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†Max Kade Foundation Fellow, on leave of absence from Technical University, Vienna, Austria.

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Singular Cores in the Three-Body Problem. I. Theory*

D. D. Brayshaw

Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742

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The three-body formalism for singular cores previously introduced by the author is considered in some detail. A new derivation is presented which clearly demonstrates the uniqueness of this formalism and clarifies its relationship to appropriate boundary conditions on the three-body wave function. It is shown that an auxiliary boundary condition must be imposed to uniquely specify a solution; this leads to an integral equation with a square-integrable kernel. A detailed proof of three-particle unitarity is given for the amplitudes defined by this equation, and explicit formulas are presented for a representative model.

I. INTRODUCTION

In a recent letter,[1] the present author introduced a generalization of the Faddeev formalism to include two-body interactions whose extremely short-range behavior is characterized by a hard core, or by a boundary condition on the wave function (BCM). Using the special properties of the BCM t matrix developed earlier,[2] it was shown that the usual Faddeev equations do not yield a unique solution for such interactions, but that a particular solution can be defined which yields the desired physical properties. In particular, the resultant three-body wave function vanishes whenever any pair of particles are within their respective core radius, while its asymptotic behavior corresponds to a unitary three-particle t matrix. In this paper we give detailed proofs of these assertions, present a new derivation of our equation which clearly demonstrates its uniqueness, and consider in some detail the special case of BCM alone (no external potential). This provides the theoretical groundwork for subsequent articles in this series dealing with the actual solution of our equations for specific models.

The principal motivation for this development is the versatility afforded by being able to utilize this additional class of interactions in the three-body problem. For example, calculations to date in the three-nucleon system with realistic interactions have been almost exclusively restricted to soft-core models, the single exception being the long and difficult variational calculation on the Hamada-Johnston hard core by Delves *et al.*[3] The results of these computations have generated some doubt as to the ability of such models to fit the experimental data. For example, it appears that any soft-core model which fits the two-nucleon phase shifts reasonably well will underbind the triton by about 2 MeV. It has also been suggested that a significant discrepancy exists in the case of the triton charge form factor.[4] Of course, it is quite possible that the source of such disagreement does not lie with the nature of the potential model, but with the neglect of corrections due to three-body forces and relativistic effects, which could well be significant. However, the inclusion of such corrections would be largely *ad hoc*, and would greatly diminish the predictive power of the present theory. It thus seems highly desirable to explore such possibilities as might be afforded by an enlarged class of two-body interactions before resigning oneself to this situation.

From this point of view there are two excellent reasons for utilizing the singular-core models. The first is simply that both the hard core and the BCM have been employed as the basis for excellent fits to the two-nucleon data.[5] Secondly, it is not unreasonable to expect such models to produce qualitatively different results in the three-body problem; functionally, the corresponding off-shell t matrices are quite different from those of soft cores, exhibiting the typical oscillatory properties of entire functions. Moreover, as shall be demonstrated in Sec. II of this paper, the inclusion of singular cores requires significant changes in the whole theoretical structure upon which current calculations have been based, i.e., the Faddeev equations.[6] Whether or not singular cores can reduce the discrepancy with experiment is of course speculation; the single piece of information one has (the Delves calculation which also underbinds by 2 MeV) is not too encouraging. On the other hand, there is still plenty of room for modifications in the form of the potential external to the hard core, i.e., a "revised Hamada-Johnston"; and the BCM model of Feshbach and Lomon[5] is totally unexplored. In any case, calculations with singular core models are bound to produce either one of two very interesting results: (1) Such a description of the very-short-range nucleon-nucleon interaction is indeed more "realistic,"[7] as evidenced by better predictions of three-nucleon observables; or (2) the inclusion of corrections due to three-body forces or relativistic effects is absolutely essential in order to understand the data.

Nuclear physics aside, the formalism also leads to a number of applications of interest to statistical and chemical physics. An example is the third virial coefficient for a (quantum-mechanical) system of hard spheres. This can be obtained knowing the wave function for three particles interacting via hard cores,[8] a special case of our formalism. In fact, with no increase in difficulty, one could also perform such a calculation with hard cores plus weak attractive forces characterized by the BCM. Such computations would be facilitated by a fact pointed out in B1; namely, that for the BCM (or hard core) alone, our equation can be reduced to integral equations in only one variable.

We begin in Sec. II with a brief review of the development given in B1. By observing a special property of the BCM t matrix unnoticed in our earlier work, we are able to present a new derivation for our equation which emphasizes the fact that it is unique. We also clarify the relationship between our formalism and direct imposition of the boundary condition on the three-body wave function. In Sec. III we introduce a "supervector" notation in order to simplify evaluation of the operator product IQ appearing in our kernel.

The structure of our equation is analyzed in Sec. IV, where we consider the simplest possible case in some detail. We thereby demonstrate that an auxiliary boundary condition must be added to the previous development in order to uniquely specify a solution; our equation then reduces to a less complex form with a square-integrable kernel. As an illustration, we present explicit formulas for the driving term and kernel relevant to this model. The structure of corresponding equations for the general problem is outlined at the end of the section.

Section V is devoted to explicit proofs of the three-particle unitarity relations for our amplitudes. At the same time, the operator notation introduced in B1 (and recapitulated in Sec. II) is employed to construct particularly transparent derivations of unitarity for the usual Faddeev amplitudes.

Finally, in Sec. VI we discuss various aspects of the formalism and its relation to the work of other authors. In the Appendix we give a derivation of the operator Q which plays a crucial role in our development.

II. THREE-BODY FORMALISM FOR SINGULAR CORES

In this section we briefly review the theoretical development given in B1, recapitulating some useful notational conventions. We also present a new derivation of the integral equation introduced in B1. This derivation supplements the previous (more physical) argument by clearly demonstrating the fact that our new equation is unique. As in B1, we make the nonessential but simplifying assumption that our three particles are spinless.

We denote the mass of particle α by m_α and the total three-body c.m. energy by W. Three-particle states are described by the usual Jacobi variables $\vec{p}_\alpha, \vec{q}_\alpha$, with the corresponding reduced masses μ_α, M_α:

$$\begin{aligned} \mu_\alpha^{-1} &= m_\beta^{-1} + m_\gamma^{-1}, \\ M_\alpha^{-1} &= m_\alpha^{-1} + (m_\beta + m_\gamma)^{-1}, \end{aligned} \tag{1}$$

and $(\alpha\beta\gamma)$ are cyclic permutations of (123). In the usual channel decomposition, the three-body state vector is $|\Psi\rangle = \sum_\alpha |\psi_\alpha\rangle$, where the $|\psi_\alpha\rangle$ satisfy

$$|\psi_\alpha\rangle = (1 - G_0 t_\alpha)|\phi\rangle - G_0 t_\alpha \sum_{\beta\neq\alpha} |\psi_\beta\rangle . \qquad (2)$$

Here t_α represents the two-body t matrix as an operator in the three-body Hilbert space, $|\phi\rangle$ is a plane-wave state, and $G_0 = G_0(W)$ is the free Green's function. Equation (2) is one expression of the Faddeev equations.[6]

It is convenient to introduce the states $|\alpha\vec{p}\,\vec{q}\rangle$, where

$$\langle\alpha\vec{p}\,'\vec{q}\,'|\beta\vec{p}\,\vec{q}\rangle = \delta_{\alpha\beta}\delta(\vec{p}\,' - \vec{p})\delta(\vec{q}\,' - \vec{q}),$$
$$\sum_\alpha \int d\vec{p}\,d\vec{q}\,|\alpha\vec{p}\,\vec{q}\rangle\langle\alpha\vec{p}\,\vec{q}| = 1 . \qquad (3)$$

We can then define the operators t, I such that

$$\langle\alpha\vec{p}\,'\vec{q}\,'|t|\beta\vec{p}\,\vec{q}\rangle = \delta_{\alpha\beta}\delta(\vec{q}\,' - \vec{q})t_\alpha(\vec{p}\,', \vec{p};\, W - q^2/2M_\alpha),$$

$$\langle\alpha\vec{p}\,'\vec{q}\,'|I|\beta\vec{p}\,\vec{q}\rangle = -\delta\left(\vec{p} + \frac{\mu_\beta}{m_\gamma}\vec{p}\,' - \frac{\mu_\beta}{M_\alpha}\vec{q}\,'\right)\delta\left(\vec{q} + \vec{p}\,' + \frac{\mu_\alpha}{m_\gamma}\vec{q}\,'\right)$$
if $\alpha\beta\gamma$ are cyclic,

$$= -\delta\left(\vec{p} + \frac{\mu_\beta}{m_\gamma}\vec{p}\,' + \frac{\mu_\beta}{M_\alpha}\vec{q}\,'\right)\delta\left(\vec{q} - \vec{p}\,' + \frac{\mu_\alpha}{m_\gamma}\vec{q}\,'\right)$$
if $\beta\alpha\gamma$ are cyclic. (4)

Here $t_\alpha(\vec{p}\,', \vec{p}; s)$ is the off-shell two-body t matrix for particles β and γ, energy s; the diagonal elements of I vanish. With the identification[9]

$$\psi_\alpha(\vec{p}_\alpha, \vec{q}_\alpha) = \langle\vec{p}_\alpha\vec{q}_\alpha|\psi_\alpha\rangle \equiv \langle\alpha\vec{p}_\alpha\vec{q}_\alpha|\psi\rangle , \qquad (5)$$

and letting $|\psi\rangle = M|\phi\rangle$, we can rewrite Eq. (2) in the form

$$M = 1 - G_0 t + G_0 t I M . \qquad (6)$$

It is important to keep in mind that the operators in Eq. (6) act on the states of Eq. (3); in particular

$$\langle\alpha\vec{p}\,'\vec{q}\,'|G_0|\beta\vec{p}\,\vec{q}\rangle = \frac{\delta_{\alpha\beta}\delta(\vec{p}\,' - \vec{p})\delta(\vec{q}\,' - \vec{q})}{p^2/2\mu_\alpha + q^2/2M_\alpha - W - i\epsilon} . \qquad (7)$$

One can easily verify that I and G_0 commute.

The development up to this point is completely general, with the object of obtaining the operator equation for M, Eq. (6). Since Eq. (6) is exactly equivalent to the equations of Faddeev, one can immediately infer that it serves to uniquely define M for a large class of two-body potentials. However, it was shown in B1 that this is not the case in the presence of singular cores. The proof is based on the fact that for such interactions, the two-body t matrix has the special property that

$$\tilde{V}G_0 t = tG_0\tilde{V} = \tilde{V} , \qquad (8)$$

where the projection operator $\tilde{V}$ corresponds to a square-well potential of unit strength and a range a_α for the matrix element

$$\langle\alpha\vec{p}\,'\vec{q}\,'|\tilde{V}|\beta\vec{p}\,\vec{q}\rangle = \delta_{\alpha\beta}\delta(\vec{q}\,' - \vec{q})\tilde{V}_\alpha(\vec{p}\,' - \vec{p}) . \qquad (9)$$

That is, $\tilde{V}_\alpha(\vec{p})$ is the Fourier transform of the unit step function $\theta(a_\alpha - r)$. Moreover, one can construct an operator Q of the form $Q = 1 + \tilde{V}B(I - 1)$ with the following properties:

$$\begin{aligned} &QQ = Q, \\ &\tilde{V}(1 - I)Q = (1 - I)Q\tilde{V} = 0, \\ &(1 - \tilde{V}I)Q = 1 - \tilde{V} , \\ &Q\tilde{V} = \tilde{V}Q\tilde{V} . \end{aligned} \qquad (10)$$

(An explicit derivation of Q is given in the Appendix.) Using Eqs. (8) and (10), one observes that

$$(1 - G_0 tI)G_0 Q\tilde{V} = 0 , \qquad (11)$$

and hence that $(1 - G_0 tI)^{-1}$ does not exist. Therefore, one cannot use the ordinary Faddeev equations [Eq. (6)] to uniquely determine M in the presence of singular cores.

To overcome this difficulty, a generalization of the Faddeev formalism was presented in B1. We consider a new operator $\tilde{t}$ chosen such that

$$1 - G_0 t = (1 - \tilde{V})(1 - G_0\tilde{t}) . \qquad (12)$$

A particular solution M to Eq. (6) can then be defined as $M = QM_e$, where M_e satisfies the new equation

$$M_e = 1 - G_0\tilde{t} + G_0\tilde{t}IQM_e . \qquad (13)$$

This new equation was motivated in B1 by imposing reasonable physical requirements on the resultant three-body wave function; namely, that it should vanish whenever any two particles are within their respective core radius, and must correspond to a unitary three-body t matrix.

We now consider a somewhat different derivation which employs another special relation concerning the two-body t-matrix: the fact that $\tilde{t}$ can be chosen such that

$$\tilde{t}\tilde{V} = 0 . \qquad (14)$$

Postponing a proof of this assertion until the end of this section, we proceed by assuming that M is any solution of Eq. (6). Employing Eq. (8), it follows that

$$\begin{aligned} \tilde{V}M &= \tilde{V}(1 - G_0 t + G_0 tIM) \\ &= \tilde{V}IM . \end{aligned} \qquad (15)$$

The form of Q then implies that $QM = M$. Noting that with our choice of states [Eq. (3)] the relationship between M and the three-body state vector is given by $|\Psi\rangle = (1 - I)M|\phi\rangle$, we have that

$$|\Psi\rangle = (1 - I)QM|\phi\rangle . \qquad (16)$$

We next observe that, as a consequence of Eq.

(14) and the properties of Q,

$$\begin{aligned}\tilde{t}IQ\tilde{V} &= \tilde{t}Q\tilde{V} \\ &= \tilde{t}\tilde{V}Q\tilde{V} \\ &= 0 .\end{aligned} \tag{17}$$

Hence, substituting Eq. (12) into Eq. (6), we deduce that

$$\tilde{t}IQM = \tilde{t}IQ(1 - G_0\tilde{t} + G_0\tilde{t}IM) . \tag{18}$$

Defining

$$\begin{aligned}X &= \tilde{t}IQM \\ &= \tilde{t}IM,\end{aligned} \tag{19}$$

we obtain an integral equation for X:

$$X = \tilde{t}IQ(1 - G_0\tilde{t}) + \tilde{t}IQG_0X . \tag{20}$$

Comparing this equation to Eq. (13), we infer that $X = \tilde{t}IQM_e$, i.e., the two equations are totally equivalent.

Moreover, we observe that X is all that is required to form $|\Psi\rangle$, since Eqs. (6) and (12) imply that

$$M = (1 - \tilde{V})(1 - G_0\tilde{t} + G_0X) + \tilde{V}IM . \tag{21}$$

Hence, due to Eq. (16), we have

$$\begin{aligned}|\Psi\rangle &= (1 - I)Q(1 - G_0\tilde{t} + G_0X)|\phi\rangle \\ &= (1 - I)QM_e|\phi\rangle .\end{aligned} \tag{22}$$

Finally, we note that although $\tilde{t}$ is not uniquely defined by Eqs. (12) and (14), any change in $\tilde{t}$ must be of the form $\Delta\tilde{t} = G_0^{-1}\tilde{V}A$. If we suppose that M_e' is the solution of Eq. (13) under the replacement $\tilde{t} \to \tilde{t}' = \tilde{t} + \Delta\tilde{t}$, it follows from Eq. (17) that

$$M_e' = M_e + \tilde{V}A(-1 + IQM_e') . \tag{23}$$

However, $|\Psi\rangle$ is invariant under such a change. We thus conclude that our equation is to all intents unique.

We conclude this section by considering the nature of $\tilde{t}$ and the proof of Eq. (14). To do so it is clearly adequate to drop subscripts and work in a two-body space. Denoting the core radius by a, we shall first deal with the case of the BCM alone ($\tilde{t} = \tilde{t}^{\text{BC}}$); the subsequent generalization to BCM plus external potential is trivial. We look for $\tilde{t}_l^{\text{BC}}$, the projection of $\tilde{t}^{\text{BC}}$ on partial wave l in the form

$$\tilde{t}_l^{\text{BC}}(p', p; s) = G_l(p', s)t_l^{\text{BC}}(\kappa, p; s) . \tag{24}$$

Here $\kappa = (2M_r s)^{1/2}$ is the on-shell momentum value; $\tilde{t}_l^{\text{BC}}$ is thus proportional to the half-on-shell BCM amplitude. We assert that G_l may be constructed in the form

$$\begin{aligned}G_l(p, s) &= 1 + (p^2 - \kappa^2) \sum_{n=0,2,\ldots}^{l} \alpha_n(\kappa^2) j_{n-2}(ap) , \qquad l \text{ even} \\ &= \frac{1}{\kappa}\left\{p + (p^2 - \kappa^2) \sum_{n=1,3,\ldots}^{l} \beta_n(\kappa^2) j_{n-2}(ap)\right\} , \qquad l \text{ odd}\end{aligned} \tag{25}$$

with $\alpha_0 = \beta_1 \equiv 0$, and the remaining α_n, β_n chosen such that $G_l(p, s) \propto p^l$ as $p \to 0$. To prove the latter statement we note that the α_n (β_n) can be determined inductively, i.e., suppose that the statement is true for a given l (say l is even for definiteness), then

$$G_l(p, s) \underset{p\to 0}{\sim} \frac{G_l^{(l)}(0, s)}{l!} p^l , \tag{26}$$

with $G_l^{(l)}(0, s)$ completely determined by the α_n, $n \leq l$. Noting that

$$G_{l+2}(p, s) - G_l(p, s) = (p^2 - \kappa^2)\alpha_{l+2}(\kappa^2) j_l(ap) , \tag{27}$$

we can clearly satisfy the condition for $l+2$ by taking

$$\alpha_{l+2}(\kappa^2) = \frac{(2l+1)!!}{\kappa^2 l!} \frac{G_l^{(l)}(0, s)}{a!} . \tag{28}$$

Since the condition holds for $l = 0$ we are done (the proof for odd l follows similarly). Given G_l, we can now apply Eq. (39) of B2 to evaluate the operator product $\tilde{V}G_0\tilde{t}^{\text{BC}}$; together with the explicit form for t^{BC} given in Eq. (48) of B2, this immediately verifies Eq. (12) for the BCM alone.

In order to generalize this result to the case of BCM plus external potential, we recall Eq. (71) of B2, which states that

$$t = t^{\text{BC}} + (1 - t^{\text{BC}}G_0)V_e(1 - G_0t) , \tag{29}$$

in which V_e is the external potential. In view of the pure BCM result, we simply observe that the choice

$$\tilde{t} = \tilde{t}^{\text{BC}} + (1 - \tilde{t}^{\text{BC}}G_0)V_e(1 - G_0t) \tag{30}$$

satisfies Eq. (12). Given Eqs. (24) and (30) it is straightforward to verify that $\tilde{t}$ satisfies the unitarity relation

$$\tilde{t}_l(p', p; s + i\epsilon) - \tilde{t}_l(p', p; s - i\epsilon) = -i\pi 2M_r\kappa\tilde{t}_l(p', \kappa; s + i\epsilon)\tilde{t}_l(\kappa, p; s - i\epsilon) . \tag{31}$$

In the subsequent sections we shall denote this symbolically by

$$\begin{aligned}\Delta\tilde{t} \equiv \tilde{t}^+ - \tilde{t}^- &= -\tilde{t}^+\Delta G_0\tilde{t}^- \\ &= -\tilde{t}^-\Delta G_0\tilde{t}^+ ,\end{aligned} \tag{32}$$

ΔG_0 being the discontinuity of the free Green's function.

Finally, having established the form of $\tilde{t}$, we turn to the consideration of Eq. (14). It is clear from Eq. (30) that $\tilde{t}$ is of the form $\tilde{t} = A\tilde{t}^{BC} + BV_e$. However, since $V_e = (1-\tilde{V})V_e = V_e(1-\tilde{V})$, we infer that $\tilde{t}\tilde{V} \propto \tilde{t}^{BC}\tilde{V}$; hence it is only necessary to treat the case of BCM alone. From the formulas developed in B2 one can easily show that

$$t_l^{BC}(\kappa, p; s) = \frac{g_l(p)}{D_l(\kappa)}, \tag{33}$$

with

$$\begin{aligned} g_l(p) &= (a\lambda_l - l)j_l(ap) + apj_{l+1}(ap), \\ D_l(\kappa) &= i\pi M_r \kappa[(a\lambda_l - l)h_l(a\kappa) + a\kappa h_{l+1}(a\kappa)]. \end{aligned} \tag{34}$$

(With our convention $\psi_l'/\psi_l = \lambda_l$ at the core radius.) Note that the verification of the above is greatly aided by the alternative formula

$$f_l(p, a, \kappa) = ia\kappa[a\kappa h_{l+1}(a\kappa)j_l(ap) - h_l(a\kappa)apj_{l+1}(ap)], \tag{35}$$

for the quantity f_l defined in B2.

Consequently, the proof that $\tilde{t}\tilde{V}=0$ rests on showing that

$$I_l \equiv \int_0^\infty dp\, p^2 g_l(p)\tilde{V}_l(p, p') = 0. \tag{36}$$

This, however, is somewhat delicate since I_l is ill defined. To see this it is convenient to employ the representation

$$\begin{aligned} g_l(p) &= \int_0^\infty dr\, r^2 j_l(pr)\hat{g}_l(r), \\ \hat{g}_l(r) &= \frac{(a\lambda_l + 1)}{a^2}\delta(r-a) + \frac{\delta'(r-a)}{r}. \end{aligned} \tag{37}$$

Thus

$$\begin{aligned} I_l &= \int_0^\infty dr\, r^2 \hat{g}_l(r)\theta(a-r)j_l(rp') \\ &= \theta(0)g_l(p') - a\delta(0)j_l(ap'), \end{aligned} \tag{38}$$

and hence is dependent on the ambiguous quantities $\theta(0)$, $\delta(0)$.

In this circumstance we argue that I_l must be evaluated as a limit in which the radial parameter related to g_l is taken to be $b > a$, the integral is performed, and the limit $b \to a$ is taken at the end; this prescription clearly gives zero as a result. This interpretation can be justified by elucidating the relationship between our formalism and the basic statement of the boundary condition. Let $\psi_l(r)$ be the partial-wave amplitude for the two-body system, and consider the integral

$$\frac{1}{a}\int_0^\infty dr\, r^2 \hat{g}_l(r)\psi_l(r) = \lambda_l\psi_l(a) - \psi_l'(a). \tag{39}$$

This clearly vanishes if one evaluates the right-hand expression in terms of the suggested limit. The boundary condition can thus be expressed as an operator condition on the two-body state vector $|\psi\rangle$, viz.,

$$\tilde{t}^{BC}(1-\tilde{V})|\psi\rangle = 0. \tag{40}$$

Here we observe that $(1-\tilde{V})|\psi\rangle$ is the projection of $|\psi\rangle$ onto the region exterior to the core. Since $|\psi\rangle = (1-G_0 t)|\phi\rangle$, we infer that

$$\begin{aligned} \tilde{t}^{BC}(1-G_0 t) &= 0, \\ \tilde{t}^{BC}(1-G_0\tilde{t}) &= 0; \end{aligned} \tag{41}$$

one can prove these relations directly from the formulas given in B2.

If we now consider the three-body state vector $|\Psi\rangle$, the boundary-condition can be expressed in the form

$$\tilde{t}^{BC}P_e|\Psi\rangle = 0, \tag{42}$$

where P_e projects $|\Psi\rangle$ onto the region exterior to all three cores. In an obvious notation,

$$P_e = (1-\tilde{V})(1-\theta_1)(1-\theta_2) \tag{43}$$

(the operators θ_1, θ_2 are defined explicitly in Sec. III). In terms of our formalism, Eq. (42) requires that

$$\tilde{t}^{BC}P_e(1-I)M_e = 0. \tag{44}$$

On the other hand, we prove in Sec. III that

$$(1-\tilde{V})(1-IQ) = P_e(1-I). \tag{45}$$

Thus Eq. (42) is equivalent to the condition

$$\tilde{t}^{BC}(1-\tilde{V})(1-IQ)M_e = 0. \tag{46}$$

In view of Eqs. (13) and (41), this is clearly satisfied by our formalism, provided that one accepts Eq. (14). The latter is equivalent to the replacement $\tilde{t}^{BC} \to \tilde{t}^{BC}(1-\tilde{V})$ wherever it occurs; this interpretation is quite natural in view of the connection we have established between $\tilde{t}^{BC}$ and the boundary condition [e.g., Eq. (40)].

In concluding this section, it is worth noting that in the particularly simple case of $V_e = 0$, where $P_e|\Psi\rangle$ is just a superposition of eigenstates of the kinetic energy operator, one can derive an integral equation for the superposition function by directly imposing the boundary condition; this equation is identical to our formalism.

III. EVALUATION OF IQ

In order to apply our formalism, Eq. (13), one must first evaluate the operator product IQ which appears in the kernel. In view of Eq. (14), it is sufficient to consider only $(1-\tilde{V})IQ$; our result has already been stated as Eq. (45). While direct verification of this formula is quite tedious, this may be avoided through the introduction of a "supervector" notation to describe our operators. We shall thus find it convenient to work in the coordinate representation, and to represent the pair of three-vectors $\vec{x}$, $\vec{y}$ by the "supervector" $\vec{\rho}$,

$$\vec{\rho}=\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}, \tag{47}$$

represented as a two-component spinor, each component being a three-vector.

We also note that Eq. (1) can be shown to imply the following relations between the reduced masses:

$$\begin{aligned}&\frac{\mu_\beta}{M_\alpha}=\frac{\mu_\alpha}{M_\beta},\\ &\frac{\mu_\alpha}{M_\beta}+\frac{\mu_\alpha\mu_\beta}{m_\gamma^{\,2}}=1,\\ &\frac{1}{M_\alpha}=\frac{1}{m_\alpha}+\frac{\mu_\alpha}{m_\beta m_\gamma}.\end{aligned} \tag{48}$$

Defining the rotation matrices $(\alpha\neq\beta\neq\gamma)$

$$R_{\beta\alpha}=\begin{pmatrix}\dfrac{-\mu_\alpha}{m_\gamma} & 1\\ \dfrac{-\mu_\beta}{M_\alpha} & \dfrac{-\mu_\beta}{m_\gamma}\end{pmatrix}, \tag{49}$$

Eq. (48) implies that $\det R_{\beta\alpha}=1$,

$$R^{-1}{}_{\beta\alpha}=\begin{pmatrix}\dfrac{-\mu_\beta}{m_\gamma} & -1\\ \dfrac{\mu_\alpha}{M_\beta} & \dfrac{-\mu_\alpha}{m_\gamma}\end{pmatrix}. \tag{50}$$

It is also easy to verify that

$$R_{\gamma\alpha}R_{\alpha\beta}R_{\beta\gamma}=1. \tag{51}$$

In what follows we shall mean by $R_{\beta\alpha}\vec{\rho}$ the "supervector" $\vec{\rho}\,'$, where

$$\vec{\rho}\,'=\begin{pmatrix}\dfrac{-\mu_\alpha}{m_\gamma}\vec{x}+\vec{y}\\ \dfrac{-\mu_\beta}{M_\alpha}\vec{x}-\dfrac{\mu_\beta}{m_\gamma}\vec{y}\end{pmatrix}. \tag{52}$$

Three-particle states can now be specified as $|\alpha\vec{\rho}\rangle\equiv|\alpha\vec{x}\vec{y}\rangle$, and the effect of the I operator can be expressed in the relations

$$\begin{aligned}\langle\alpha\vec{\rho}|I|F\rangle&=-\langle\sigma R_{\sigma\alpha}\vec{\rho}|F\rangle-\langle\epsilon R^{-1}{}_{\alpha\epsilon}\vec{\rho}|F\rangle,\\ \langle G|I|\alpha\vec{\rho}\rangle&=-\langle G|\sigma R_{\sigma\alpha}\vec{\rho}\rangle-\langle G|\epsilon R^{-1}{}_{\alpha\epsilon}\vec{\rho}\rangle,\end{aligned} \tag{53}$$

where $\alpha\sigma\epsilon$ are cyclic. Thus the operator I which connects the Faddeev channels has the effect of a rotation on $\vec{\rho}$.

If one defines operators θ_1, θ_2 such that

$$\begin{aligned}\langle\alpha\vec{\rho}|\theta_1|\beta\vec{\rho}\,'\rangle&=\delta_{\alpha\beta}\delta(\vec{\rho}-\vec{\rho}\,')\,\theta\left(a_\sigma-\left|\frac{\mu_\alpha}{m_\epsilon}\vec{x}-\vec{y}\right|\right),\\ \langle\alpha\vec{\rho}|\theta_2|\beta\vec{\rho}\,'\rangle&=\delta_{\alpha\beta}\delta(\vec{\rho}-\vec{\rho}\,')\,\theta\left(a_\epsilon-\left|\frac{\mu_\alpha}{m_\sigma}\vec{x}+\vec{y}\right|\right),\end{aligned} \tag{54}$$

the operator B derived in the Appendix can be expressed as

$$B=1-\tfrac{1}{2}(\theta_1+\theta_2)+\tfrac{1}{3}\theta_1\theta_2. \tag{55}$$

We recall for convenience that the relationship between Q and B is such that

$$1-IQ=(1+I\tilde{V}B)(1-I). \tag{56}$$

To proceed, we use Eqs. (9) and (53) to obtain

$$\begin{aligned}\langle\alpha\vec{\rho}|I\tilde{V}(1-I)|\beta\vec{\rho}\,'\rangle&=-\theta\left(a_\sigma-\left|\frac{\mu_\alpha}{m_\epsilon}\vec{x}-\vec{y}\right|\right)\langle\sigma R_{\sigma\alpha}\vec{\rho}|1-I|\beta\vec{\rho}\,'\rangle\\ &\quad-\theta\left(a_\epsilon-\left|\frac{\mu_\alpha}{m_\sigma}\vec{x}+\vec{y}\right|\right)\langle\epsilon R^{-1}{}_{\alpha\epsilon}\vec{\rho}|1-I|\beta\vec{\rho}\,'\rangle.\end{aligned} \tag{57}$$

However, by again employing Eq. (53), it is easy to show that

$$\begin{aligned}\langle\alpha\vec{\rho}|1-I|\beta\vec{\rho}\,'\rangle&=\langle\sigma R_{\sigma\alpha}\vec{\rho}|1-I|\beta\vec{\rho}\,'\rangle\\ &=\langle\epsilon R^{-1}{}_{\alpha\epsilon}\vec{\rho}|1-I|\beta\vec{\rho}\,'\rangle.\end{aligned} \tag{58}$$

Recalling Eq. (54), we thus obtain

$$I\tilde{V}(1-I)=-(\theta_1+\theta_2)(1-I). \tag{59}$$

It is easy to verify that the effect of I on the operators $\tilde{V}$, θ_1, θ_2 is to permute them among themselves; hence one deduces that

$$\begin{aligned}&[I,\ \tilde{V}\theta_1\theta_2]=0,\\ &[I,\ \tilde{V}\theta_1+\tilde{V}\theta_2+\theta_1\theta_2]=0,\\ &(1-\tilde{V})I\theta_1\theta_2=0.\end{aligned} \tag{60}$$

If we now employ the above relations in evaluating the right-hand side of Eq. (56), we obtain

$$(1-\tilde{V})(1-IQ)=(1-\tilde{V})[1-\theta_1-\theta_2-\tfrac{1}{2}\theta_1\theta_2 I](1-I). \tag{61}$$

At this point we observe that the definition of I, Eq. (4), implies that

$$\begin{aligned}&I^{-1}=\tfrac{1}{2}(1+I),\\ &(1-I)^2=3(1-I),\\ &(1-I)(2+I)=0.\end{aligned} \tag{62}$$

With this input Eq. (61) reduces to the desired result,

$$(1-\tilde{V})(1-IQ)=(1-\tilde{V})(1-\theta_1)(1-\theta_2)(1-I). \tag{63}$$

IV. STRUCTURE OF EQUATIONS FOR A SPECIAL CASE

In Sec. II it was shown that while the Faddeev equations do not determine a unique solution in the presence of singular cores, there exists a one-to-one correspondence between the three-body wave function and the solutions of a new integral equation [Eq. (13)]. In this section we analyze the properties of this equation by considering the simplest possible case in some detail. We thereby demonstrate that this equation by itself is not sufficient to uniquely specify a solution, although it embodies the full content of the BCM. We further show that this ambiguity may be eliminated by imposing an auxiliary boundary condition pertaining to the behavior of the channel wave functions in the interior region. When this has been done we arrive at a well-defined formalism with a square-integrable kernel, the solutions of which correspond to a unitary three-particle T matrix (as is shown in the next section). This development may be extended to the most general form of Eq. (13); we quote the structure of the resulting equations at the end of this section.

We shall thus consider the case of identical particles of mass M, interacting only in relative s waves, with no external potential (BCM alone). We then have $\bar{t}=\bar{t}^{\,\mathrm{BC}}$, and

$$\langle \alpha\vec{p}\vec{q}\,|\bar{t}^{\,\mathrm{BC}}|\,\beta\vec{p}\,'\vec{q}\,'\rangle=\frac{\delta_{\alpha\beta}}{4\pi}\,\delta(\vec{q}\,'-\vec{q})\,\frac{g_0(p')}{D_0(\kappa)}, \tag{64}$$

with g_0, D_0 defined in Eq. (34), and $\kappa=(MW-\frac{3}{4}q^2)^{1/2}$. It is obvious that Eq. (13) reduces to an equation in a single vector variable $(\vec{q})$; however, for convenience in manipulation we choose to embed this equation in the full $|\alpha\vec{p}\vec{q}\rangle$ space. We therefore define operators $F, \hat{t}$ such that

$$\langle \alpha\vec{p}\vec{q}\,|\hat{t}\,|\,\beta\vec{p}\,'\vec{q}\,'\rangle=\frac{\delta_{\alpha\beta}}{4\pi}\,\delta(\vec{q}\,'-\vec{q})g_0(p'),$$
$$\langle \alpha\vec{p}\vec{q}\,|F|\,\beta\vec{p}\,'\vec{q}\,'\rangle=\delta_{\alpha\beta}\delta(\vec{q}\,'-\vec{q})\,\frac{f(\vec{p})}{D_0(\kappa)}, \tag{65}$$

where $f(\vec{p})$ is an arbitrary function such that

$$\int d\vec{p}\,f(\vec{p})=1\,. \tag{66}$$

We may then express $\bar{t}^{\,\mathrm{BC}}$ as the operator product

$$\bar{t}^{\,\mathrm{BC}}=F\hat{t} \tag{67}$$

in the full Hilbert space. As a consequence of Eq. (41) it follows that

$$\hat{t}\,G_0F\hat{t}=\hat{t}\,. \tag{68}$$

For this model Eq. (13) implies that we can write M_e in the form

$$M_e=1+G_0FY,$$
where
$$Y=\hat{t}\,(-1+IQM_e)\,. \tag{69}$$

Thus Y must satisfy the equation

$$(1-\hat{t}IQG_0F)Y=-\hat{t}\,(1-IQ)\,,$$
or
$$\hat{t}\,(1-IQ)G_0FY=-\hat{t}\,(1-IQ)\,, \tag{70}$$

using Eq. (68). We observe that this is in actuality an integral equation in $\vec{q}$ alone, since Eq. (69) implies that

$$\langle \alpha\vec{p}\vec{q}\,|\,Y|\,\beta\vec{p}\,'\vec{q}\,'\rangle=Y_{\alpha\beta}(\vec{q}\,|\,\vec{p}\,'\vec{q}\,')\,. \tag{71}$$

Since we have assumed identical particles, in this case it is sufficient to consider the symmetrized sum

$$Y(\vec{q}\,|\vec{p}\,'\vec{q}\,')=\sum_{\beta}Y_{\alpha\beta}(\vec{q}\,|\vec{p}\,'\vec{q}\,')\,. \tag{72}$$

The significance of Y is obvious if we consider its relationship to the three-body wave function or T matrix. For example, the channel wave function in the exterior region (each pair outside the range of its interaction) is in this case

$$\psi_{\alpha}^{\mathrm{ext}}(\vec{x},\vec{y})=e^{-i\vec{x}\cdot\vec{p}\,'}\,e^{-i\vec{y}\cdot\vec{q}\,'}+2\pi^2M\int d\vec{q}\,\frac{e^{i\kappa x}}{x}\,e^{-i\vec{y}\cdot\vec{q}}\,\frac{Y(\vec{q}\,|\vec{p}\,'\vec{q}\,')}{D_0(\kappa)}, \tag{73}$$

with $p'^2+\frac{3}{4}q'^2=W$. That is, a superposition of plane waves in $\vec{y}$ and outgoing waves in x weighted by $Y(\vec{q}\,|\vec{p}\,'\vec{q}\,')$. Clearly $(H_0-W)|\psi_{\alpha}^{\mathrm{ext}}\rangle=0$, and hence (since I and H_0 commute) $(H_0-W)|\Psi^{\mathrm{ext}}\rangle=0$, as it must. In fact, Eq. (73) is the most general form for the exterior solution of any finite-range s-wave potential model. Equivalently, the three-particle T matrix for this problem has the form

$$T(\vec{p}\vec{q}\,|\vec{p}\,'\vec{q}\,')=T^{c}(\vec{q}\,|\vec{p}\,'\vec{q}\,')+T^{c}(-\vec{p}-\tfrac{1}{2}\vec{q}\,|\vec{p}\,'\vec{q}\,')+T^{c}(\vec{p}-\tfrac{1}{2}\vec{q}\,|\vec{p}\,'\vec{q}\,'), \tag{74}$$

where

$$T^{c}(\vec{q}\,|\vec{p}\,'\vec{q}\,')=-\frac{Y(\vec{q}\,|\vec{p}\,'\vec{q}\,')}{D_0(\kappa)}\,,$$
$$p^2+\tfrac{3}{4}q^2=p'^2+\tfrac{3}{4}q'^2=W\,.$$

Here the physical values of q, q' are the domain $[0,(\frac{4}{3}MW)^{1/2}]$ for $W>0$, and the point $Q_B=[\frac{4}{3}M(W+E_B)]^{1/2}$ for $W+E_B>0$, where E_B is the binding energy of the two-body bound state (if any). In the usual fashion, appropriate combinations of these values correspond to the amplitudes for elastic scattering from the bound-state, breakup, and $(3)\to(3)$ scattering processes.

Returning to Eq. (70), we observe that Eqs. (14) and (45) imply that

$$\hat{t}(1-IQ)=\hat{t}P_e(1-I)=\hat{t}(1-\theta_1)(1-\theta_2)(1-I). \tag{75}$$

In evaluating this product it is convenient to work in a mixed momentum and coordinate representation, where for example

$$\langle \alpha\vec{x}\vec{y}|\beta\vec{p}\vec{q}\rangle=\delta_{\alpha\beta}e^{-i\vec{x}\cdot\vec{p}}e^{-i\vec{y}\cdot\vec{q}}. \tag{76}$$

Recalling Eqs. (37) and (54), we have

$$\begin{aligned}\langle \alpha\vec{p}\vec{y}|\hat{t}(1-\theta_1)(1-\theta_2)|\beta\vec{x}'\vec{y}'\rangle &= \frac{\delta_{\alpha\beta}}{16\pi^2}\delta(\vec{y}-\vec{y}')\hat{g}_0(x')\\ &\times\theta(|\tfrac{1}{2}\vec{x}'+\vec{y}'|-a)\theta(|\tfrac{1}{2}\vec{x}'-\vec{y}'|-a).\end{aligned} \tag{77}$$

Since the properties of $\hat{g}_0(x')$ require that x' is effectively equal to a, we observe that this matrix element vanishes identically for $y<\frac{1}{2}\sqrt{3}\,a$.[10] Thus, if we define the projection operator θ such that

$$\langle \alpha\vec{x}\vec{y}|\theta|\beta\vec{x}'\vec{y}'\rangle=\delta_{\alpha\beta}\delta(\vec{x}'-\vec{x})\delta(\vec{y}'-\vec{y})\theta(\tfrac{1}{2}\sqrt{3}\,a-y), \tag{78}$$

we deduce that

$$\theta\hat{t}(1-\theta_1)(1-\theta_2)=0, \tag{79}$$

a fact which has important consequences for our formalism. Employing Eq. (77), it is straightforward to show that

$$\hat{t}(1-\theta_1)(1-\theta_2)G_0F=(1-K_L)\hat{t}G_0F, \tag{80}$$

where K_L is the local operator

$$\langle\alpha\vec{x}\vec{y}|K_L|\beta\vec{x}'\vec{y}'\rangle=\delta_{\alpha\beta}\delta(\vec{x}'-\vec{x})\delta(\vec{y}'-\vec{y})K_L(y);$$

$$K_L(y)=\begin{cases}1, & y\le\frac{1}{2}\sqrt{3}\,a\\ 1-\dfrac{(y^2-\frac{3}{4}a^2)}{ay}, & \frac{1}{2}\sqrt{3}\,a\le y\le\frac{3}{2}a\\ 0, & y\ge\frac{3}{2}a.\end{cases} \tag{81}$$

It is thus convenient to define an operator H such that

$$\begin{aligned}&\theta\hat{t}H=0,\\ &(1-K_L)\hat{t}(1-H)=\hat{t}(1-IQ).\end{aligned} \tag{82}$$

Equation (70) then takes the form

$$(1-K_L)\hat{t}(1-H)G_0FY=(1-K_L)\hat{t}(H-1). \tag{83}$$

Defining a kernel K such that $\hat{t}HG_0F\hat{t}=K\hat{t}$, or

$$\begin{aligned}&\langle\alpha\vec{p}\vec{q}|K|\beta\vec{p}'\vec{q}'\rangle=\delta(\vec{p}-\vec{p}')K_{\alpha\beta}(\vec{q},\vec{q}'),\\ &K_{\alpha\beta}(\vec{q},\vec{q}')=\int d\vec{p}'\langle\alpha\vec{p}\vec{q}|\hat{t}HG_0F|\beta\vec{p}'\vec{q}'\rangle,\end{aligned} \tag{84}$$

we may write Eq. (83) in the form

$$(1-K_L)[(1-K)Y-\hat{t}(H-1)]=0. \tag{85}$$

The general solution of this equation is

$$Y=Z[(1-\theta)\hat{t}(H-1)+\theta Y], \tag{86}$$

where $Z=(1-K)^{-1}$, and θY is arbitrary (note that by definition $\theta K=0$, and hence $\theta Z=\theta$).

In writing Eq. (86) we have casually assumed that Z exists. With the possible exception of a few discrete values of W, this follows from the fact that $K(W)$ is square-integrable.[11] In the special case under consideration, we need only

$$\begin{aligned}K(\vec{q},\vec{q}';W)&\equiv\sum_\beta K_{\alpha\beta}(\vec{q},\vec{q}'),\\ &=\sum_l\frac{2l+1}{4\pi}K_l(\vec{q},\vec{q}';W)P_l(\hat{q}\cdot\hat{q}').\end{aligned} \tag{87}$$

A straightforward but tedious calculation yields

$$\begin{aligned}&K_l(q,q';W)=\frac{N_l(q,q';W)}{D_0(\kappa')},\\ &N_l(q,q';W)=\overline{N}_l(q,q';W)\\ &\qquad+\int_{\sqrt{3a/2}}^{b}\frac{dy\,y^2j_l(yq)}{z_0(y)}N_l(y,q';W),\end{aligned} \tag{88}$$

with $b>\frac{3}{2}a$. Here

$$\overline{N}_l(q,q';W)=-M\int_{-1}^{1}dzP_l(z)\frac{f_l(q,b,Q)}{q^2-Q^2}\frac{Qg_0(K)}{[MW-(1-\frac{1}{4}z^2)q'^2]^{1/2}},$$

with

$$\begin{aligned}&Q=-\tfrac{1}{2}zq'+[MW-(1-\tfrac{1}{4}z^2)q'^2]^{1/2},\\ &K=[MW-\tfrac{3}{4}Q^2]^{1/2};\end{aligned} \tag{89}$$

while

$$N_l(y,q';W)=-Ma\int_{-z_0}^{z_0}dz\frac{e^{ix_0\kappa'}}{x_0}I_l(y,z,q';W),$$

$$z_0(y)=\min\left\{1,\frac{y^2-\frac{3}{4}a^2}{ay}\right\},$$

$$x_0=(\tfrac{1}{4}a^2-ayz+y^2)^{1/2},$$

$$\begin{aligned}I_l(y,z,q';W)&=C_l(y,z)j_l(y_0q')\\ &+\left\{\left[\lambda_0+\frac{(a-2yz)}{4x_0^2}(1-ix_0\kappa')\right]j_l(y_0q')\right.\\ &\left.+\frac{(9a+6yz)}{16y_0}q'j_{l+1}(y_0q')\right\}P_l(\alpha),\end{aligned} \tag{90}$$

$$\alpha = \frac{-\frac{3}{4}az - \frac{1}{2}y}{y_0},$$

$$y_0 = (\tfrac{9}{16}a^2 + \tfrac{3}{4}ayz + \tfrac{1}{4}y^2)^{1/2},$$

$$C_l(y, z) = -l\frac{(9a+6yz)}{16y_0^2}P_l(\alpha) + \frac{(l+1)y}{2ay_0}[P_{l+1}(\alpha) - \alpha P_l(\alpha)].$$

We note that the integrand of the expression for $\overline{N}_l$ is complex for fixed z; for $W<0$ its value at $-z$ gives the conjugate, and K_l is thus real for negative W. On the other hand, for $W>0$ the denominator $(q^2 - Q^2)$ can vanish and the contribution to K_l is complex. The $\overline{N}_l$ term is analogous to the ordinary Faddeev kernel in the case of separable potentials. Despite the profusion of equations, the resultant kernel is quite simple in structure and easy to handle numerically; power-counting estimates are adequate to demonstrate that $\mathrm{Tr}K^2 < \infty$, establishing the above assertion.

However, although Z is well defined, the presence of θY in Eq. (86) illustrates the following important consequence of Eq. (79): *The BCM does not uniquely specify the three-body wave function.*[12] Therefore, in order to eliminate the resulting ambiguity, one must specify an auxiliary boundary condition to determine θY. Since, for $x \sim a$, θY corresponds to the channel wave function in a region where at least one pair of particles is within their core radius, it might seem reasonable to require that $\theta Y \equiv 0$. However, it is easy to demonstrate that this choice is not compatible with three-particle unitarity. Instead, it turns out that one may require

$$(\theta Y)_{W=W_0} = 0 \tag{91}$$

for some particular energy W_0 by imposing the condition

$$\theta \mathfrak{D} Y = 0 . \tag{92}$$

Here

$$\langle \alpha \vec{p}'\vec{q}' | \mathfrak{D} | \beta \vec{p}\vec{q} \rangle = \delta_{\alpha\beta}\delta(\vec{p}' - \vec{p})\delta(\vec{q}' - \vec{q})\frac{D_0(\kappa_0)}{D_0(\kappa)}, \tag{93}$$

$\kappa_0 = (MW_0 - \frac{3}{4}q^2)^{1/2}$, and Eq. (92) is to be satisfied at all energies W, reducing to Eq. (91) at $W = W_0$.

Applying this condition to the general solution given in Eq. (86), we find that

$$\theta Y = -\eta\theta\mathfrak{D}Z(1-\theta)\hat{t}(H-1), \quad \eta = [\theta\mathfrak{D}Z\theta]^{-1}, \tag{94}$$

with $[\ ]^{-1}$ denoting the inverse on the finite θ subspace (η is an operator on that space to itself).

While this form is compact and convenient for the unitarity proof given in the next section, a more useful form for computation (and proof that η exists) may be obtained by defining $R(q, W)$ such that $\mathfrak{D} = 1 - R$,

$$1 - R(q, W) = \frac{D_0(\kappa_0)}{D_0(\kappa)}; \tag{95}$$

one may infer from this that $R = O(q^{-2})$ as $q \to \infty$. Our constraint then takes the form

$$\theta Y = \theta RZ(1-\theta)\hat{t}(H-1) + \theta RZ\theta Y . \tag{96}$$

Furthermore, there is no loss in generality in letting $\theta Y = (1-K)X$, where X must satisfy

$$X = \theta RZ(1-\theta)\hat{t}(H-1) + \overline{K}X, \quad \overline{K} = K + \theta R . \tag{97}$$

In this particular case,

$$\overline{K}_l(q, q'; W) = K_l(q, q'; W) + \theta_l(q, q'; \tfrac{1}{2}\sqrt{3}\,a)R(q', W), \tag{98}$$

$$\theta_l(q, q'; R) = \frac{2R^2}{\pi}\frac{qj_{l+1}(Rq)j_l(Rq') - q'j_{l+1}(Rq')j_l(Rq)}{q^2 - q'^2}$$

It is clear that $\overline{K}$ is as well behaved as K; we can thus define $\overline{Z} = (1 - \overline{K})^{-1}$, and write

$$X = \overline{Z}\theta RZ(1-\theta)\hat{t}(H-1). \tag{99}$$

Moreover, we have $\theta Y = \theta X$, and hence we arrive at the result

$$Y = Z(1 + \theta\overline{Z}\theta RZ)(1-\theta)\hat{t}(H-1). \tag{100}$$

We have thus established that the extension of the two-body boundary condition to the three-body system, plus the auxiliary boundary condition of Eq. (92), serve to uniquely determine Y and hence the exterior wave function (the interior wave function vanishes identically). By choosing $W_0 < 0$, we maintain the reality of our kernel $\overline{K}$ for $W<0$, and guarantee the unitarity of the resulting three-particle T matrix. Of course, nothing in the BCM tells us how to choose W_0; we shall return to this point in Sec. VI.

In order to determine the various amplitudes of physical interest, one needs to compute matrix elements of the form $\langle \alpha\vec{p}\vec{q} | Y | \phi \rangle$. In general, we take $|\phi\rangle = |\alpha JMl\lambda p'q'\rangle$, coupling $\lambda(\vec{p}')$ to $l(\vec{q}')$ to form a state of total angular momentum J; the on-shell condition is that $p' = \kappa'$. In the present case, Eq. (100) implies that we need the quantity

$$\Omega_J(q | l\lambda p'q') = \tfrac{1}{3}\sum_\beta \langle \alpha\vec{p}qJM | (1-\theta)\hat{t}(H-1) | \beta JMl\lambda p'q' \rangle . \tag{101}$$

Of particular interest is the value of Ω for $\lambda = 0$,

$J=l$; we denote this by $\Omega_l(q|p'q')$ and present the evaluated expression below. In terms of the notation stated in Eqs. (89), (90), and (98),

$$\Omega_l(q|p'q')=\bar{\Omega}_l(q|p'q')+\int_{\sqrt{3a}/2}^{b}\frac{dy\,y^2 j_l(yq)}{z_0(y)}\Omega_l(y|p'q'),$$

$$\bar{\Omega}_l(q|p'q')=-\frac{1}{6\sqrt{\pi}}\left\{g_0(p')\left[\frac{\delta(q'-q)}{q^2}-\theta_l(q,q';\tfrac{1}{2}\sqrt{3}a)\right]+\int_{-1}^{1}dz\,P_l(\beta)g_0(P_0)\left[\frac{\delta(q-Q_0)}{q^2}-\theta_l(q,Q_0;b)\right]\right\},$$

$$\beta=\frac{-p'z-\frac{1}{2}q'}{Q_0},\quad Q_0=(p'^2+p'q'z+\tfrac{1}{4}q'^2)^{1/2},\quad P_0=(\tfrac{1}{4}p'^2-\tfrac{3}{4}p'q'z+\tfrac{9}{16}q'^2)^{1/2},\tag{102}$$

$$\Omega_l(y|p'q')=-\frac{a}{3\pi^{3/2}}\int_{-z_0}^{z_0}dz\,J_l(y,z;p',q'),$$

$$J_l(y,z;p',q')=C_l(y,z)j_0(x_0p')j_l(y_0q')$$
$$+\left\{\left[\lambda_0 j_0(x_0p')+\frac{(a-2yz)}{4x_0}p'j_1(x_0p')\right]j_l(y_0q')+\frac{(9a+6yz)}{16y_0}j_0(x_0p')q'j_{l+1}(y_0q')\right\}P_l(\alpha).$$

In Eqs. (88)–(90), (98), and (102), we have all the information necessary to investigate the consequences of Eq. (100) numerically for the simple model under consideration. The results of such calculations will be presented in a subsequent article.

We conclude this section by exhibiting the structure of our formalism in the most general situation. It is helpful to define

$$U=(1-\tilde{t}^{\,BC}G_0)V_e(1-G_0t),\tag{103}$$

so that $\tilde{t}=\tilde{t}^{\,BC}+U$. With a simple generalization of the operators $\hat{t}$ and F to nonidentical particles and an arbitrary number of partial waves, we can retain Eqs. (67) and (68); as a result $\hat{t}G_0U=0$. From Eq. (13) we deduce that

$$M_e=1+G_0\Lambda,$$
$$\Lambda=\rho[U(IQ-1)+FY],\tag{104}$$
$$\rho=(1-UIQG_0)^{-1}.$$

The existence of ρ can be demonstrated for typical short-range potentials V_e. Applying Eq. (44), Y must satisfy

$$\hat{t}(1-IQ)G_0\rho FY=-\hat{t}(1-IQ)[1+G_0\rho U(IQ-1)].\tag{105}$$

Extending our definition of K to $\hat{t}HG_0\rho F\hat{t}=K\hat{t}$, we find

$$Y=Z\{(1-\theta)\hat{t}(H-1)[1+G_0\rho U(IQ-1)]+\theta Y\}.\tag{106}$$

In order to determine θY we need the generalization of Eq. (92); we therefore require that

$$\theta\hat{t}G_0(W_0)\Lambda=0.\tag{107}$$

This condition reduces to Eq. (92) in the simple case considered previously, and again implies Eq. (91). In similar fashion to the above, we finally obtain

$$\theta Y=-\theta\bar{Z}\theta\hat{t}G_0(W_0)\rho\Sigma,$$
$$\Sigma=FZ(1-\theta)\hat{t}(H-1)[1+G_0\rho U(IQ-1)]+U(IQ-1).\tag{108}$$

Here

$$\bar{K}\hat{t}=(K+\theta R-\theta\hat{t}G_0(W_0)\tilde{\rho}F)\hat{t},$$
$$\tilde{\rho}\equiv\rho-1.\tag{109}$$

V. THREE-BODY UNITARITY

In this section we give an explicit proof of the three-particle unitarity relations for our new formalism. In doing so, it will be convenient to adopt a notation of the type illustrated in Eq. (32) in order to express the discontinuities of an amplitude across its cut. As is well known, the discontinuities of the off-shell three-body t matrix T as a function of the total energy W arise from two sources: (1) scattering to states consisting of three free particles, with a threshold $W=0$, (2) elastic scattering of a single particle from a bound state of two others. In the latter case thresholds are found at $W=\nu_{\alpha j}$, where $-\nu_{\alpha j}$ is the binding energy for the jth bound state of particles β and γ. The cuts from both sources are taken to lie to the right of the corresponding threshold along the real W axis.

As an illustration, we first consider the relation for cut (1) in the ordinary Faddeev formalism. We note that the relationship between M and T is given by

$$1-G_0T=(1-I)M. \quad (110)$$

By assumption, we have that in this case the operator

$$Z=(1-G_0tI)^{-1} \quad (111)$$

exists. The unitarity condition for t is that $\Delta t=-t^-\Delta G_0t^+$; thus

$$\Delta Z=Z^-(1-G_0^-t^-)\Delta G_0t^+IZ^+. \quad (112)$$

From Eq. (6) we have $M=Z(1-G_0t)$; it follows that

$$\begin{aligned}\Delta M&=\Delta Z(1-G_0^+t^+)-Z^-(1-G_0^-t^-)\Delta G_0t^+\\&=M^-\Delta G_0t^+[IZ^+(1-G_0^+t^+)-1]\\&=M^-\Delta G_0t^+(IM^+-1).\end{aligned} \quad (113)$$

However, Eq. (6) implies that

$$t(IM-1)=G_0^{-1}(M-1), \quad (114)$$

while $\Delta G_0G_0^{-1}A=0$ unless a corresponding factor of G_0 occurs in A (ΔG_0 puts the operator to the right on shell). Thus

$$\Delta M=M^-\Delta G_0G_0^{-1}M^+. \quad (115)$$

On the other hand, Eq. (110) says that

$$T=-G_0^{-1}[(1-I)M-1], \quad (116)$$

and hence

$$\Delta T=-G_0^{-1}(1-I)\Delta M. \quad (117)$$

Since

$$\begin{aligned}M&=Z(1-G_0tI+I-1)I^{-1}\\&=I^{-1}+Z(I-1)I^{-1},\end{aligned} \quad (118)$$

using Eq. (62) we find that

$$(1-I)M=\tfrac{1}{3}(1-I)M(1-I). \quad (119)$$

Thus

$$\begin{aligned}\Delta T&=-\tfrac{1}{3}G_0^{-1}(1-I)M^-\Delta G_0G_0^{-1}(1-I)M^+\\&=-\tfrac{1}{3}T^-\Delta G_0T^+,\end{aligned} \quad (120)$$

where we have used the fact that I and G_0 commute. Note that the factor of $\frac{1}{3}$ appearing in Eq. (120) arises from triple counting due to our choice of intermediate states; Eq. (120) is exactly equivalent to the usual statement of three-particle unitarity.

We now turn to an analogous derivation based on our formalism. For simplicity, we will assume[13] that $V_e=0$; the general case requires more lengthy manipulations but is not qualitatively different. We thus have $M=QM_e$, with

$$\begin{aligned}M_e&=1+G_0FY,\\Y&=Z(1-\eta\theta\mathfrak{D}Z)(1-\theta)\hat{t}(H-1).\end{aligned} \quad (121)$$

Let A be any operator such that $\langle\alpha\vec{p}\vec{q}|A|\rho\vec{p}'\vec{q}'\rangle=\delta(\vec{p}'-\vec{p})A_{\alpha\rho}(\vec{q},\vec{q}')$. From the definitions given in Sec. IV, it is easy to show that

$$\begin{aligned}\mathfrak{D}A\hat{t}&=\hat{t}G_0(W_0)FA\hat{t},\\KA\hat{t}&=\hat{t}HG_0FA\hat{t}.\end{aligned} \quad (122)$$

Thus, due to the final $\hat{t}$ in the above formula for Y, we may effectively take $\mathfrak{D}=\hat{t}G_0(W_0)F$, $K=\hat{t}HG_0F$ in calculating the discontinuity of that expression. With this understanding, we employ the formula

$$\Delta F=-F^-\hat{t}\Delta G_0F^+, \quad (123)$$

which follows trivially from Eqs. (67) and (32), to deduce that

$$\begin{aligned}\Delta Z&=Z^-\hat{t}H\Delta(G_0F)Z^+\\&=[\hat{t}+Z^-\hat{t}(H-1)]\Delta G_0F^+Z^+.\end{aligned} \quad (124)$$

Similarly, we arrive at

$$\begin{aligned}\Delta Y&=(\hat{t}+Y^-)\Delta G_0F^+Y^+,\\\Delta M_e&=M_e^-\Delta G_0F^+Y^+.\end{aligned} \quad (125)$$

Substituting the latter expression into Eq. (117), we have determined that

$$\Delta T=-G_0^{-1}(1-I)QM_e^-\Delta G_0F^+Y^+. \quad (126)$$

We now observe that the definitions of Q, M_e, Y imply that

$$(1-I)QM_e=\tfrac{1}{3}(1-I)QM_e(1-I). \quad (127)$$

Thus,

$$\begin{aligned}\Delta T&=-\tfrac{1}{3}G_0^{-1}(1-I)QM_e^-\Delta G_0(1-I)F^+Y^+\\&=\tfrac{1}{3}T^-\Delta G_0(1-I)F^+Y^+\\&=-\tfrac{1}{3}T^-\Delta G_0T^+,\end{aligned} \quad (128)$$

as desired.

We next consider cuts of type (2), recalling that

$$t_\alpha(\vec{p}',\vec{p};s)\underset{s\to\nu_{\alpha j}}{\sim}\frac{2l+1}{4\pi}P_l(\hat{p}'\cdot\hat{p})\frac{g_{\alpha j}(p')g_{\alpha j}(p)}{s-\nu_{\alpha j}}, \quad (129)$$

l being the partial wave in which the bound state occurs. It is helpful to define the operators $r_{\alpha j}$, $S_{\alpha j}$ such that

$$\begin{aligned}\langle\beta\vec{p}'\vec{q}'|r_{\alpha j}|\gamma\vec{p}\vec{q}\rangle&=\delta_{\beta\alpha}\delta_{\gamma\alpha}\delta(\vec{q}'-\vec{q})\,\frac{2l+1}{4\pi}\\&\quad\times P_l(\hat{p}'\cdot\hat{p})g_{\alpha j}(p')g_{\alpha j}(p),\\\langle\beta\vec{p}'\vec{q}'|S_{\alpha j}|\gamma\vec{p}\vec{q}\rangle&=\delta_{\beta\alpha}\delta_{\gamma\alpha}\frac{\delta(\vec{p}'-\vec{p})\delta(\vec{q}'-\vec{q})}{W-q^2/2M_\alpha-\nu_{\alpha j}+i\epsilon}.\end{aligned} \quad (130)$$

The cut of t arising from the bound-state pole $\nu_{\alpha j}$ then has the discontinuity

$$\Delta t = r_{\alpha j} \Delta S_{\alpha j} . \tag{131}$$

Clearly, $\Delta S_{\alpha j} \propto \delta(q - q_{\alpha j})$, where

$$q_{\alpha j}^2 = 2M_\alpha (W - \nu_{\alpha j}) . \tag{132}$$

For the usual Faddeev theory it follows that

$$\begin{aligned} \Delta M &= \Delta Z(1 - G_0 t^+) - Z^- G_0 \Delta t \\ &= Z^- G_0 \Delta t (IM^+ - 1) ; \end{aligned} \tag{133}$$

thus

$$\Delta T = -G_0^{-1}(1 - I) Z^- G_0 \Delta t (IM^+ - 1) . \tag{134}$$

On the other hand, we note that the effect of the operator $\Delta S_{\alpha j} S_{\alpha j}^{-1}$ is to pick out the residue at the $\nu_{\alpha j}$ pole of the operator it acts on; hence

$$\Delta S_{\alpha j} S_{\alpha j}^{-1} t = \Delta S_{\alpha j} r_{\alpha j} , \tag{135}$$

for example. Therefore

$$\begin{aligned} \Delta S_{\alpha j} S_{\alpha j}^{-1} T &= -\Delta S_{\alpha j} S_{\alpha j}^{-1} G_0^{-1}(1 - G_0 t + G_0 t IM) \\ &= -\Delta S_{\alpha j} r_{\alpha j} (IM - 1) \\ &= -\Delta t (IM - 1) . \end{aligned} \tag{136}$$

Similarly,

$$\begin{aligned} T S_{\alpha j}^{-1} \Delta S_{\alpha j} &= -G_0^{-1}(1 - I) Z (1 - G_0 t) S_{\alpha j}^{-1} \Delta S_{\alpha j} \\ &= G_0^{-1}(1 - I) Z G_0 \Delta t . \end{aligned} \tag{137}$$

We also note that $r_{\alpha j}^2 = \rho_{\alpha j} r_{\alpha j}$, where[14]

$$\rho_{\alpha j} = \int_0^\infty dp\, p^2 g_{\alpha j}^2(p) . \tag{138}$$

Thus

$$\begin{aligned} \Delta T &= -G_0^{-1}(1 - I) Z^- G_0 \frac{r_{\alpha j}}{\rho_{\alpha j}} \Delta t (IM^+ - 1) \\ &= G_0^{-1}(1 - I) Z^- G_0 \frac{r_{\alpha j}}{\rho_{\alpha j}} \Delta S_{\alpha j} S_{\alpha j}^{-1} T^+ \\ &= T^- S_{\alpha j}^{-1} \frac{\Delta S_{\alpha j}}{\rho_{\alpha j}} S_{\alpha j}^{-1} T^+ . \end{aligned} \tag{139}$$

Defining

$$\Delta_{\alpha j} = S_{\alpha j}^{-1} \frac{\Delta S_{\alpha j}}{\rho_{\alpha j}} S_{\alpha j}^{-1} , \tag{140}$$

we finally obtain

$$\Delta T = T^- \Delta_{\alpha j} T^+ . \tag{141}$$

When Eq. (141) is inserted between the proper initial and final states one obtains the usual unitarity relations connecting the breakup, elastic scattering, and rearrangement amplitudes.

Finally, considering the same cut for our singular core formalism (again with $V_e = 0$ for simplicity), we observe that F contains the bound-state poles; thus $F \to R_{\alpha j} S_{\alpha j}$, or

$$\Delta S_{\alpha j} S_{\alpha j}^{-1} F = R_{\alpha j} . \tag{142}$$

Recalling Eq. (12), we deduce that

$$G_0 r_{\alpha j} \Delta S_{\alpha j} = (1 - \tilde{V}) G_0 R_{\alpha j} \hat{t} \Delta S_{\alpha j} . \tag{143}$$

It is helpful to define $\bar{r}_{\alpha j}$ such that

$$\bar{r}_{\alpha j} \hat{t} = r_{\alpha j} ; \tag{144}$$

this corresponds to the definition of F in Eq. (67), and we use the same dummy function. Thus

$$G_0 \bar{r}_{\alpha j} \Delta S_{\alpha j} = (1 - \tilde{V}) G_0 R_{\alpha j} \Delta S_{\alpha j} . \tag{145}$$

If we again invoke Eq. (122) and the subsequent discussion, we can effectively take

$$\begin{aligned} \Delta K &= \hat{t} H G_0 \Delta F \\ &= \hat{t} H G_0 R_{\alpha j} \Delta S_{\alpha j} \\ &= \hat{t} H G_0 \bar{r}_{\alpha j} \Delta S_{\alpha j} . \end{aligned} \tag{146}$$

Here we have employed $\hat{t} H \tilde{V} = 0$, which may be deduced from Eq. (82). In this fashion it is straightforward to obtain

$$\begin{aligned} \Delta T &= -G_0^{-1}(1 - I) Q \{ G_0 F^- Z^- \theta \Delta Y \\ &\qquad + [1 + G_0 F^- Z^- \hat{t} H] G_0 \Delta S_{\alpha j} \bar{r}_{\alpha j} Y^+ \} , \\ \theta \Delta Y &= -\theta Y^- \Delta S_{\alpha j} S_{\alpha j}^{-1} \frac{\bar{r}_{\alpha j}}{\rho_{\alpha j}} Y^+ , \end{aligned} \tag{147}$$

where we have used $r_{\alpha j} \bar{r}_{\alpha j} = \rho_{\alpha j} \bar{r}_{\alpha j}$.

In order to put this expression for ΔT into the form of Eq. (141), we apply similar reasoning to deduce the relations

$$\begin{aligned} &\Delta S_{\alpha j} S_{\alpha j}^{-1} T^+ = -\Delta S_{\alpha j} \bar{r}_{\alpha j} Y^+ , \\ &(Z - 1) \Delta S_{\alpha j} S_{\alpha j}^{-1} = Z \hat{t} H G_0 \bar{r}_{\alpha j} \Delta S_{\alpha j} , \\ &T^- \Delta S_{\alpha j} S_{\alpha j}^{-1} = -G_0^{-1}(1 - I) Q \Lambda , \\ &\Lambda = G_0 F^- Z^- \theta Y^- \Delta S_{\alpha j} S_{\alpha j}^{-1} \\ &\qquad - [1 + G_0 F^- Z^- \hat{t} H] G_0 r_{\alpha j} \Delta S_{\alpha j} . \end{aligned} \tag{148}$$

Appropriate substitutions then give us the required result.

VI. DISCUSSION

In the preceding sections we have considered in some detail a specific prescription for introducing singular cores into the three-body problem. It is important to note that we have made the explicit assumption that our three-body wave function must vanish whenever any pair of particles are within their core radius. This is equivalent to assuming that the BCM is present in each two-body partial wave, i.e., that there is some minimum radius r_0 within which all two-body partial waves vanish. However, it is quite possible to introduce models in which the hard core or BCM appears in only a finite number of partial waves. Our proof that the usual Faddeev formalism does not yield a unique solution does not apply to this case; on the other hand, the Faddeev kernel is not square-integrable, and hence one cannot *prove* the existence of solutions. Of course, this does not mean that such solutions do not exist, and numerical solutions have in fact been obtained for the case of hard core plus square well (two-body s waves only) by Kim and Tubis.[15]

Due to the centrifugal barrier, it does not appear likely that one will be able to distinguish between these two possibilities from the experimental information contained in higher partial waves; their relative usefulness will hinge on the nature of the three-body predictions generated and the ease of calculation they afford. From this point of view, our approach has the advantage of possessing a square-integrable kernel, which both guarantees unique solutions and simplifies numerical analysis. Moreover, in the special case of BCM alone (no external potential) our formalism reduces to a one-dimensional integral equation, a simplification analogous to that occurring in the usual Faddeev formalism for separable interactions. Although this is of no direct help in performing computations with "realistic" singular core models, it does facilitate initial calculations designed to explore the possible consequences of this approach.

On the other hand, we have had to introduce a boundary parameter W_0 which is not specified by the BCM. Depending on one's point of view, there are several procedures available for selecting W_0. One may, for example, take it to be the ground-state energy of the three-body system; our formalism then defines an analytic continuation of the bound-state wave function to the scattering region, while W_0 is the largest negative value of W for which Z has a pole (and hence is determined uniquely by the BCM parameters). Conversely, one may take W_0 as a free parameter and adjust it to produce a three-particle binding energy in agreement with experiment; this is then the largest negative value of W for which $\bar{Z}$ has a pole. In the latter case one is clearly extending the phenomenological treatment of singular cores (e.g., the BCM) to the three-body system. In fact, this approach may be generalized to provide a complete phenomenology of three-particle final states, a topic we shall discuss in a related article now under preparation.

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APPENDIX: CONSTRUCTION OF Q

In this appendix we derive an explicit form for the operator Q employed in the text. We want Q to be of the form

$$Q=1+\tilde{V}B(I-1), \tag{A1}$$

where $\tilde{V}$ and B commute, and Q satisfies the properties summarized in Eq. (10). We first observe that it is sufficient that B satisfies

$$\tilde{V}(1-I)\tilde{V}B(1-I)=\tilde{V}(1-I), \tag{A2}$$

and is diagonal in the coordinate representation. Since $\tilde{V}$ is also diagonal the commutativity follows trivially, while

$$\tilde{V}(1-I)Q=\tilde{V}(1-I)[1-\tilde{V}B(1-I)]=0; \tag{A3}$$

hence

$$(1-\tilde{V}I)Q=(1-\tilde{V})Q=1-\tilde{V}.$$

Also, it is easy to verify that $I^T=I$; thus, taking the transpose of Eq. (A2),

$$(1-I)\tilde{V}=(1-I)\tilde{V}B(1-I)\tilde{V}. \tag{A4}$$

This implies that

$$(1-I)Q\tilde{V}=(1-I)\tilde{V}[1-B(1-I)\tilde{V}]=0, \tag{A5}$$

while the remaining properties of Eq. (10) follow trivially.

Therefore, it is only necessary to find a diagonal operator B such that Eq. (A2) is satisfied. To do so it is convenient to make the double Fourier transformation $\vec{p}\to\vec{x}$, $\vec{q}\to\vec{y}$ and to consider Eq. (A2) in coordinate space. It is also convenient to utilize the "supervector" notation introduced in Sec. III, such that

$$\vec{\rho}=\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}.$$

Let

$$\delta_{\alpha\beta}F(\vec{\rho},\vec{\eta})=\langle\alpha\vec{x}\vec{y}\,|\,\beta\vec{p}\vec{q}\rangle$$
$$=\delta_{\alpha\beta}e^{-i\vec{x}\cdot\vec{p}}e^{-i\vec{y}\cdot\vec{q}}. \tag{A6}$$

One can then easily verify that

$$\langle\alpha\vec{x}\vec{y}|I|\beta\vec{p}\vec{q}\rangle = 0, \quad \alpha=\beta$$
$$= -F(R_{\beta\alpha}\vec{\rho},\vec{\eta}), \quad \alpha\beta \text{ cyclic}$$
$$= -F(R^{-1}{}_{\alpha\beta}\vec{\rho},\vec{\eta}), \quad \beta\alpha \text{ cyclic}. \tag{A7}$$

Similarly, representing

$$\langle\alpha\vec{x}\vec{y}|B|\beta\vec{x}'\vec{y}'\rangle = \delta_{\alpha\beta}\delta(\vec{x}-\vec{x}')\delta(\vec{y}-\vec{y}')B_\alpha(\vec{\rho}), \tag{A8}$$

it follows that

$$\langle\alpha\vec{x}\vec{y}|\tilde{V}(1-I)\tilde{V}B|\beta\vec{x}'\vec{y}'\rangle$$
$$=\theta(a_\alpha - x)\langle\alpha\vec{x}\vec{y}|(1-I)|\beta\vec{x}'\vec{y}'\rangle\theta(a_\beta - x')B_\beta(\vec{\rho}'). \tag{A9}$$

Using Eqs. (A7) and (A9), one can easily show that

$$\langle\alpha\vec{x}\vec{y}|\tilde{V}(1-I)\tilde{V}B(1-I)|\gamma\vec{p}\vec{q}\rangle$$
$$=\left[\theta(a_\alpha - x)B_\alpha(\vec{\rho}) + \theta\left(a_\sigma - \left|\frac{\mu_\alpha}{m_\epsilon}\vec{x}-\vec{y}\right|\right)B_\sigma(R_{\sigma\alpha}\vec{\rho})\right.$$
$$\left.+\,\theta\left(a_\epsilon - \left|\frac{\mu_\alpha}{m_\sigma}\vec{x}+\vec{y}\right|\right)B_\epsilon\left(R^{-1}{}_{\alpha\epsilon}\vec{\rho}\right)\right]$$
$$\times\langle\alpha\vec{x}\vec{y}|\tilde{V}(1-I)|\gamma\vec{p}\vec{q}\rangle, \tag{A10}$$

with $\alpha\sigma\epsilon$ cyclic.

Comparing this result to Eq. (A2), it is clear that our purpose can be achieved if we choose $B_\alpha(\vec{\rho})$ such that the bracket in Eq. (A10) is unity for $x<a_\alpha$. To do so, we consider in turn four separate domains. Suppose first that

$$x<a_\alpha,$$
$$\left|\frac{\mu_\alpha}{m_\epsilon}\vec{x}-\vec{y}\right| > a_\sigma, \tag{A11}$$
$$\left|\frac{\mu_\alpha}{m_\sigma}\vec{x}+\vec{y}\right| > a_\epsilon;$$

let us call this region I_α. In this region the last two θ functions in the bracket vanish and we may obviously choose

$$B_\alpha(\vec{\rho}) = 1, \quad \vec{\rho}\in I_\alpha. \tag{A12}$$

We next consider region II_α, defined by

$$x<a_\alpha,$$
$$\left|\frac{\mu_\alpha}{m_\epsilon}\vec{x}-\vec{y}\right| < a_\sigma, \tag{A13}$$
$$\left|\frac{\mu_\alpha}{m_\sigma}\vec{x}+\vec{y}\right| > a_\epsilon.$$

For this case the first two terms in the bracket contribute, but we must be careful in handling $B_\sigma(R_{\sigma\alpha}\vec{\rho})$ since its argument lies in a different domain. Letting $\vec{\rho}' = R_{\sigma\alpha}\vec{\rho}$, we have that

$$\left|\frac{\mu_\sigma}{m_\alpha}\vec{x}'-\vec{y}'\right| = \left|\frac{\mu_\alpha}{m_\sigma}\vec{x}+\vec{y}\right| > a_\epsilon,$$
$$\left|\frac{\mu_\sigma}{m_\epsilon}\vec{x}'+\vec{y}'\right| = |\vec{x}| < a_\alpha, \tag{A14}$$

for $\vec{\rho}\in\text{II}_\alpha$. Hence, defining region III_α to be the domain

$$x<a_\alpha,$$
$$\left|\frac{\mu_\alpha}{m_\epsilon}\vec{x}-\vec{y}\right| > a_\sigma, \tag{A15}$$
$$\left|\frac{\mu_\alpha}{m_\sigma}\vec{x}+\vec{y}\right| < a_\epsilon,$$

it is straightforward to verify that for $\vec{\rho}\in\text{II}_\alpha$, $R_{\sigma\alpha}\vec{\rho}\in\text{III}_\sigma$. Similarly, one finds that for $\vec{\rho}\in\text{III}_\alpha$, $R^{-1}{}_{\alpha\epsilon}\vec{\rho}\in\text{II}_\epsilon$. Therefore, we can satisfy our requirement in the regions II_α and III_α by taking

$$B_\alpha(\vec{\rho}) = \tfrac{1}{2}, \quad \vec{\rho}\in\text{II}_\alpha, \quad \text{or} \quad \vec{\rho}\in\text{III}_\alpha. \tag{A16}$$

Finally, we consider region IV_α, defined by

$$x<a_\alpha,$$
$$\left|\frac{\mu_\alpha}{m_\epsilon}\vec{x}-\vec{y}\right| < a_\sigma, \tag{A17}$$
$$\left|\frac{\mu_\alpha}{m_\sigma}\vec{x}+\vec{y}\right| < a_\epsilon.$$

Here all three terms in the bracket contribute, but one can show that for $\vec{\rho}\in\text{IV}_\alpha$, $R_{\sigma\alpha}\vec{\rho}\in\text{IV}_\sigma$ and $R^{-1}{}_{\alpha\epsilon}\vec{\rho}\in\text{IV}_\epsilon$. Thus all of the B functions are in the same relative domain and we may simply take

$$B_\alpha(\vec{\rho}) = \tfrac{1}{3}, \quad \vec{\rho}\in\text{IV}_\alpha. \tag{A18}$$

The above requirements on B_α may be summarized in the explicit formula

$$B_\alpha(\vec{\rho}) = 1 - \tfrac{1}{2}\theta\left(a_\sigma - \left|\frac{\mu_\alpha}{m_\epsilon}\vec{x}-\vec{y}\right|\right) - \tfrac{1}{2}\theta\left(a_\epsilon - \left|\frac{\mu_\alpha}{m_\sigma}\vec{x}+\vec{y}\right|\right)$$
$$+\tfrac{1}{3}\theta\left(a_\sigma - \left|\frac{\mu_\alpha}{m_\epsilon}\vec{x}-\vec{y}\right|\right)\theta\left(a_\epsilon - \left|\frac{\mu_\alpha}{m_\sigma}\vec{x}+\vec{y}\right|\right). \tag{A19}$$

[Since B only occurs multiplied by $\tilde{V}$ one need not put in the explicit factor $\theta(a_\alpha - x)$.]

We have thus demonstrated the existence of our Q operator by actual construction.

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[1]D. D. Brayshaw, Phys. Rev. Letters 26, 659 (1971), hereafter referred to as B1.

[2]D. D. Brayshaw, Phys. Rev. C 3, 35 (1971), hereafter referred to as B2. All of our remarks concerning the boundary-condition model (BCM) will also apply to the hard core as a special case.

[3]L. M. Delves, J. M. Blatt, I. Pask, and B. Davies, Phys. Letters 28B, 472 (1969), and earlier work quoted therein.

[4]J. A. Tjon, B. F. Gibson, and J. S. O'Connell, Phys. Rev. Letters 25, 540 (1970). This is supported by more recent calculations by C. Gignoux and A. Laverne, Phys. Rev. Letters 29, 436 (1972); also E. P. Harper, Y. E. Kim, and A. Tubis, Phys. Rev. Letters 28, 1533 (1972). However, another recent calculation with the same (Reid) potential by Hennell and Delves (unpublished) appears to be in disagreement with these, obtaining values in better agreement with experiment.

[5]See, for example, T. Hamada and I. D. Johnston, Nucl. Phys. 34, 382 (1962); and H. Feshbach and E. Lomon, Phys. Rev. Letters 6, 635 (1961).

[6]L. D. Faddeev, Zh. Eksp. Teor. Fiz. 39, 1459 (1960) [Soviet Phys. JETP 12, 1014 (1961)].

[7]One should keep in mind that any representation of the very-short-range N-N interaction within the nonrelativistic framework is, at best, a phenomenological tool. We mean "realistic" in the sense of a superior mathematical formulation, not necessarily the most physically appealing description. One would not assert literally that nature employs singular cores; such a concept may, however, be the most efficient idealization.

[8]See, for example, T. D. Lee and C. N. Yang, Phys. Rev. 113, 1165 (1959); also A. Pais and G. E. Uhlenbeck, *ibid*. 116, 250 (1959).

[9]As is well known, any of the usual states $|\vec{p}_\alpha \vec{q}_\alpha\rangle$ ($\alpha = 1, 2, 3$) completely specify the system; however, t_α takes on a particularly simple form in the α representation. In a sense, we have introduced a threefold copy of the space in order to take advantage of this fact, simplifying the notation and operator manipulations. The only important distinction between $|\alpha\vec{p}_\alpha \vec{q}_\alpha\rangle$ and $|\vec{p}_\alpha \vec{q}_\alpha\rangle$ comes about in normalizations; thus with our choice of intermediate states $\langle\Psi|\Psi\rangle = 3$.

[10]In the equal-mass problem, with particles 1 and 2 at their core radius a, the domain $y < \frac{1}{2}\sqrt{3}\,a$ corresponds to at least one of the pairs $\{13\}$, $\{23\}$ being within their core. Conversely, $y > \frac{3}{2}a$ implies that no pair is within their core.

[11]Strictly speaking, this is true only for $W < 0$; for $W > 0$ the usual $i\epsilon$ limit must be taken. However, only $\bar{N}_1$ is singular in this limit, and, since this term is identical in structure to the Faddeev kernel for a separable interaction, it is clear that this distinction is of little significance. More precise analysis shows that Z exists except for a finite number of real negative values of W. Numerical calculations for the simple model discussed in the text and parameters relevant to the N-N triplet state yield a single such value.

[12]The essential point here is that the BCM requires the *sum* of the channel wave functions to vanish in the interior and have prescribed logarithmic derivative on the boundary; this does not specify each channel uniquely. However, the precise choice of channels *does* affect the exterior wave function in certain regions.

[13]We do not, however, make the other simplifying assumptions imposed in Sec. IV. With suitable generalizations of Eqs. (65) and (81), the results of that section which are stated in operator form remain valid; e.g., Eqs. (86) and (94).

[14]The introduction of $\rho_{\alpha\gamma}$ is merely a formal convenience. It is possible that the definite integral defining it does not converge in some cases, but this in no way affects the results as it explicitly cancels when one evaluates Eq. (141).

[15]Y. E. Kim and A. Tubis, Phys. Rev. C 1, 1627 (1970). The same authors have more recently performed a similar calculation with s-wave BCM only; see Y. E. Kim and A. Tubis, Phys. Letters 38B, 354 (1972).