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Wick Equation, the Infinite-Momentum Frame, and Perturbation Theory*

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The eigenvalues of the Wick equation in the weak-binding limit are found in perturbation theory employing two different approaches: (1) a covariant approach using an integral representation for the Bethe-Salpeter wave function and (2) quantization in the infinite-momentum frame using the technique of Kogut and Soper. The eigenvalues agree to order $\alpha^3 \ln\alpha$.

I. INTRODUCTION

Recently there has been considerable interest in the application of the infinite-momentum frame to quantum field theory. Weinberg[1] first considered the infinite-momentum limit of "old-fashioned" perturbation diagrams for φ^3 and φ^4 theories. He showed that the vacuum diagrams present in these theories become simpler in the high-momentum limit. Chang and Ma[2] then examined Feynman diagrams in a φ^3 theory in terms of the new variables $\tau = (t+z)/\sqrt{2}$ and $\vartheta = (t-z)/\sqrt{2}$ and their conjugate momenta. Using these variables they were able to reproduce Weinberg's results without the necessity of going to the limit of infinite momentum. Kogut and Soper[3] extended their results by considering the formal foundations of quantum electrodynamics in the infinite-momentum frame. Their hope was that an exact theory in this frame would make the calculation of amplitudes simpler for various high-energy electromagnetic processes.

In this paper we wish to show that the infinite-momentum frame can be used to deal with a low-energy process, namely the calculation of the invariant mass squared of a bound system. Although such an approach necessarily sacrifices manifest Lorentz covariance from the start, the hope is that the infinite-momentum frame might lead to simplifications in the calculation of binding energies in systems that are more complicated than the one considered here.

The Bethe-Salpeter equation[4] is, of course, the conventional tool for dealing with the relativistic bound-state problem. The presence of a relative time variable in the Bethe-Salpeter wave function makes its physical significance unclear. Moreover, the perturbation theories one uses to calculate the binding energies have a generally noncovariant character with this relative time or energy variable singled out.

The one problem which is dealt with by explicitly covariant methods is the solution of the Wick equation,[5] that is the Bethe-Salpeter equation describing two spinless particles bound by a massless scalar field in the ladder approximation. For this case, due to the symmetries that are present and the possibility of performing a Wick rotation, Wick[5] and Cutkosky[6] were able to introduce a spectral or integral representation for the wave function in momentum space. The eigenvalue problem could then be reduced to one of the Sturm-Liouville type for this spectral function.

This paper is divided into two independent parts. In Sec. II, starting from the results of Wick and Cutkosky, we construct a covariant perturbation theory for the Wick equation in the weak-binding limit. We use this perturbation theory to calculate the bound-state energies to order $\alpha^3 \ln\alpha$. In Sec. III we consider the same problem in the framework of the infinite-momentum frame. We quantize the usual Lagrangian describing two equal-mass particles interacting through exchange of a scalar photon in this frame. From this Lagrangian we derive the Hamiltonian and apply a Tamm-Dancoff approximation to the equation $H|\psi\rangle = E|\psi\rangle$, which retains at most one scalar photon in the intermediate state. We solve this equation in perturbation theory and obtain eigenvalues in agreement with the results of the covariant Wick equation.

In Appendix A we present a brief derivation of Wick's results for the ground state. In Appendix B we give an alternative derivation of the infinite-momentum bound-state equation starting from the covariant Bethe-Salpeter Green's function. Last, in Appendix C we present the details of the perturbation theory for the eigenvalues of the infinite-momentum bound-state equation of Sec. III.

II. COVARIANT PERTURBATION THEORY

The Bethe-Salpeter equation for two scalar particles of equal mass interacting through a massless scalar field is written in momentum space as[5,7]

$$[(p+i\xi)^2+1][(p-i\xi)^2+1]\varphi(p) = \frac{(\alpha/\pi)}{\pi^2}\int \frac{d^4p'\varphi(p')}{(p-p')^2}\,, \tag{2.1}$$

where α is the fine-structure constant, the mass

of each particle has been set equal to unity, and ξ_μ is one-half of the total four-momentum of the bound system and equals $(\vec{0}, \mu/2)$ in the center-of-mass frame. In Eq. (2.1) the Wick rotation has been performed so that the metric is Euclidean with

$$p^2 = p_1^2 + p_2^2 + p_3^2 + p_4^2 . \tag{2.2}$$

Wick was able to find a set of solutions to Eq. (2.1) in terms of an integral over an unknown function $g(z)$. By writing

$$\varphi(p) = \int_{-1}^{1} dz\, g(z)(p^2 + 2iz\xi\cdot p + 1 - \xi^2)^{-3}, \tag{2.3}$$

he was able to obtain a solution to Eq. (2.1) provided $g(z)$ satisfied the differential equation

$$g''(z) + (\alpha/\pi)(1-z^2)^{-1}(1-\xi^2+\xi^2z^2)^{-1}g(z) = 0, \tag{2.4}$$

with the boundary condition that $g(+1) = g(-1) = 0$. In Appendix A we present a simple alternative derivation of Wick's results.

Cutkosky extended Wick's results by trying to find solutions which were linear combinations of functions of the form

$$\varphi_n^{lm}(p, z) = \frac{\mathcal{Y}_{lm}(\vec{p})}{[p^2 + 2izp\cdot\xi + 1 - \xi^2]^{n+2}}, \tag{2.5}$$

where $\mathcal{Y}_{lm}(\vec{p})$ is a solid harmonic of the three space components of p. He showed that the eigenvalues could be obtained by solving the differential equation

$$\begin{aligned} &g_n''(z) + 2(n-1)z(1-z^2)^{-1}g_n'(z) \\ &\quad - n(n-1)(1-z^2)^{-1}g_n(z) \\ &\quad + (\alpha/\pi)(1-z^2)^{-1}(1-\xi^2+\xi^2z^2)^{-1}g_n(z) = 0, \end{aligned} \tag{2.6}$$

with the boundary conditions $g_n(\pm1) = 0$. Equation (2.4) is a special case of Eq. (2.6) with $n = 1$. The eigenvalue equation is independent of l so that the degeneracy is the same as that of the nonrelativistic hydrogen atom.

In the weak-binding limit, $\xi \to 1$, since

$$\xi = \mu/2 = 1 - B/2, \tag{2.7}$$

where B is the binding energy of the system. In this limit Eq. (2.6) can be solved in perturbation theory. We consider the binding energy as fixed and expand $g(z)$ and α in terms of ω, where

$$\omega^2 = \xi^{-2} - 1. \tag{2.8}$$

If Eq. (2.6) is multiplied by $(1-z^2)^{1-n}$, the result is

$$\begin{aligned} &\frac{d}{dz}\left[(1-z^2)^{1-n}\frac{d}{dz}g_n(z)\right] \\ &\quad - n(n-1)(1-z^2)^{-n}g_n(z) \\ &\quad + (\alpha/\pi)(1-z^2)^{-n}(1-\xi^2+\xi^2z^2)^{-1}g_n(z) = 0 . \end{aligned} \tag{2.9}$$

This has the effect of making the differential operator

$$\mathcal{L} = \frac{d}{dz}(1-z^2)^{1-n}\frac{d}{dz} - n(n-1)(1-z^2)^{-n} \tag{2.10}$$

Hermitian for the space of functions to be considered. Following Wick, we write for $\xi \to 1$ $(\omega \to 0)$

$$(\alpha/\pi)(1-z^2)^{-n}(1-\xi^2+\xi^2z^2)^{-1} = \lambda(V_0 + V_1), \tag{2.11}$$

where

$$\begin{aligned} &\lambda = \alpha/\omega, \quad V_0 = \delta(z), \\ &V_1 = \frac{\omega}{\pi}\left[\frac{1+\omega^2}{(1-z^2)^n(\omega^2+z^2)}\right] - \delta(z). \end{aligned} \tag{2.12}$$

Using these definitions and writing

$$\begin{aligned} &g_n(z) = g_n^{(0)} + g_n^{(1)} + \cdots, \\ &\lambda_n = \lambda_n^{(0)} + \lambda_n^{(1)} + \cdots, \end{aligned} \tag{2.13}$$

Eq. (2.9) becomes

$$\begin{aligned} &\mathcal{L}(g_n^{(0)} + g_n^{(1)} + \cdots) \\ &\quad + (\lambda_n^{(0)} + \lambda_n^{(1)} + \cdots)(V_0 + V_1)(g_n^{(0)} + g_n^{(1)} + \cdots) = 0 . \end{aligned} \tag{2.14}$$

The boundary conditions are now $g_n^{(i)}(\pm1) = 0$, $i = 0, 1, 2, \ldots$.

The lowest-order equation,

$$\mathcal{L}g_n^{(0)} + \lambda_n^{(0)}V_0g_n^{(0)} = 0, \tag{2.15}$$

has solutions

$$g_n^{(0)} = c_n(1-|z|)^n, \quad \lambda_n^{(0)} = 2n, \tag{2.16}$$

with c_n an arbitrary constant. The other solutions to Eq. (2.15), the so-called "abnormal" solutions, are discarded because they do not have the correct nonrelativistic limit. The first-order equation contained in Eq. (2.14) is

$$(\mathcal{L} + \lambda_n^{(0)}V_0)g_n^{(1)} + \lambda_n^{(1)}V_0g_n^{(0)} + \lambda_n^{(0)}V_1g_n^{(0)} = 0 . \tag{2.17}$$

If this equation is multiplied by $g_n^{(0)}$ and integrated from -1 to 1, the result is

$$\lambda_n^{(1)} = \frac{-\lambda_n^{(0)}\int_{-1}^{1} g_n^{(0)}V_1g_n^{(0)}dz}{\int_{-1}^{1} g_n^{(0)}V_0g_n^{(0)}dz} . \tag{2.18}$$

In obtaining this result use has been made of the fact that

$$\begin{aligned} \int_{-1}^{1} g_n^{(0)}(\mathcal{L} + \lambda_n^{(0)}V_0)g_n^{(1)}dz &= \int_{-1}^{1} g_n^{(1)}(\mathcal{L} + \lambda_n^{(0)}V_0)g_n^{(0)}dz \\ &= 0. \end{aligned} \tag{2.19}$$

Retaining terms of order $\omega \ln\omega$ in Eq. (2.18), we find that

$$\lambda_n^{(1)} = -8n^2\omega(\ln\omega)/\pi + O(\omega^2)\,. \tag{2.20}$$

Thus

$$\begin{aligned}\alpha &= \omega\lambda_n \approx \omega(\lambda_n^{(0)} + \lambda_n^{(1)}) \\ &= 2n\omega - 8n^2\omega^2(\ln\omega)/\pi + O(\omega^3).\end{aligned} \tag{2.21}$$

With the aid of Eqs. (2.7) and (2.8), Eq. (2.21) can be solved for the binding energy B in terms of α. To order $\alpha^3\ln\alpha$ we find that

$$B_n = \frac{\alpha^2}{4n^2} + \frac{\alpha^3}{\pi n^2}\ln\alpha + O(\alpha^3)\,. \tag{2.22}$$

We note, incidentally, that because of the nature of $\mathfrak{L}$ for the ground state ($n=1$) and the fact that $V_0 = \delta(z)$, the perturbation theory discussed above is particularly simple for this case. From Eq. (2.17), the first-order correction to $g_1(z)$ satisfies the equation

$$g_1^{(1)\prime\prime} = -[\lambda_1^{(0)} V_0 g_1^{(1)} + \lambda_1^{(0)} V_1 g_1^{(0)} + \lambda_1^{(1)} V_0 g_1^{(0)}]\,. \tag{2.23}$$

Since the right-hand side of this equation is apart from the constant $g_1^{(1)}(0)$, a known function of z, Eq. (2.23) can be integrated directly and $g_1^{(1)}(z)$ obtained. This result holds for all higher-order corrections as well, and thus a knowledge of only the zeroth-order ground-state wave function $g_1^{(0)}(z)$ is required to calculate the ground-state wave function $g_1(z)$ to all orders.

III. INFINITE-MOMENTUM FRAME

In this section we consider the same bound-state problem as treated in Sec. II, but from the viewpoint of the infinite-momentum frame. We begin with the usual Lagrangian describing the interaction of two scalar particles of equal mass m and a massless scalar particle:

$$L = (\hat\partial_\mu\varphi^\dagger)(\hat\partial^\mu\varphi) - m^2\varphi^\dagger\varphi + \tfrac{1}{2}(\hat\partial_\mu\varphi_0)(\hat\partial^\mu\varphi_0) - g\varphi^\dagger\varphi\varphi_0\,. \tag{3.1}$$

Here the $\hat x^\mu$ refer to the usual space-time coordinates. Expressed in terms of the new coordinates [8] of the infinite-momentum frame,

$$\begin{aligned}x^\mu &= (\tau, x^1, x^2, \mathfrak{z}) \\ &= C^\mu{}_\nu \hat x^\nu \\ &= g^{\mu\nu}x_\nu\,,\end{aligned} \tag{3.2}$$

with

$$C^\mu{}_\nu = \begin{pmatrix} 2^{-1/2} & 0 & 0 & 2^{-1/2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2^{-1/2} & 0 & 0 & -2^{-1/2} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

The Lagrangian becomes

$$\begin{aligned}L = {} & (\partial_\tau\varphi^\dagger)(\partial_{\mathfrak{z}}\varphi) + (\partial_{\mathfrak{z}}\varphi^\dagger)(\partial_\tau\varphi) \\ & - \vec\nabla\varphi^\dagger\cdot\vec\nabla\varphi - m^2\varphi^\dagger\varphi + (\partial_\tau\varphi_0)(\partial_{\mathfrak{z}}\varphi_0) \\ & - \tfrac{1}{2}(\vec\nabla\varphi_0)^2 - g\varphi^\dagger\varphi\varphi_0\,,\end{aligned} \tag{3.4}$$

where [9] $\vec{\mathbf{x}} = (x^1, x^2)$. The fields φ and φ_0 are quantized according to the equal-τ commutation relations

$$\begin{aligned}[\varphi(x), \varphi^\dagger(x')]_{\tau=\tau'} &= \frac{1}{4i}\epsilon(\mathfrak{z}-\mathfrak{z}')\delta^2(\vec{\mathbf{x}}-\vec{\mathbf{x}}')\,, \\ [\varphi_0(x), \varphi_0(x')]_{\tau=\tau'} &= \frac{1}{4i}\epsilon(\mathfrak{z}-\mathfrak{z}')\delta^2(\vec{\mathbf{x}}-\vec{\mathbf{x}}')\,.\end{aligned} \tag{3.5}$$

Using Noether's theorem and the Lagrangian of Eq. (3.1) or (3.4), P_μ and $J_{\mu\nu}$ can be constructed in terms of the fields. These generators satisfy the usual commutation relations of the Poincaré group. [The $g^{\mu\nu}$ of Eq. (3.3) must be used, where needed.] In particular

$$P^\mu = (P^0, P^1, P^2, P^3) \equiv (\eta_{op}, P^1, P^2, H) \tag{3.6}$$

satisfy the commutation relations

$$[P^\mu, P^\nu] = 0. \tag{3.7}$$

In this frame $P_0 = P^3 = H$ is the variable conjugate to τ and plays the role of the Hamiltonian. Expressed in terms of the fields, H is given by

$$H = H_0 + H_I\,,$$

where

$$\begin{aligned}H_0 &= \int d\vec{\mathbf{x}}\,d\mathfrak{z}\,[\vec\nabla\varphi^\dagger\cdot\vec\nabla\varphi + m^2\varphi^\dagger\varphi + \tfrac{1}{2}(\vec\nabla\varphi_0)^2]\,, \\ H_I &= g\int d\vec{\mathbf{x}}\,d\mathfrak{z}\,\varphi^\dagger\varphi\varphi_0\,.\end{aligned} \tag{3.8}$$

In the Schrödinger picture, the Fourier expansions of the fields at $\tau = 0$ take the form

$$\begin{aligned}\varphi(\vec{\mathbf{x}}, \mathfrak{z}) &= (2\pi)^{-3}\int d\vec{\mathbf{q}}\int_0^\infty \frac{d\eta}{2\eta}[a(\vec{\mathbf{q}}, \eta)e^{-i(\eta\mathfrak{z}-\vec{\mathbf{q}}\cdot\vec{\mathbf{x}})} + b^\dagger(\vec{\mathbf{q}}, \eta)e^{i(\eta\mathfrak{z}-\vec{\mathbf{q}}\cdot\vec{\mathbf{x}})}], \\ \varphi_0(\vec{\mathbf{x}}, \mathfrak{z}) &= (2\pi)^{-3}\int d\vec{\mathbf{q}}\int_0^\infty \frac{d\eta}{2\eta}[a_0(\vec{\mathbf{q}}, \eta)e^{-i(\eta\mathfrak{z}-\vec{\mathbf{q}}\cdot\vec{\mathbf{x}})} + a_0^\dagger(\vec{\mathbf{q}}, \eta)e^{i(\eta\mathfrak{z}-\vec{\mathbf{q}}\cdot\vec{\mathbf{x}})}]\,.\end{aligned} \tag{3.9}$$

The operators $a(\vec{q}, \eta)$, $a^\dagger(\vec{q}, \eta)$, etc. satisfy the commutation relations

$$[a(\vec{q}, \eta), a^\dagger(\vec{q}', \eta')] = (2\pi)^3 2\eta\delta(\eta-\eta')\delta^2(\vec{q}-\vec{q}'),$$

$$[b(\vec{q}, \eta), b^\dagger(\vec{q}', \eta')] = (2\pi)^3 2\eta\delta(\eta-\eta')\delta^2(\vec{q}-\vec{q}'), \tag{3.10}$$

$$[a_0(\vec{q}, \eta), a_0^\dagger(\vec{q}', \eta')] = (2\pi)^3 2\eta\delta(\eta-\eta')\delta^2(\vec{q}-\vec{q}'),$$

with all other commutators vanishing.

In terms of these creation and annihilation operators,

$$\eta_{op} = (2\pi)^{-3}\int d\vec{q}\int_0^\infty \frac{d\eta}{2\eta}\,\eta[a^\dagger(\vec{q}, \eta)a(\vec{q}, \eta) + b^\dagger(\vec{q}, \eta)b(\vec{q}, \eta) + a_0^\dagger(\vec{q}, \eta)a_0(\vec{q}, \eta)],$$

$$\vec{P} = (2\pi)^{-3}\int d\vec{q}\int_0^\infty \frac{d\eta}{2\eta}\vec{q}\,[a^\dagger(\vec{q}, \eta)a(\vec{q}, \eta) + b^\dagger(\vec{q}, \eta)b(\vec{q}, \eta) + a_0^\dagger(\vec{q}, \eta)a_0(\vec{q}, \eta)], \tag{3.11}$$

$$H_0 = (2\pi)^{-3}\int d\vec{q}\int_0^\infty \frac{d\eta}{2\eta}\left\{\left(\frac{\vec{q}^2+m^2}{2\eta}\right)[a^\dagger(\vec{q}, \eta)a(\vec{q}, \eta) + b^\dagger(\vec{q}, \eta)b(\vec{q}, \eta)] + \left(\frac{\vec{q}^2}{2\eta}\right)a_0^\dagger(\vec{q}, \eta)a_0(\vec{q}, \eta)\right\}.$$

Thus the operators $a^\dagger(\vec{q}, \eta)$, $b^\dagger(\vec{q}, \eta)$, and $a_0^\dagger(\vec{q}, \eta)$ when acting on the bare vacuum $|0\rangle$ create eigenstates of the bare Hamiltonian H_0, η_{op}, and $\vec{P}$.

The calculation of the bound-state spectrum requires the determination of the eigenvalues of the operator

$$P^2 = P_\mu P^\mu = 2\eta_{op}H - \vec{P}^2, \tag{3.12}$$

that is,

$$P^2|\psi\rangle = \mu^2|\psi\rangle, \tag{3.13}$$

where for a bound state of two a or two b particles $\mu = 2m - B$ with B the binding energy. In the transverse center-of-mass frame Eq. (3.12) assumes the simple form

$$2\eta_{op}H|\psi_{\vec{P}=0}\rangle = \mu^2|\psi_{\vec{P}=0}\rangle. \tag{3.14}$$

Since $[\eta_{op}, H] = 0$, we can consider simultaneous eigenstates of both η_{op} and H. If η_{op} has eigenvalue M, then

$$H|\psi_{\vec{P}=0,M}\rangle = \frac{\mu^2}{2M}|\psi_{\vec{P}=0,M}\rangle. \tag{3.15}$$

Thus, solution of the bound-state problem is equivalent to finding the eigenvalues of the Hamiltonian.

In order to solve Eq. (3.15) we consider its overlap with the eigenstates of H_0. These states must, of course, have their total transverse momentum equal to zero and have eigenvalue M of the operator η_{op}. This leads to a coupled set of equations involving the projection of $|\psi_{\vec{P}=0,M}\rangle$ onto the eigenstates of H_0. We now make a Tamm-Dancoff[10] approximation of retaining only the two-particle state

$$a^\dagger(\vec{q}, \eta)a^\dagger(-\vec{q}, M-\eta)|0\rangle,$$

and the two-particle, one-scalar-photon state

$$a^\dagger(\vec{q}, \eta)a^\dagger(\vec{q}', \eta')a_0^\dagger(\vec{q}_0, \eta_0)|0\rangle,$$

where $\vec{q}+\vec{q}'+\vec{q}_0 = 0$ and $\eta+\eta'+\eta_0 = M$, in this coupled set. Reducing these two coupled equations to a single equation for the wave function,

$$\langle 0|a(\vec{q}, \eta)a(-\vec{q}, M-\eta)|\psi\rangle \equiv \langle \vec{q}, \eta|\psi\rangle, \tag{3.16}$$

leads to the result

$$\left[\frac{\vec{q}^2+m^2}{2\eta} + \frac{\vec{q}^2+m^2}{2(M-\eta)} - \frac{\mu^2}{2M}\right]\langle \vec{q}, \eta|\psi\rangle = \frac{g^2}{(2\pi)^3}\int d\vec{q}'\,d\eta'\,\frac{\theta(\eta')\theta(M-\eta')}{2\eta' 2(M-\eta')}\langle \vec{q}', \eta'|\psi\rangle V(\vec{q}, \eta; \vec{q}', \eta'), \tag{3.17}$$

where

$$V(\vec{q}, \eta; \vec{q}', \eta') = \frac{1}{2|\eta-\eta'|}\left\{\theta(\eta-\eta')\left[\frac{\vec{q}'^2+m^2}{2\eta'} + \frac{\vec{q}^2+m^2}{2(M-\eta)} + \frac{(\vec{q}-\vec{q}')^2}{2(\eta-\eta')} - \frac{\mu^2}{2M}\right]^{-1}\right.$$
$$\left. + \theta(\eta'-\eta)\left[\frac{\vec{q}^2+m^2}{2\eta} + \frac{\vec{q}'^2+m^2}{2(M-\eta')} + \frac{(\vec{q}-\vec{q}')^2}{2(\eta'-\eta)} - \frac{\mu^2}{2M}\right]^{-1}\right\}.$$

The θ functions in Eq. (3.17) arise because η is restricted to positive values for all intermediate states. Furthermore, in obtaining Eq. (3.17) we have neglected self-energy terms. We will return to this point later. In terms of a series of τ-ordered perturbation diagrams, the approximations made correspond to keeping the terms illustrated by Fig. 1 in the interaction kernel. This equation is effectively the same as Weinberg's[1] Eq. (32) for the T matrix, which retains one-meson exchange in the interaction kernel.[11] Weinberg obtained his equation by taking the limit as the momentum goes to infinity of the corresponding "old-fashioned" perturbation diagrams.

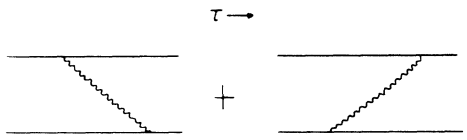

FIG. 1. The one-scalar-photon intermediate states which are retained as the interaction kernel in Eq. (3.17).

Since the literature contains derivations of the Bethe-Salpeter equation in the high-momentum limit which have the covariant Bethe-Salpeter equation as a starting point,[12] in Appendix B we also present a derivation of this kind. We obtain a result identical to Eq. (3.19) below in the order of approximation considered.

We now define

$$\eta=\frac{M}{2}(1+k), \quad \frac{\langle \vec{q}, \eta|\psi\rangle}{1-k^2}=\psi(\vec{q}, k). \tag{3.18}$$

With this substitution and the mass of the two mesons set equal to unity as in Sec. II, Eq. (3.17) becomes

$$(\vec{q}^{\,2}+k^2+\bar{B}-\bar{B}k^2)\psi(\vec{q}, k)=\frac{g^2}{4(2\pi)^3}\int d\vec{q}'dk'\theta(1-k')\theta(1+k')\bar{V}(\vec{q}, k;\vec{q}', k')\psi(\vec{q}', k'),$$

where

$$\bar{V}(\vec{q}, k;\vec{q}', k')=|k-k'|^{-1}\Bigg\{\theta(k-k')\left[\frac{\vec{q}^{\,2}+1}{1-k}+\frac{\vec{q}^{\,\prime 2}+1}{1+k'}+\frac{(\vec{q}-\vec{q}')^2}{k-k'}-2(1-\bar{B})\right]^{-1}$$

$$+\theta(k'-k)\left[\frac{\vec{q}^{\,\prime 2}+1}{1-k'}+\frac{\vec{q}^{\,2}+1}{1+k}+\frac{(\vec{q}-\vec{q}')^2}{k'-k}-2(1-\bar{B})\right]^{-1}\Bigg\}. \tag{3.19}$$

In this equation

$$\bar{B}=B-B^2/4, \tag{3.20}$$

where B, the binding energy, is defined below Eq. (3.13). If we define a three-vector $\vec{p}$ by[13]

$$\vec{p}^{\,2}=\vec{q}^{\,2}+k^2, \tag{3.21}$$

then Eq. (3.19) has a structure analogous to that of the Schrödinger equation for the nonrelativistic hydrogen atom with the reduced mass set equal to $\frac{1}{2}$:

$$(p^2+B)\psi(\vec{p})=\frac{g^2}{4(2\pi)^3}\int\frac{d^3p'\psi(\vec{p}')}{(\vec{p}-\vec{p}')^2}\ . \tag{3.22}$$

This equation has bound-state solutions $\psi^{(0)}_{nlm}(\vec{p})$,[14] with eigenvalues $B_n=\alpha^2/4n^2$ if we set $g^2/4(2\pi)^3=\alpha/2\pi^2$, as for the covariant Wick equation.

In order to verify that the $\psi^{(0)}_{nlm}(\vec{p})$ can serve as lowest-order solutions to Eq. (3.19), it is necessary to show that the terms that have been neglected in obtaining Eq. (3.22) from Eq. (3.19) make a small additional contribution to the binding energy. Here we will outline the steps leading to Eq. (3.30) below. A more complete derivation is given in Appendix C.

Equation (3.19) can be rewritten as

$$[h_0+\bar{B}-k^2\bar{B}-(J-I)]\psi=0, \tag{3.23}$$

where h_0 is the nonrelativistic Coulomb Hamiltonian of Eq. (3.22), $J\psi$ is the right-hand side of Eq. (3.19), and $I\psi$ is the right-hand side of Eq. (3.22). Let ψ and $\bar{B}$ be expanded in powers of the coupling constant:

$$\psi=\psi^{(0)}+\psi^{(1)}+\cdots,$$
$$\bar{B}=\bar{B}^{(0)}+\bar{B}^{(1)}+\cdots. \tag{3.24}$$

The lowest-order solutions are taken to be solutions of Eq. (3.22), that is,[15]

$$\psi_n^{(0)} = \sum_{lm} C_{lm}^n \psi_{nlm}^{(0)}(\vec{p}) .$$

Then from Eq. (C9) of the appendix,

$$\bar{B}_n^{(1)} = \bar{B}_n^{(0)} \langle k^2 \rangle_n + \langle (\bar{J} - I) \rangle_n . \tag{3.25}$$

In this equation $\bar{J}$ is obtained from J by allowing the k' integration contained in J to run from $-\infty$ to ∞ and by replacing $\bar{B}$ by $\bar{B}^{(0)}$. These changes alter the result only in higher orders. The first term of Eq. (3.25) makes an α^4 contribution to the binding energy. Since we are only interested in $\langle \bar{J} - I \rangle_n$ to the lowest order, it is possible to make use of the sharply peaked nature of the [16] $\psi_{nlm}^{(0)}(\vec{p})$ to approximate this expression by Eq. (C12):

$$\langle \bar{J} - I \rangle_n \approx \frac{\alpha}{2\pi^2} \int d\vec{p} d\vec{p}' \psi_n^{(0)*}(\vec{p}) \psi_n^{(0)}(\vec{p}') \{ [(\vec{p} - \vec{p}')^2 + |k - k'| (\alpha^2/n^2)]^{-1} - (\vec{p} - \vec{p}')^{-2} \} . \tag{3.26}$$

Since the term in curly brackets, i.e., the effective-interaction term, is just a function of $\vec{p} - \vec{p}'$, in coordinate space Eq. (3.26) takes the simple form

$$\langle \bar{J} - I \rangle_n \approx \alpha \int d\vec{r} |\Psi_n^{(0)}(\vec{r})|^2 \Delta V(\vec{r}), \tag{3.27}$$

where $\Psi_n^{(0)}(\vec{r})$ is the Fourier transform of $\psi_n^{(0)}(\vec{p})$ and

$$\Delta V(\vec{r}) = \frac{1}{2\pi^2} \int \frac{d\vec{p} e^{-i\vec{p}\cdot\vec{r}}}{p^2 + |k|(\alpha^2/n^2)} - \frac{1}{r} . \tag{3.28}$$

Equation (3.27) can then be evaluated. The result is

$$\langle \bar{J} - I \rangle_n = \frac{\alpha^3}{n^2 \pi} \ln\alpha + O(\alpha^3) . \tag{3.29}$$

Thus from Eq. (C21),

$$B_n = \frac{\alpha^2}{4n^2} + \frac{\alpha^3}{\pi n^2} \ln\alpha + O(\alpha^3) . \tag{3.30}$$

In order to show the equivalence of the bound-state problem formulated in the infinite-momentum frame to the covariant Wick equation, it is necessary to consider the contribution that multi-scalar-photon states of the form of Fig. 2 make to the binding energy. This is so because diagrams of this type are included in the ladder approximation from which the binding energies of Sec. II were calculated. A consistent Tamm-Dancoff approximation carried to the two-scalar-photon level, in addition to the first two diagrams of Fig. 2, would also include diagrams with crossed photons. We do not consider them here since they would not be included in Wick's ladder approximation either. The largest contribution to the binding energy from multi-scalar-photon states comes from the two-photon states, and this can be shown to be at least of order α^3. Thus the approaches of Sec. II and III are consistent to order $\alpha^3 \ln\alpha$ [cf. Eqs. (2.22) and (3.30)].

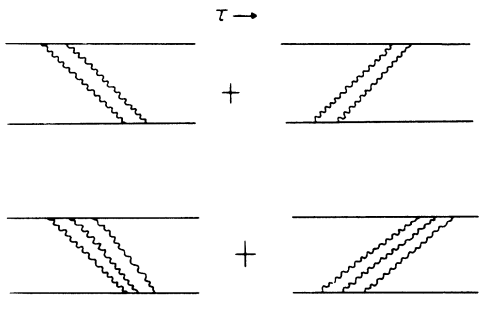

FIG. 2. Multi-scalar-photon states which contribute to the Wick equation in the ladder approximation but have been neglected in obtaining Eq. (3.17).

IV. CONCLUSION

In this paper we have shown that the technique of quantizing a field theory in the infinite-momentum frame can be extended from its use in the calculation of S-matrix elements to the bound-state problem. We have applied this theory to the system of two scalar mesons bound by a massless scalar meson and have seen that it yields results consistent with the conventional Bethe-Salpeter formalism. In so doing, we have neglected self-energy terms and other renormalization effects. If this quantization procedure, however, is to be employed to calculate bound-state energies to higher orders in this case or in another of greater physical interest, such effects must necessarily be taken into account. This extension will be discussed in a future publication.

In order to make a detailed comparison possible between the conventional covariant Bethe-Salpeter equation and the infinite-momentum frame equation, we have developed a covariant perturbation theory for the Wick equation. This method is of some interest on its own account. Previously, explicit solutions of Wick's equation have been exhibited only in the strong-binding limit ($\xi = 0$) or to order

α^2 in the weak-binding limit. One curious feature of this calculation is the appearance of an $\alpha^3 \ln\alpha$ correction to the bound-state Balmer energies. Such a term is peculiar to the type of $\varphi^2\varphi_0$ theory considered here with exchange of a scalar photon and does not arise in electrodynamics where the photon has spin.[17]

ACKNOWLEDGMENTS

We would like to thank S. Brown, G. Domokos, and S. Mintz for helpful conversations.

APPENDIX A

We attempt here to find a solution of the Wick equation

$$W\varphi = I\varphi\,, \tag{A1}$$

where $W\varphi$ and $I\varphi$ are the left- and right-hand sides, respectively, of Eq. (2.1), by employing an integral representation of the form

$$\varphi(p) = \int_a^b dz\, g(z) R(p,z)^{-3}\,. \tag{A2}$$

In this equation a, b, and $g(z)$ are to be determined so that $\varphi(p)$ satisfies Eq. (A1) and

$$R(p,z) = (p^2 + 2izp\cdot\xi + 1 - \xi^2)\,. \tag{A3}$$

Under the action[5] of I,

$$I\varphi = (\alpha/2\pi)\int_a^b dz\, g(z) R(p,z)^{-1} Q^{-1}(z)\,, \tag{A4}$$

where

$$Q(z) = (1 - \xi^2 + z^2\xi^2)\,. \tag{A5}$$

Since

$$W = R^2 - 4izp\cdot\xi R - (1-z^2)(2ip\cdot\xi)^2\,,$$

in terms of Eq. (A2), Eq. (A1) assumes the form

$$\int_a^b dz\, g(z)[R^{-1} - 2z(2ip\cdot\xi)R^{-2} - (1-z^2)(2ip\cdot\xi)^2 R^{-3} - (\alpha/2\pi)Q^{-1}R^{-1}] = 0\,. \tag{A6}$$

Using the fact that

$$\frac{d}{dz}R^{-1} = -R^{-2}2i(p\cdot\xi)$$

and

$$\frac{d^2}{dz^2}R^{-1} = 2R^{-3}(2ip\cdot\xi)^2\,, \tag{A7}$$

Eq. (A6) becomes, after integrating by parts,

$$\int_{-1}^{1} dz\, R(p,z)^{-1}[(1-z^2)g''(z) + (\alpha/\pi)Q^{-1}(z)g(z)] = 0\,, \tag{A8}$$

provided a and b are chosen to be ± 1 and $g(\pm 1) = 0$. Since this equation must hold for arbitrary p, Eq. (2.4) follows immediately.

APPENDIX B

In this appendix we will obtain Eq. (3.19) by explicitly taking the infinite-momentum limit with the covariant Wick equation as a starting point. Our purpose in part is to derive Eq. (3.19) in the Weinberg[1] approach, but without reference to the T matrix, and in part to carry out a development parallel to that of Kim and Zaoui.[12] The results we obtain agree with those derived from quantization in the infinite-momentum frame as a starting point and with the Weinberg T-matrix approach. We have, however, not been able to find a point of contact with the approach of Kim and Zaoui.

Our starting point is Eq. (2.1), but in a form it has in Minkowski space prior to Wick rotation. The corresponding Green's function equation in relative momentum space can be written as

$$G^{(12)}(\mathcal{P},p,p') = G_0^{(12)}(\mathcal{P},p,p'')\lambda I(p'',p''') \times G^{(12)}(\mathcal{P},p''',p')\,, \tag{B1}$$

where $\mathcal{P}$, p, and p' are c-number four-vectors, integration is performed over variables with repeated indices, and

$$\mathcal{P}_\mu = (P_0, 0, 0, P), \quad \mathcal{P}^2 = \mu^2\,.$$

We further have

$$G_0^{(12)}(\mathcal{P},p,p') = \delta^4(p-p')G_0^{(1)}(p_1)G_0^{(2)}(p_2),$$
$$G_0^{(l)}(p_l) \equiv G_0^{(l)} = i(p_l{}^2 - m^2 + i\epsilon)^{-1}, \quad l = 1,2 \tag{B2a}$$
$$p_1 = p + \mathcal{P}/2, \quad p_2 = -p + \mathcal{P}/2.$$

The interaction term is given by

$$I(p,p') = \frac{i}{\pi^2}\,\frac{1}{(p-p')^2}\,. \tag{B2b}$$

We now define the quantities

$$\omega_1 = (\vec{p}_1{}^2 + m^2)^{1/2} = \{[\tfrac{1}{2}P(1+k)]^2 + \vec{q}^{\,2} + m^2\}^{1/2},$$
$$\omega_2 = (\vec{p}_2{}^2 + m^2)^{1/2} = \{[\tfrac{1}{2}P(1-k)]^2 + \vec{q}^{\,2} + m^2\}^{1/2}, \tag{B3}$$

where $|k| \leq 1$ and $\vec{q}$ is transverse to $\vec{P}$. We will eventually have $P \to \infty$. Using Eq. (B3) we set

$$p_{1+} = \tfrac{1}{2}P_0 - \omega_1 + i\epsilon\,,$$
$$p_{2+} = \tfrac{1}{2}P_0 - \omega_2 + i\epsilon\,,$$
$$p_{1-} = \tfrac{1}{2}P_0 + \omega_1 - i\epsilon\,, \tag{B4}$$
$$p_{2-} = \tfrac{1}{2}P_0 + \omega_2 - i\epsilon\,.$$

We can now take partial fractions and express G_0 in terms of what will later turn out to be positive- and negative-energy contributions:

$$G_0^{(1)} = G_+^{(1)} + G_-^{(1)} = \frac{-i}{2\omega_1}\left(\frac{-1}{p_0+p_{1+}} + \frac{1}{p_0+p_{1-}}\right),$$
$$G_0^{(2)} = G_+^{(2)} + G_-^{(2)} = \frac{-i}{2\omega_2}\left(\frac{1}{p_0-p_{2+}} - \frac{1}{p_0-p_{2-}}\right). \tag{B5}$$

As suggested by Kim and Zaoui, the behavior of $G_0^{(1)}G_0^{(2)}$ for $P\to\infty$ is crucial to the analysis. Taking this limit, we obtain

$$G_0^{(12)} = G_+^{(1)}G_+^{(2)} + G_+^{(1)}G_-^{(2)} + G_-^{(1)}G_+^{(2)} + G_-^{(1)}G_-^{(2)} \underset{P\to\infty}{\sim} G_+^{(1)}G_+^{(2)} . \tag{B6}$$

This is the analog of Weinberg's statement that pair terms disappear in T-matrix elements in the $P\to\infty$ limit. This can be shown explicitly by carrying out the same sort of analysis as given below for $G_+^{(1)}G_+^{(2)}$ for the remaining terms of Eq. (B6). Taking partial fractions again, we obtain

$$G_0^{(1)}G_0^{(2)} \underset{P\to\infty}{\sim} G_+^{(1)}G_+^{(2)} = \frac{-1}{(P_0-\omega_1-\omega_2)}\,\frac{1}{4\omega_1\omega_2}\left(\frac{1}{p_0+p_{1+}} - \frac{1}{p_0-p_{2+}}\right). \tag{B7}$$

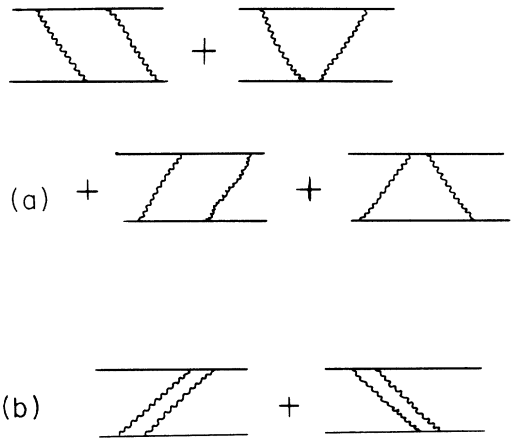

FIG. 3. Two-photon exchanges which are (a) retained and (b) neglected in the Green's-function expansion of Eq. (B10).

If we now substitute this expression into Eq. (B1) in its iterated form and integrate over p_0 and p_0' we obtain

$$\tilde{G}^{(12)}(\mathcal{P},\vec{p},\vec{p}') \equiv \int dp_0\,dp_0'\,G^{(12)} = \tilde{G}_0^{(12)}(\mathcal{P},\vec{p},\vec{p}') + \tilde{G}_0^{(12)}(\mathcal{P},\vec{p}'',\vec{p}'')\lambda\tilde{I}(\mathcal{P},\vec{p}'',\vec{p}''')\tilde{G}_0^{(12)}(\mathcal{P},\vec{p}''',\vec{p}') + \cdots , \tag{B8}$$

with

$$\tilde{G}_0^{(12)} = \frac{\delta^3(\vec{p}-\vec{p}')\pi i}{2\omega_1\omega_2(P_0-\omega_1-\omega_2)} ,$$
$$\tilde{I}(\mathcal{P},\vec{p},\vec{p}') = \frac{-i}{2\pi^2\tilde{\omega}}\left(\frac{1}{p_{2+}+p_{1+}'-\tilde{\omega}} + \frac{1}{p_{1+}+p_{2+}'-\tilde{\omega}}\right), \tag{B9}$$
$$\tilde{\omega} = \{[(k-k')\tfrac{1}{2}P]^2 + (\vec{q}-\vec{q}')^2\}^{1/2} .$$

The second term of Eq. (B8) can be represented diagrammatically by Fig. 1 of the text, but with somewhat different vertex rules and the replacement $\eta = \frac{1}{2}M(1+k) \to \frac{1}{2}P(1+k)$ and $\tau \to t$. The next iteration in the Green's function expansion can be represented by the diagrams of Figs. 3(a) and 3(b).

If we now neglect the contributions in Fig. 3(b) (i.e., in this order and all higher orders consider intermediate states with at most one scalar photon present), we can resum the Green's function expansion into the form

$$\tilde{G}^{(12)}(\mathcal{P},\vec{p},\vec{p}') \approx \tilde{G}_0^{(12)}(\mathcal{P},\vec{p},\vec{p}') + \tilde{G}_0^{(12)}(\mathcal{P},\vec{p},\vec{p}'')\lambda\tilde{I}(\mathcal{P},\vec{p}'',\vec{p}''')\tilde{G}^{(12)}(\mathcal{P},\vec{p}''',\vec{p}') . \tag{B10}$$

The neglect of interaction terms involving two-or-more-photon intermediate states corresponds precisely to the Tamm-Dancoff approximation made in the infinite-momentum-frame quantization approach.

The homogeneous equation which corresponds to Eq. (B10) is

$$\psi(\mathcal{P},\vec{p}) = \int \tilde{G}_0^{(12)}(\mathcal{P},\vec{p},\vec{p}')d\vec{p}'\lambda\tilde{I}(\mathcal{P},\vec{p}',\vec{p}'')d\vec{p}''\psi(\mathcal{P},\vec{p}''), \tag{B11}$$

where we have at this point explicitly inserted the integrations implied by the repeated variables.

We are now ready to carry out the $P\to\infty$ limit. Expanding as Weinberg does, we get the leading terms [$m^2=1$; μ, B, and $\bar{B}$ defined in Eqs. (3.13) and (3.20)]:

$$\omega_1 \approx \left[\tfrac{1}{2}P(1+k) + \frac{\vec{q}^{\,2}+1}{P(1+k)}\right],$$

$$\omega_2 \approx \left[\tfrac{1}{2}P(1-k)+\frac{\vec{q}^{\,2}+1}{P(1-k)}\right], \quad -1 \leq k \leq 1 \tag{B12}$$

$$\bar{\omega} = \tfrac{1}{2}P|k-k'|,$$

and

$$\frac{1}{p_{1+}+p'_{2+}-\bar{\omega}} \approx P\left[2(1-\bar{B})-\frac{\vec{q}^{\,2}+1}{1+k}-\frac{\vec{q}^{\,\prime 2}+1}{1-k'}-\frac{(\vec{q}-\vec{q}')^2}{k'-k}\right]^{-1}\theta(k'-k),$$

$$\frac{1}{p'_{1+}+p_{2+}-\bar{\omega}} \approx P\left[2(1-\bar{B})-\frac{\vec{q}^{\,2}+1}{1-k}-\frac{\vec{q}^{\,\prime 2}+1}{1+k'}-\frac{(\vec{q}-\vec{q}')^2}{k-k'}\right]^{-1}\theta(k-k'), \tag{B13}$$

$$\frac{1}{P_0-\omega_1-\omega_2} \approx -\tfrac{1}{2}P(1-k^2)[\bar{B}-\bar{B}k^2+\vec{q}^{\,2}+k^2]^{-1}.$$

If we substitute these results into Eq. (B11), noting that $d\vec{p} = d\vec{q}(\frac{1}{2}P)dk$, we obtain Eq. (3.19).

Had we carried out a derivation similar to that following Eq. (B6) but inserting $G_a^{(1)}G_b^{(2)}$, where either a or b or both is a minus sign (pair terms), the P dependence would not drop out and the contribution would vanish as $P \to \infty$.

APPENDIX C

In this appendix we calculate the first-order correction to the Balmer eigenvalues from Eq. (3.19). We begin with this equation as it is rewritten in Eq. (3.23):

$$\theta(1-k)\theta(1+k)[h_0+\bar{B}-k^2\bar{B}-(J-I)]\psi = 0. \tag{C1}$$

The θ functions are used here to indicate that Eq. (3.19) is originally defined for $-1 \leq k \leq 1$ only. In Eq. (C1),[13] h_0 is the nonrelativistic Coulomb Hamiltonian of Eq. (3.22), i.e.,

$$\begin{aligned} h_0\psi &= \vec{p}^{\,2}\psi(\vec{p}) - \frac{g^2}{4(2\pi)^3}\int \frac{d\vec{p}'\psi(\vec{p}')}{(\vec{p}-\vec{p}')^2} \\ &= \vec{p}^{\,2}\psi - I\psi, \end{aligned} \tag{C2}$$

and $J\psi$ is given by the right-hand side of Eq. (3.19):

$$J\psi = \frac{g^2}{4(2\pi)^3}\int d\vec{q}'dk'\,\theta(1-k')\theta(1+k')\bar{V}(\vec{q},k;\vec{q}',k')\psi(\vec{q}',k'), \tag{C3}$$

where

$$\bar{V}(\vec{q},k;\vec{q}',k') = |k-k'|^{-1}\left\{\theta(k-k')\left[\frac{\vec{q}^{\,2}+1}{1-k}+\frac{\vec{q}^{\,\prime 2}+1}{1+k'}+\frac{(\vec{q}-\vec{q}')^2}{k-k'}-2(1-\bar{B})\right]^{-1}\right.$$
$$\left.+\,\theta(k'-k)\left[\frac{\vec{q}^{\,\prime 2}+1}{1-k'}+\frac{\vec{q}^{\,2}+1}{1+k}+\frac{(\vec{q}-\vec{q}')^2}{k'-k}-2(1-\bar{B})\right]^{-1}\right\}.$$

The bound-state energies can be determined in perturbation theory if ψ and $\bar{B}$ are expanded in powers of the coupling constant:

$$\begin{aligned} \psi &= \psi^{(0)}+\psi^{(1)}+\cdots, \\ \bar{B} &= \bar{B}^{(0)}+\bar{B}^{(1)}+\cdots. \end{aligned} \tag{C4}$$

The lowest-order equation is taken to be

$$(h_0+\bar{B}^{(0)})\psi^{(0)} = 0, \tag{C5}$$

which has solutions $\psi^{(0)} = \psi^{(0)}_{nlm}$, where $\psi^{(0)}_{nlm}(\vec{p})$ are the bound-state solutions of the nonrelativistic hydrogen atom. Since the states $\psi^{(0)}_{nlm}$ are degenerate for fixed n, we can choose as our zero-order solutions the states

$$\psi^{(0)}_n = \sum_{lm} C^n_{lm}\psi^{(0)}_{nlm}, \tag{C6}$$

where the C^n_{lm} are constants to be determined by the perturbation theory. Since Eq. (C1) does not possess the O(4) nor the O(3) symmetry of Eq. (C2), we should treat this problem as one in degenerate perturbation

theory and expect that for the exact solutions of Eq. (C1) l and m will not be good quantum numbers. As we will see, however, to the order in which we are interested, l and m remain good quantum numbers and the degeneracy is not removed. Hence we can apply nondegenerate perturbation theory to this order.

With the substitution of the zero-order solutions of Eq. (C6), the first-order equation becomes

$$(h_0+\bar{B}_n^{(0)})\psi_n^{(1)}+\theta(1-k)\theta(1+k)[\bar{B}_n^{(1)}-k^2\bar{B}_n^{(0)}-(J-I)]\psi_n^{(0)}=0. \tag{C7}$$

In this equation $k^2\bar{B}_n\psi_n^{(0)}$ and $(J-I)\psi_n^{(0)}$ are assumed to be small perturbations on the zero-order solutions, so that we can replace $\bar{B}_n$ by $\bar{B}_n^{(0)}$ in both these expressions. We will see that this is a consistent approximation.

We now take the inner product of Eq. (C7) with $\psi_n^{(0)}$. We obtain

$$\bar{B}_n^{(1)}\int d\vec{\mathrm{p}}\,\theta(1-k)\theta(1+k)|\psi_n^{(0)}|^2=\bar{B}_n^{(0)}\int d\vec{\mathrm{p}}\,\theta(1-k)\theta(1+k)k^2|\psi_n^{(0)}|^2+\int d\vec{\mathrm{p}}\,\theta(1-k)\theta(1+k)\psi_n^{(0)*}(J-I)\psi_n^{(0)}. \tag{C8}$$

Since the zero-order solutions $\psi_{nlm}^{(0)}(\vec{\mathrm{p}})$ are sharply peaked for small values (of order α) of p,[16] Eq. (C8) can be evaluated to lowest order by extending the range of the k and k' integrations from $-\infty$ to ∞, i.e., dropping the θ functions. We will call J with these replacements $\bar{J}$. Since the $\psi_n^{(0)}$ are normalized to unity,

$$\bar{B}_n^{(1)}=\bar{B}_n^{(0)}\langle k^2\rangle_n+\langle\bar{J}-I\rangle_n. \tag{C9}$$

The first term of the right-hand side of Eq. (C9) makes an α^4 contribution to $\bar{B}^{(1)}$ since

$$\langle k^2\rangle_n=O(\alpha^2),\quad \bar{B}_n^{(0)}=\alpha^2/4n^2. \tag{C10}$$

In order to evaluate the second term, $\langle\bar{J}-I\rangle_n$, we again make use of the nature of the zero-order solutions. If we expand the factors of $(1+k)^{-1}$, $(1-k')^{-1}$, etc. in powers of k and k' to first order and neglect $\vec{\mathrm{q}}^{\,2}$ and $\vec{\mathrm{q}}^{\,\prime 2}$ with respect to 1 in the expression for J [Eq. (C3)], we see that $\langle\bar{J}-I\rangle_n$ is identically zero. This is the reason for taking the $\psi_{nlm}^{(0)}$ as zero-order solutions. As a next order approximation, we retain the factors of $\vec{\mathrm{q}}^{\,2}$ and $\vec{\mathrm{q}}^{\,\prime 2}$ and expand the denominators involving $1-k$, $1-k'$, etc. to second order in powers of k and k'. The resulting equation can then be written as

$$\langle\bar{J}-I\rangle_n\approx\frac{\alpha}{2\pi^2}\int d\vec{\mathrm{p}}\,d\vec{\mathrm{p}}'\psi_n^{(0)*}(\vec{\mathrm{p}})\psi_n^{(0)}(\vec{\mathrm{p}}')\{[(\vec{\mathrm{p}}-\vec{\mathrm{p}}')^2+|k-k'|(\vec{\mathrm{p}}^{\,2}+\vec{\mathrm{p}}^{\,\prime 2}+2\bar{B}_n^{(0)})]^{-1}-(\vec{\mathrm{p}}-\vec{\mathrm{p}}')^{-2}\}. \tag{C11}$$

As it stands Eq. (C11) is difficult to evaluate. As a lowest-order approximation, we replace $\vec{\mathrm{p}}^{\,2}$ and $\vec{\mathrm{p}}^{\,\prime 2}$ in the denominator of the effective-interaction term with $\langle\vec{\mathrm{p}}^{\,2}\rangle_n=\langle\vec{\mathrm{p}}^{\,\prime 2}\rangle_n=\alpha^2/4n^2$, respectively. To verify the validity of this approximation we can rewrite the term in curly brackets as

$$\frac{-|k-k'|(\vec{\mathrm{p}}^{\,2}+\vec{\mathrm{p}}^{\,\prime 2}+2\bar{B}_n^{(0)})}{(\vec{\mathrm{p}}-\vec{\mathrm{p}}')^2[(\vec{\mathrm{p}}-\vec{\mathrm{p}}')^2+|k-k'|(\vec{\mathrm{p}}^{\,2}+\vec{\mathrm{p}}^{\,\prime 2}+2\bar{B}_n^{(0)})]}.$$

If we make the above replacement in the denominator but retain the exact form in the numerator, then Eq. (C11) can also be evaluated using the techniques outlined below. To lowest order the result is the same as for the equation

$$\langle\bar{J}-I\rangle_n\approx\frac{\alpha}{2\pi^2}\int d\vec{\mathrm{p}}\,d\vec{\mathrm{p}}'\psi_n^{(0)*}(\vec{\mathrm{p}})\psi_n^{(0)}(\vec{\mathrm{p}}')\{[(\vec{\mathrm{p}}-\vec{\mathrm{p}}')^2+|k-k'|(\alpha^2/n^2)]^{-1}-(\vec{\mathrm{p}}-\vec{\mathrm{p}}')^{-2}\}. \tag{C12}$$

Since the effective-interaction term of Eq. (C12) is a function of $\vec{\mathrm{p}}-\vec{\mathrm{p}}'$, the equation can be conveniently written in coordinate space as

$$\langle\bar{J}-I\rangle_n\approx\alpha\int d\vec{\mathrm{r}}|\Psi_n^{(0)}(\vec{\mathrm{r}})|^2\Delta V(\vec{\mathrm{r}}), \tag{C13}$$

where $\Psi_n^{(0)}(\vec{\mathrm{r}})$ is the Fourier transform of $\psi_n^{(0)}(\vec{\mathrm{p}})$ and

$$\Delta V(\vec{\mathrm{r}})=\frac{1}{2\pi^2}\int\frac{d\vec{\mathrm{p}}\,e^{-i\vec{\mathrm{p}}\cdot\vec{\mathrm{r}}}}{\vec{\mathrm{p}}^{\,2}+|k|(\alpha^2/\pi^2)}-\frac{1}{r}. \tag{C14}$$

If we define $\chi^2=k^2+|k|(\alpha^2/n^2)$, then the $d\vec{\mathrm{q}}$ integration can be carried out[18] giving

$$\Delta V(\vec{\mathrm{r}})=\frac{2}{\pi}\int_0^\infty\cos(kz)K_0(\chi R)dk-\frac{1}{r}, \tag{C15}$$

where $R=(x^2+y^2)^{1/2}$ and K_0 is the modified Bessel function of the second kind. Making use of the identity[19]

$$\frac{1}{r} = \frac{2}{\pi}\int_0^\infty \cos(kz)K_0(kR)dk\,, \tag{C16}$$

we write

$$\Delta V(\vec{r}) = \frac{2}{\pi}\int_0^\infty \cos(kz)[K_0(\chi R) - K_0(kR)]dk\,. \tag{C17}$$

The major contribution to this integral occurs for small k. As far as the evaluation of the integral in Eq. (C13) is concerned, the replacement of $K_0(\chi R)$ and $K_0(kR)$ by their leading terms, $-\ln(\chi R)$ and $-\ln(kR)$, respectively, is a sufficient approximation. The terms that are neglected here can be shown to contribute to a higher order only. Therefore [20]

$$\begin{aligned}\langle \bar{J} - I\rangle_n &\approx -\frac{\alpha}{\pi}\int d\vec{r}\,|\Psi_n^{(0)}(\vec{r})|^2\int_0^\infty dk\cos(kz)\ln\frac{(k+\alpha^2/n^2)}{k}\\ &= \frac{\alpha}{\pi}\int d\vec{r}\,|\Psi_n^{(0)}(\vec{r})|^2\frac{1}{z}\int_0^\infty dk\sin(kz)[(k+\alpha^2/n^2)^{-1} - k^{-1}]\\ &= \frac{\alpha}{\pi}\int d\vec{r}\,|\Psi_n^{(0)}(\vec{r})|^2\frac{1}{|z|}\left\{\sin(|z|\alpha^2)\,\mathrm{Ci}(|z|\alpha^2) + \tfrac{1}{2}\pi[\cos(|z|\alpha^2) - 1] - \cos(|z|\alpha^2)\,\mathrm{Si}(|z|\alpha^2)\right\},\end{aligned} \tag{C18}$$

where the second line follows after an integration by parts. Making use of the series expansions for the Ci and Si functions as well as the sine and cosine, we evaluate this last integral. To lowest order the result is

$$\langle \bar{J} - I\rangle_n = \frac{\alpha^3}{\pi n^2}\ln\alpha + O(\alpha^3), \tag{C19}$$

and thus from Eq. (C9)

$$\bar{B}_n^{(1)} = \frac{\alpha^3}{\pi n^2}\ln\alpha + O(\alpha^3) \tag{C20}$$

Combining this result with Eqs. (3.20) and (C4) and the fact that $\bar{B}_n^{(0)} = \alpha^2/4n^2$, we have the final result:

$$B_n = \frac{\alpha^2}{4n^2} + \frac{\alpha^3}{\pi n^2}\ln\alpha + O(\alpha^3)\,. \tag{C21}$$

As was pointed out previously, to this approximation ($\alpha^3\ln\alpha$) O(4) and O(3) symmetry has been restored, i.e., the effective interaction $\Delta V(\vec{r})$ is independent of z. If we were to evaluate Eq. (C18) beyond lowest order, then the factors of z would appear in the integration and the $\psi_{nlm}^{(0)}$ would be mixed by the perturbation. The covariant Wick equation maintains O(4) and O(3) symmetry and presumably if we were to include the two-scalar-photon states of Fig. 2, this symmetry would again be restored to order α^3.

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[7] The ξ appearing here is the same as Wick's η. Wick's coupling constant λ has been set equal to (α/π) so that to lowest order in α the eigenvalues of Eq. (2.1) reduce to the nonrelativistic Balmer result.

[8] We are using here the notation of Ref. 3.

[9] From here on, unless otherwise stated, symbols with arrows will refer to transverse vectors.

[10] I. Tamm, J. Phys. U.S.S.R. 9, 449 (1945); S. M. Dancoff, Phys. Rev. 78, 382 (1950); M. M. Lévy, *ibid.* 88, 72 (1952); 88, 725 (1952); A. Klein, *ibid.* 90, 1101 (1953).

[11] Weinberg's T-matrix equation can be written symbolically in the form $T = V + VG_0T$. If one drops the inhomogeneous term and sets $G_0T = \psi$, one obtains our Eq. (3.17).

[12] Y. S. Kim and R. Zaoui, Phys. Rev. D 4, 1764 (1971).

[13] The symbol $\tilde{p}$ will refer to the three-dimensional vector $\tilde{p} = (\tilde{q}, k) = (q^1, q^2, k)$. Thus $(\tilde{p} - \tilde{p}')^2 = (\tilde{q} - \tilde{q}')^2 + (k - k')^2$.

[14] B. Podolsky and L. Pauling, Phys. Rev. 34, 109 (1929).

[15] The reason that any linear combination of the degenerate zero-order solutions is adequate to calculate the first-order corrections to the binding energy is explained in Appendix C following Eq. (C6).

[16] The ground-state solution in momentum space, for example, is given by $\alpha^{5/2}(2\pi)^{-1}(\tilde{p}^2 + \alpha^2/4)^{-2}$. See Ref. 14.

[17] R. Karplus and A. Klein, Phys. Rev. 87, 848 (1952).

[18] W. Gröbner and N. Hofreiter, *Integraltafel* (Springer, New York, 1966), Vol. 2, p. 133, Eq. (51a) and p. 197, Eq. (6a).

[19] Reference 18, Vol. 2, p. 201, Eq. (8a).

[20] Reference 18, Vol. 2, p. 127, Eq. (64a); Ref. 18, Vol. 1, p. 129, Eq. (5).