to constrain the parameters of the model. To this end, we examine the eigenvalues of the charged meson mass matrix (for $\langle\chi_L\rangle \equiv \chi \neq 0$, $\langle\chi_R\rangle = 0$):

$$M_\pm^2 = \tfrac{1}{8}\left((g_L^2 + g_R^2)\sigma^2 + g_L^2\chi^2 \pm \{[(g_L^2 + g_R^2)\sigma^2 + g_L^2\chi^2]^2 - 4g_R^2 g_L^2\chi^2\sigma^2\}^{1/2}\right). \tag{4.9}$$

Substituting these values of $M^\pm$ and the corresponding values of α, β into (4.8), one obtains a very involved inequality relating the independent parameters. One possible approach would be to substitute numerical values for the parameters to determine whether any set satisfying the inequality could be found. We have not done this. Instead, we have looked at the three regions in the parameter manifold where it would appear that the numerator of (4.8) could be made much smaller than the denominator: namely, (1) $|g_R/g_L| \ll 1$, (2) $M_+^2 \simeq M_-^2$, and (3) $|\alpha/\beta|$ (or $|\beta/\alpha|$) $\ll 1$.

Case 1. For $|g_R/g_L| \ll 1$,

$$M_+^2 \simeq \tfrac{1}{4}g_L^2(\sigma^2 + \chi^2)$$

and

$$M_-^2 \simeq \tfrac{1}{4}g_R^2\frac{\chi^2\sigma^2}{\chi^2 + \sigma^2}$$

from (4.9). So from (4.2),

$$\frac{\alpha}{\beta} \simeq \frac{g_L}{g_R}\frac{\sigma^2 + \chi^2}{\sigma^2}.$$

Substituting into (4.8), $(G_\beta - G_\mu)/G_\mu$ is found to be of order unity. In other words, although g_R/g_L is small, α/β is large so that the universality constraint is not satisfied.

Case 2. For $M_+ \simeq M_-$ inspection of (4.9) indicates that one needs

$$\left|\frac{g_R}{g_L}\right| \gg 1$$

and

$$g_R^2\sigma^2 \simeq g_L^2\chi^2 + \lambda g_L^2\sigma^2,$$

where λ is a parameter of order unity. Equation (4.2) gives

$$\frac{\alpha}{\beta} \simeq 1 - \frac{g_L(1+\lambda)}{g_R}.$$

Substitution in (4.8) again gives $(G_\beta - G_\mu)/G_\mu$ of order unity.

Case 3. In the case where $|\alpha/\beta| \ll 1$ or $|\beta/\alpha| \ll 1$, the off diagonal elements of the mass matrix are small. Again, simple arguments analogous to the first two cases result in $(G_\beta - G_\mu)/G_\mu$ being of order unity.

In short, it appears that one cannot incorporate weak-interaction universality into the present scheme without some modifications. Probably the easiest solution is to give a nonzero vacuum expectation value to the field χ_R, $\langle\chi_R\rangle_0 \equiv \bar{\chi} \neq 0$. Furthermore, in calculating the vector-meson masses $\langle\chi_L\rangle_0$ can be neglected since it is needed only to produce the lepton masses $m_l = G_l\langle\chi_L\rangle_0$. $\langle\chi_L\rangle_0$ can therefore be chosen arbitrarily small compared to $\bar{\chi}$ and σ. The effect of all this on the mass matrix is to systematically interchange left and right and to replace χ by $\bar{\chi}$. With this new mass matrix it becomes easy to incorporate universality. In particular, if $|g_R/g_L| \ll 1$, one finds $|(G_\beta - G_\mu)/G_\mu| \ll 1$; this is true in the limits

$$\eta \equiv \frac{g_L^2\sigma^2}{g_R^2\bar{\chi}^2} \gg 1, \quad \eta \ll 1, \quad \text{and also for} \quad \eta = 1.$$

A comment may be relevant here concerning the constraint $g_R \ll g_L$. We know from (4.1) that $g_R, g_L > e$. Therefore $g_L \gg e$ presumably implies $g_L > 1$.

This means that whereas the theory is renormalizable, for any process for which the coupling constant is simply g_L the usefulness of lowest-order perturbation theory is probably limited. Of course, the usual weak and electromagnetic processes of the known particles involve expansions in combinations of the coupling constants, e.g.,

$$[g_L^{-2}+g_R^{-2}+(2g')^{-2}]^{-1/2},$$

which are less than unity.

We now substitute the parameters of the model (incorporating weak universality) into the general expression for the second-order mass shift derived in Sec. II. Equation (2.18) can be written

$$\Delta m=\frac{m}{16\pi^2}[\gamma_0(t^{(\bar{\pi})})^2(\tfrac{5}{2}+\ln x_{\bar{\pi}})\gamma_0-4(\gamma_0 t^{(\bar{\pi})})^2(1+\ln x_{\bar{\pi}})]. \tag{4.10}$$

We recall that in this model m commutes with all the generators. We next observe that the only way a mass splitting can arise is through a cross term involving a τ matrix and the generator Y, since both Y^2 and τ_i^2 are proportional to the identity, and since τ_i does not couple to τ_j in the mass matrix unless $i=j$. As a result, only those diagonalized generators which contain the hypercharge can contribute. These are precisely the neutral mesons. In this case $t_i=\alpha_i t_{3L}+\beta_i t_{3R}+\gamma_i t_Y$, and one finds that

$$t_i^2 \doteq 2\gamma_i g' Y\tau_3[(\alpha_i g_L+\beta_i g_R)+\gamma_5(\beta_i g_R-\alpha_i g_L)], \tag{4.11}$$

$$(\gamma_0 t_i)^2 \doteq 2\gamma_i g' Y\tau_3(\alpha_i g_L+\beta_i g_R), \tag{4.12}$$

where the $\doteq$ reminds us that terms proportional to the identity are dropped. We recall (cf. Sec. II) that terms linear in γ_5 can be dropped in the calculation of the lowest-order mass splitting. Hence the formula for the mass shift simplifies to

$$\Delta m \doteq \frac{3me^2\tau_3}{16\pi^2}\left[\tfrac{1}{2}+\ln\left(\frac{\lambda_1^2}{m^2}\right)\right]+\frac{3m}{16\pi^2}\ln\left(\frac{\lambda_1^2}{\lambda_2^2}\right)t_2^2. \tag{4.13}$$

$\lambda_{1,2}$ and $t_{1,2}$ are the masses and generators of the two heavy neutral mesons. Note that if $\lambda_1, \lambda_2\to\infty$ with λ_1/λ_2 finite (which is quite possible in our framework, if for instance $g_L\to\infty$, $g_R\to\infty$, $g'\to\frac{1}{2}e$, g_L/g_R fixed), then

$$\lim_{\lambda_i\to\infty}(m_p-m_n)=\frac{3me^2}{16\pi^2}\ln\frac{\lambda_i^2}{m^2}, \tag{4.14}$$

which is just the conventional divergent result of quantum electrodynamics[10] (QED). One sees how the neutral meson masses have entered naturally to become the cutoffs needed in QED.

We next observe that the contribution to m_p-m_n of the first term of (4.13) (which might be called the "photon" term) is always positive. The second term is

$$\frac{3m}{16\pi^2}\ln\left(\frac{\lambda_1^2}{\lambda_2^2}\right)t_2^2 \doteq \frac{3m}{16\pi^2}\ln\left(\frac{\lambda_1^2}{\lambda_2^2}\right)(-4g'^2\tau_3) \times\left[\alpha_2^2+\beta_2^2+\alpha_2\beta_2\left(\frac{g_L}{g_R}+\frac{g_R}{g_L}\right)\right], \tag{4.15}$$

where we have used (4.11), (4.12), and the orthogonality relation

$$\frac{\alpha_i}{g_L}+\frac{\beta_i}{g_R}+\frac{\gamma_i}{2g'}=0.$$

If, say, $\lambda_1<\lambda_2$, one can see that the only way to to achieve $m_n>m_p$ is to have the interference term

$$\frac{3m\alpha_2\beta_2}{16\pi^2}\ln\left(\frac{\lambda_1^2}{\lambda_2^2}\right)(-4g'^2\tau_3)\left(\frac{g_L}{g_R}+\frac{g_R}{g_L}\right)$$

larger in absolute value than the sum of the "photon" contribution and the noninterference term. This is difficult to do since the requirement that the photon term be small compared to the interference term (namely, $\lambda_2\gg\lambda_1$) implies that the latter is actually smaller than the noninterference term, so that we again find $m_p>m_n$. In particular, we have examined the behavior of this function in the region $g_L/g_R\gg1$ (because of universality), $\gamma\equiv 2g'/g_R\simeq 1$, and $\rho\equiv g_R^2\bar{\chi}^2/g_L^2\sigma^2$ arbitrary and in the region $g_L/g_R\gg 1$, $\rho\simeq 1$, γ arbitrary. We have not found any solutions which give the neutron a larger mass than the proton. However, we have not done a systematic computer test of all the permissible values of the parameters,[4] and the possibility of generating a positive proton-neutron mass difference cannot be definitely ruled out.

V. DISCUSSION AND SUMMARY

We have presented a fairly detailed calculation of the proton-neutron mass difference within a renormalizable model of the weak and electromagnetic interactions of the type described in Ref. 2.

As one would expect with a model of this class, the mass difference is finite. The actual value depends on several free parameters of the model. However, it appears to be quite difficult to arrange these parameters so that the neutron is more massive than the proton. The dominant "wrong-sign" contribution of the mass difference can be interpreted as the photon exchange contribution and can indeed be seen to reduce to the conventional divergent value when the weak interactions are neglected, i.e., vector-meson masses go to infinity.

The arbitrariness and limitations of some parts of this model have been emphasized throughout this paper. What sorts of improvements would be necessary to make the model more realistic?

First of all, the strong interactions have been completely neglected within this framework. We have treated the nucleons as pointlike Dirac particles which is certainly incorrect. A first step might be to calculate the Feynman integrals dispersively, replacing the point particles with their experimentally obtained form factors. This would have to be supplemented by some hypothesis, such as the parton model, to relate the electromagnetic form factors to the neutral heavy boson ones which are, of course, unknown. Such dispersive techniques have been used within the context of neutron-proton mass differences in unified renormalizable theories of the weak and electromagnetic interactions; however, these calculations did not yield finite results since the models used were not of the class we have discussed.[11,12] Another possible approach to incorporating the strong interactions is that of building a gauge theory of the strong interactions and attempting a grand synthesis of strong, weak, and electromagnetic processes.[13] One consequence of such an approach is that it would presumably lead to inclusion of the familiar vector mesons, e.g., the ρ meson, among the gauge fields. This in turn would violate the assumption that $m^2 \ll \lambda_i^2$ and require the preservation of terms proportional m^2/λ_i^2, in particular, the scalar-meson contribution and the $k^\mu k^\nu$ term in the vector-meson exchange. Perhaps these additional contributions would cancel the "wrong-sign" contribution discussed above.

Another limitation of the present model is its total omission of the strange particles. Simple generalizations, such as to SU(3)×SU(3), where the fermions are quark fields, suffer from the drawback that the fermion representation need not be pseudoreal [for example, the (3, 0) representation of SU(3)×SU(3)]. In such cases, the product representation is irreducible and the general approach we have been using is no longer applicable. If on the other hand the fermions form an octet, e.g., the baryon octet, the Yukawa-coupled scalar representation contains vectors with nonconjugate little groups. In this case, either the desired zeroth-order symmetry (here isospin) is preserved to all orders, or the renormalized Lagrangian fails to preserve the symmetry even in zeroth order.

Of course, there are many other compact simple Lie groups with pseudoreal representations to choose from (see Appendix B). Also, using product groups is not the only way to make $D_F^* \otimes D_F$ reducible, and hence make zeroth-order mass relations possible. On the other hand, finding realistic models of this form which incorporate the strange particles appears to be quite difficult. As a result, it may be wise to also explore the mechanisms of "accidental" mass relations,[3,6] mentioned above, in attempts to find realistic models of weak and electromagnetic hadronic mass differences.

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APPENDIX A

In Ref. 2, it is tacitly assumed that the symmetry is completely broken. This implies that all the vector mesons are massive and, hence, that $\mu_{\alpha\beta}{}^2$ is invertible. When there is a residual unbroken symmetry, e.g., charge conservation, the analysis goes through with the following minor modifications. First, after diagonalizing the gauge meson mass matrix, we can write it as

$$\begin{bmatrix} \lambda_1{}^2 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_r{}^2 & & & & \\ & & & 0 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

and the gauge meson propagator $D^{\mu\nu}_{\alpha\beta}$ becomes

$$\begin{bmatrix} D^{\mu\nu}(\lambda_1{}^2) & & & & \\ & \ddots & & & \\ & & D^{\mu\nu}(\lambda_r{}^2) & & \\ & & & -g^{\mu\nu}/k^2 & \\ & & & & \ddots \\ & & & & & -g^{\mu\nu}/k^2 \end{bmatrix}$$

with

$$D^{\mu\nu}(\lambda_i{}^2) = \frac{-g^{\mu\nu} + k^\mu k^\nu/\lambda_i{}^2}{k^2 - \lambda_i{}^2} .$$

Secondly, the projection operator is modified to

$$\Pi'_{ij} \equiv \delta_{ij} + \sum_{\alpha,\beta=1}^{r} (\theta_\beta \langle \phi \rangle_0)_i \, \mu'^{-2}{}_{\alpha\beta} (\theta_\alpha \langle \phi \rangle_0)_j ,$$

where $\mu'^2{}_{\alpha\beta}$ is the $r \times r$ matrix formed by the massive vector bosons; one can easily verify that Π'_{ij} annihilates all fields of the form $\theta_\gamma \langle \phi \rangle_0$ – i.e., the Goldstone bosons. Either $1 \leq \gamma \leq r$ in which case the calculation is identical to the original, total-symmetry-breaking case, or else $\gamma > r$ indicating that θ_γ is an unbroken generator and hence $\theta_\gamma\langle\phi\rangle_0=0$ anyway.

APPENDIX B

Theorem. Let α,β denote pseudoreal irreducible unitary representations of G_L, G_R respectively (by pseudoreal,[7] one means a representation equivalent to its conjugate but not transformable into a real form). Then the representation (α,β) of $G_L\times G_R$ is reducible into exactly two real irreducible components.

Proof. Let $\phi_{ij}\in(\alpha,\beta)$ be a complex matrix transforming in the usual manner under elements of $G_L\times G_R$:

$$\phi\to U_L\phi U_R^\dagger .$$

The most general linear recombination of the Hermitian fields comprising ϕ has the form

$$(L\phi)_{ij}=\phi'_{ij}=c_{ij,kl}\,\phi_{kl}+d_{ij,kl}\,\phi^*_{kl} .$$

If τ_L^α (τ_R^β) represent the skew-Hermitian generators of G_L (G_R) on the representations α (β), then L commutes with all the generators of $G_L\times G_R$ if and only if

$$(\tau_L^\alpha)_{mi}\,c_{ij,kl}\pm(\tau_L^\alpha)_{mi}\,d_{ij,kl}=c_{mj,il}(\tau_L^\alpha)_{ik}\pm d_{mj,il}(\tau_L^\alpha)^*_{ik},$$

$$(\tau_R^\alpha)_{jm}\,c_{ij,kl}\pm(\tau_R^\alpha)_{jm}\,d_{ij,kl}=c_{im,kr}(\tau_R^\alpha)_{lr}\pm d_{im,kr}(\tau_R^\alpha)^*_{lr}.$$

But the representations α,β are irreducible, so by Schur's lemma

$$c_{ij,kl}=\lambda\delta_{ik}\,\delta_{jl} .$$

Also, by assumption $\exists\, S_L, S_R$ (antisymmetric) such that

$$S_{L,R}{}^{-1}\tau^\alpha_{L,R}S_{L,R}=\tau^{\alpha*}_{L,R} .$$

Again applying Schur's lemma, one finds

$$d_{ij,kl}=\mu(S_R{}^{-1})_{lj}(S_L)_{ik} .$$

So the most general operator commuting with all the generators is

$$(L)_{ij,kl}=\lambda\delta_{ik}\,\delta_{jl}+\mu(S_R{}^{-1})_{lj}(S_L)_{ik}K ,$$

where K is the complex conjugation operator. Alternatively,

$$L=\lambda'P_+ +\mu'P_- ,$$

$$(P_\pm)_{ij,kl}\equiv\frac{\delta_{ik}\,\delta_{jl}\pm(S_R{}^{-1})_{lj}(S_L)_{ik}K}{2} .$$

$P_\pm$ are readily seen to be projection operators: They project onto the disjoint irreducible representations

$$V_\pm\equiv\{\phi'\,|\,\phi'=\tfrac{1}{2}(\phi\pm S_L\phi^*S_R{}^{-1}),\ \phi\in(\alpha,\beta)\}.\quad\text{QED}.$$

The classification given in Ref. 7 shows that the only pseudoreal representations of the compact simple Lie groups are the following:

(a) $SU(2(2m+1))$: The only pseudoreal representations are characterized by a highest weight of the form $(\lambda_1,\ldots,\lambda_{2m},\lambda_{2m+1}\text{ odd},\lambda_{2m},\ldots,\lambda_1)$.

(b) $O(8n+3)$: $(\lambda_1\text{ odd},\lambda_2,\ldots,\lambda_{4n+1})$.
$O(8n+5)$: $(\lambda_1\text{ odd},\lambda_2,\ldots,\lambda_{4n+2})$.

(c) All irreducible unitary representations (IUR's) $(\lambda_1,\ldots,\lambda_n)$ of $Sp(n)$ with $\lambda_1+\lambda_3+\ldots$ odd.

(d) All IUR's $(\lambda_1,\ldots,\lambda_{4n+2})$ of $O(4(2n+1))$ with $\lambda_1+\lambda_2$ odd.

(e) All IUR's $(\lambda_1,\ldots,\lambda_7)$ of E_7 with $\lambda_2+\lambda_3+\lambda_6$ odd.

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