which follows from the time-reversal invariance of the strong interactions [see Martin and Spearman (Ref. 13)].

[16]G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960).

[17]The amplitude $A_{\ddagger}^{0}(t,s)$ may be obtained from the s-channel amplitudes given in Graham and Johnson (Ref. 11) by means of the s-t crossing matrix [see Gasiorowicz (Ref. 12, p. 263)].

[18]In writing down Eq. (3.12) we have anticipated applications in a later section where the value of t at which the appropriate amplitudes are evaluated is of $O(\mu^2)$. This would make the terms in C and D of $O(\mu^4)$, and so they have been dropped.

[19]Our amplitudes and notation in this section are those of R. G. Moorhouse, Ann. Rev. Nucl. Sci. 19, 301 (1969).

[20]This is the combination studied by Cheng and Dashen (Ref. 1). Their result is that $F(0, 2\mu^2)=F_\pi^{-2}\sigma_{NN}+O(\mu^4)$ with F_π the pion decay constant (=93 MeV) and the limit $t\to 2\mu^2$ taken before $\nu\to 0$.

[21]Actually, Altarelli *et al.* (Refs. 3 and 10) use an expansion which allows the pions to be off mass shell. Our coefficients, while not the same as theirs, are related to them.

[22]In arriving at Eq. (4.7) it is important to note that the coefficient of $R_{\frac{1}{2},\frac{1}{2};00}(\nu,t)$ depends only on t.

[23]H. Pagels and W. J. Pardee, Phys. Rev. D 4, 3335 (1971).

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Three-Particle Scattering with Three-Particle Interactions*

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The scattering formalism of Alt, Grassberger, and Sandhas is extended to include a possible three-particle interaction. This is then employed to find a set of scattering integral equations in which no unphysical auxiliary amplitudes related to the three-body force appear, as well as to develop a practical method for treating perturbatively the effect of a weak three-body force. The general K-matrix formalism and connected-kernel Heitler equations are also developed. This yields some indication of the structure that might be expected in a relativistic connected-kernel K-matrix formalism.

I. INTRODUCTION

Virtually all of the existing calculations of both the bound and the continuum states of the three-nucleon system (or any other three-particle system, such as that consisting of three pions) include only two-particle interactions. The role of a three-nucleon force is unclear at present,[1] and its possible nature is obscured by the uncertainties in the off-shell behavior of the two-nucleon transition amplitude. Given the validity of a nonrelativistic dynamics for the three-nucleon system, it is obvious that the questions of off-shell behavior and of the magnitude and character of the three-nucleon force are intimately correlated, assuming that the bound-state and on-shell scattering parameters of both the two- and the three-nucleon systems have been accounted for. These questions are also rather ill-defined if one adopts the stance of phenomenological potential scattering, and should they prove to be quantitatively significant a somewhat more fundamental approach to the entire problem may be in order.

Nonetheless, there are several reasons for examining how a three-particle force alters the scattering integral equations and the computational procedures derived from them. One of these is the opportunity to examine in a dynamically well-defined framework a situation somewhat related to the relativistic three-particle problem. Another is to use these equations to formulate modifications of some of the standard methods for calculating three-particle amplitudes so that three-particle forces can be introduced and their effects studied.

The rather straightforward modification of the three-particle scattering integral equations which is entailed when three-body forces are included in addition to the usual pair interactions was first pointed out by Newton.[2] Calculations using a separable three-nucleon force to simulate some of the noncentral and short-range features of the two-nucleon interaction were carried out by Phillips.[3] Both of the preceding authors employed, essentially, the Lovelace[4] version of the scattering integral equations.

Our objectives in this paper are twofold. First, we wish to embed the ideas of Newton and Phillips within the somewhat more practical form of the scattering integral equations due to Alt, Grassberger, and Sandhas.[5] Although this turns out to be a rather simple extension of the latter formalism, it will permit us to deduce some results which should be useful in carrying out practical calculations when a three-body force is included in addition to the pair interactions. We will derive a set of quasi-two-body Lippmann-Schwinger equations for the amplitudes related to scattering from an initial state of a free particle plus a bound pair (e.g., an N-d state). We also present a practical and perhaps quantitatively adequate method for studying the effects of a weak three-body interaction perturbatively. This may completely suffice for the case of the three-nucleon system.

Our second objective is to construct a K-matrix formalism in the case of a general interaction along the lines previously followed for only pair interactions.[6] This yields the sort of structure we might expect in a relativistic connected-kernel K-matrix formalism.

II. SCATTERING INTEGRAL EQUATIONS

We will now extend the formalism of Alt *et al.*[5] to the case where the interaction consists of a three-particle force in addition to the pair interactions. The total interaction potential in this case is

$$V=\sum_\alpha V_\alpha, \tag{2.1}$$

where the channel index α takes on the values 0, 1, 2, 3, 4. For $\alpha \neq 0, 4$, V_α is the pair interaction between particles β and γ (with $\beta, \gamma \neq \alpha, 0, 4$), $V_0 = 0$, and V_4 is the three-particle potential which depends upon the coordinates of particles 1, 2, 3.

V_4 is assumed to be Hermitian and short-ranged in addition to any necessary limitations on its singular character. Roughly speaking, V_4 is short-ranged if its effects are suitably small when the distance between *any* pair of particles is larger than some characteristic length. In addition we will suppose that V_4 by itself is incapable of generating a three-particle bound state.

The operator

$$U_{00}(z)=V+VG(z)V \tag{2.2}$$

can be rewritten as

$$U_{00}(z)=\sum_\alpha t_\alpha(z)+\sum_{\beta\alpha} t_\beta(z)G_0(z)U_{\beta\alpha}(z)G_0(z)t_\alpha(z), \tag{2.3}$$

where

$$U_{\beta\alpha}(z)\equiv\bar{\delta}_{\beta\alpha}G_0(z)^{-1}+V-V_\alpha-V_\beta+\delta_{\beta\alpha}V_\beta + V^\beta G(z)V^\alpha . \tag{2.4}$$

Here

$$V^\alpha \equiv V-V_\alpha=\sum_\gamma \bar{\delta}_{\alpha\gamma}V_\gamma,$$

$$\bar{\delta}_{\beta\alpha}\equiv 1-\delta_{\beta\alpha},$$

$$G_0(z)\equiv(z-H_0)^{-1},$$

$$G(z)\equiv(z-H_0-V)^{-1},$$

where H_0 is the total kinetic-energy operator and z is a (complex) parametric energy. The operators $t_\alpha(z)$ are defined as solutions of the integral equations

$$\begin{aligned} t_\alpha(z) &= V_\alpha+V_\alpha G_0(z)t_\alpha(z) \\ &= V_\alpha + t_\alpha(z)G_0(z)V_\alpha . \end{aligned} \tag{2.5}$$

For $\alpha\neq 0, 4$, t_α is just the two-particle transition operator defined on the three-particle Hilbert space. Evidently $t_0=0$. Finally, t_4 is the three-particle transition operator generated by the three-particle force V_4. We note that V_4, and consequently t_4, has no disconnected structure, so that Eqs. (2.5) are all well-defined integral equations.[2]

It is easily verified from Eqs. (2.3) and the two-particle bound-state pole residue prescription,[7-10] or directly from Eqs. (2.4), that the on-shell matrix elements $\langle\phi_\beta(E)|U_{\beta\alpha}(E+i0)|\phi_\alpha(E)\rangle$ for $\alpha,\beta\neq 4$ coincide with the physical amplitudes for scattering from the state $|\phi_\alpha(E)\rangle$ to the state $|\phi_\beta(E)\rangle$. The channel states $|\phi_\alpha(E)\rangle$ for $\alpha\neq 0,4$ refer to non-interacting two-particle states comprised of a particle α moving freely and a bound state of the other pair; $|\phi_0\rangle$ corresponds to a three-particle plane-wave state. $U_{4\alpha}$ and $U_{\beta 4}$ for all α,β are auxiliary operators whose matrix elements have no immediate physical significance.

Using the definitions (2.4) and (2.5) one finds that the operators $U_{\beta\alpha}(z)$ satisfy

$$\begin{aligned} U(z) &= \bar{\delta}G_0(z)^{-1}+\bar{\delta}t(z)G_0(z)U(z) \\ &= \bar{\delta}G_0(z)^{-1}+U(z)G_0(z)t(z)\bar{\delta}, \end{aligned} \tag{2.6}$$

where we have employed the usual matrix notation[11] with respect to the channel indices. That is, $U(z)$ represents the 5×5 matrix whose elements are the operators $U_{\beta\alpha}(z)$, $t(z)$ is a diagonal matrix whose elements are the operators $t_\alpha(z)$, and $\bar{\delta}$ is the matrix with elements $1-\delta_{\beta\alpha}$.

With the use of Eqs. (2.6) other forms of the scattering integral equations which include the three-body interaction can be found. For example, the generalization of the so-called optimal equations[7,10] is very easily derived. These equations still retain their original character in this case. Namely, they are coupled equations for the half-off-shell (or on-shell, depending upon the form chosen) elastic and rearrangement amplitudes and a set of half-off-shell amplitudes whose sum is the breakup amplitude.

Now if we decompose $t(z)$ into two parts, namely,

$$t(z) = t^{(1)}(z) + t^{(2)}(z)\,, \tag{2.7}$$

then[5]

$$\begin{aligned} U(z) &= \bar{U}(z) + \bar{U}(z)G_0(z)t^{(1)}(z)G_0(z)U(z) \\ &= \bar{U}(z) + U(z)G_0(z)t^{(1)}(z)G_0(z)\bar{U}(z)\,, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} \bar{U}(z) &= \bar{\delta}G_0(z)^{-1} + \bar{\delta}t^{(2)}(z)G_0(z)\bar{U}(z) \\ &= \bar{\delta}G_0(z)^{-1} + \bar{U}(z)G_0(z)t^{(2)}(z)\bar{\delta}\,. \end{aligned} \tag{2.9}$$

Let us choose

$$t_\alpha^{(1)}(z) = t_\alpha(z)\bar{\delta}_{\alpha 4} \tag{2.10a}$$

so that

$$t_\alpha^{(2)}(z) = t_4(z)\delta_{\alpha 4}\,. \tag{2.10b}$$

With this choice for $t^{(2)}$ Eqs. (2.9) can be solved exactly:

$$\bar{U}_{\beta\alpha}(z) = \bar{\delta}_{\beta\alpha}G_0(z)^{-1} + \bar{\delta}_{\beta 4}t_4(z)\bar{\delta}_{4\alpha}\,. \tag{2.11}$$

Equations (2.8) then become, in component form,

$$\begin{aligned} U_{\beta\alpha}(z) &= \bar{U}_{\beta\alpha}(z) + \sum_{\gamma=1}^{3}\bar{U}_{\beta\gamma}(z)G_0(z)t_\gamma(z)G_0(z)U_{\gamma\alpha}(z) \\ &= \bar{U}_{\beta\alpha}(z) + \sum_{\gamma=1}^{3}U_{\beta\gamma}(z)G_0(z)t_\gamma(z)G_0(z)\bar{U}_{\gamma\alpha}(z)\,. \end{aligned} \tag{2.12}$$

These define a closed set of equations for $U_{\beta\alpha}(z)$ with β, $\alpha \neq 4$. That is, there is now no coupling to the unphysical auxiliary operators $U_{\beta 4}$ and $U_{4\alpha}$.

If we define

$$F(z) = G_0(z)U(z)G_0(z)\,,$$

it is clear from Eqs. (2.11) and (2.12) that, for example,

$$\begin{aligned} F_{\beta\alpha}(z) &= G_0(z)[\bar{\delta}_{\beta\alpha} + t_4(z)G_0(z)] \\ &\quad + \sum_{\gamma=1}^{3}G_0(z)[\bar{\delta}_{\beta\gamma} + t_4(z)G_0(z)]t_\gamma(z)F_{\gamma\alpha}(z)\,, \end{aligned} \tag{2.13}$$

where we can restrict ourselves to β, $\alpha \neq 4$. Furthermore, only the subset of (2.13) for β, $\alpha \neq 0$ which is uncoupled from the $\beta = 0$ or $\alpha = 0$ equations need be considered for the calculation of physical scattering amplitudes. If on the appropriate two-particle subspaces the operators $t_\gamma(z)$, $\gamma = 1, 2, 3$, are of finite rank, then we can derive from (2.13) a set of multichannel quasi-two-particle equations for the so-called vertex-state matrix elements[12] of the $F_{\beta\alpha}(z)$ just as if there were no three-particle force.[5,12] Equations (2.13) are a generalization of the equations derived by Phillips.[3]

To solve Eqs. (2.13) even in the case of finite-rank pair interactions requires the knowledge of the off-shell matrix elements of $t_4(z)$. Therefore, except in some especially simple situations,[3] the use of (2.13) for the exact inclusion of the three-body interaction may be computationally prohibative. However, given the usual physical circumstance in which the three-body interaction is expected to be weak compared with the pair interactions, such an exact accounting of its effects is unnecessary and a perturbative approach will probably suffice. In the three-body bound-state problem the simplest recourse is to use first-order perturbation theory to compute the energy shift arising from V_4.

We can construct a scattering perturbation theory by interchanging the roles of $t^{(1)}(z)$ and $t^{(2)}(z)$ in Eqs. (2.8) and (2.9).[12] We then have, for example,

$$U(z) = U^{(0)}(z) + U^{(0)}(z)G_0(z)t^{(2)}(z)G_0(z)U(z)\,, \tag{2.14}$$

where

$$U^{(0)}(z) = \bar{\delta}G_0(z)^{-1} + \bar{\delta}t^{(1)}(z)G_0(z)U^{(0)}(z)\,. \tag{2.15}$$

The perturbation theory is now defined by the iteration solution of Eq. (2.14).[12] Thus, to first order in t_4 we have

$$U(z) = U^{(0)}(z) + U^{(0)}(z)G_0(z)t^{(2)}(z)G_0(z)U^{(0)}(z)\,. \tag{2.16}$$

In component form (2.16) becomes

$$\begin{aligned} U_{\beta\alpha}(z) &= U_{\beta\alpha}^{(0)}(z) + [\delta_{\beta 0} + U_{\beta 0}^{(0)}(z)G_0(z)] \\ &\quad \times t_4(z)[\delta_{0\alpha} + G_0(z)U_{0\alpha}^{(0)}(z)]\,, \end{aligned} \tag{2.17}$$

where we have used the fact that

$$U_{4\alpha}^{(0)} = U_{0\alpha}^{(0)} + \delta_{0\alpha}G_0(z)^{-1}$$

and

$$U_{\beta 4}^{(0)} = U_{\beta 0}^{(0)} + \delta_{\beta 0}G_0(z)^{-1}\,.$$

Clearly, we need consider only the submatrix of (2.17) for which β, $\alpha \neq 4$.

The use of (2.17) still entails solving the three-body integral equation for $t_4(z)$. However, since the integral equation for t_4 has a connected kernel, we expect that if V_4 is sufficiently weak we can take

$$t_4(z) \simeq V_4 \tag{2.18}$$

so that from (2.17) we find the two-potential type of expression

$$\begin{aligned} U_{\beta\alpha}(z) &\simeq U_{\beta\alpha}^{(0)}(z) + [\delta_{\beta 0} + U_{\beta 0}^{(0)}(z)G_0(z)] \\ &\quad \times V_4[\delta_{0\alpha} + G_0(z)U_{0\alpha}^{(0)}(z)]\,. \end{aligned} \tag{2.19}$$

Equation (2.19) represents the simplest expression for studying the correction to the scattering generated by arbitrary pair interactions by a comparatively weak three-body interaction. We note, however, that we have two criteria for weakness, namely that V_4 should be weak (in some sense) compared to the two-body forces and that it should be weak enough that (2.18) is approximately valid.

Let us comment upon the computational practicality of (2.19). We see from (2.19) that

$$F_{\beta\alpha}(z) \simeq F^{(0)}_{\beta\alpha}(z) + F^{(1)}_{\beta\alpha}(z)\,, \tag{2.20}$$

where

$$F^{(0)}_{\beta\alpha}(z) = G_0(z)U^{(0)}_{\beta\alpha}(z)G_0(z) \tag{2.21}$$

and

$$F^{(1)}_{\beta\alpha}(z) = \left(1 + \sum_{\gamma=1}^{3} F^{(0)}_{\beta\gamma}(z)t_\gamma(z)\right)G_0(z) \times V_4 G_0(z)\left(1 + \sum_{\lambda=1}^{3} t_\lambda(z)F^{(0)}_{\lambda\alpha}(z)\right). \tag{2.22}$$

We confine ourselves only to the submatrix of Eqs. (2.20)–(2.22) for which $\beta, \alpha = 1, 2, 3,$ and consider specifically the case of elastic N-d scattering. Suppose for simplicity that each of the pair interactions is a finite sum of separable terms. We can then regard the (on-shell and half-off-shell) vertex-state matrix elements[12] of $F^{(0)}_{\beta\alpha}(z)$ as known as a consequence of solving the quasi-two-particle integral equations for these quantities. To obtain the elastic scattering amplitudes with the three-particle force present one requires the appropriate vertex-state matrix elements of $F_{\beta\alpha}(z)$ with respect to the *same* vertex states as in the case with only two-body forces. Therefore the evaluation of these matrix elements using Eqs. (2.20)–(2.22) evidently requires no additional information other than that obtained in the solution of the problem represented by $F^{(0)}_{\beta\alpha}(z)$. Also, this evaluation will present no more difficulty than similar calculations which have already been carried out using various components of the nucleon-nucleon interaction as perturbations.[13–15]

We also remark that a somewhat more complicated perturbation theory is defined by using the approximation (2.18) directly in the kernels and inhomogeneous terms of Eqs. (2.13). If one wants to examine the effect of V_4 this procedure entails solving Eqs. (2.13) twice, namely with and without V_4.

Recently Gregorio and Avalos[16] have considered some aspects of multibody forces in the scattering problem. In Ref. 16 V_4 is employed explicitly in the scattering equations rather than introducing $t_4(z)$ as we have done. We conclude this section by using our formalism to derive such equations in more immediately applicable forms than those in Ref. 16. Despite the apparent relative simplicity of these equations it appears that they offer fewer possible computational advantages than our previous equations, in addition to being considerably more cumbersome because of their asymmetrical structure for analyses such as that carried out in Sec. III.

If we note that

$$t_4(z)G_0(z)U_{4\alpha}(z) = V_4\{1 + G_0(z)[1 + t_\alpha(z)G_0(z)]U_{\alpha\alpha}(z)\}\,, \tag{2.23}$$

then Eqs. (2.6) can again be reduced to a closed set of equations with the index-four operators eliminated:

$$U_{\beta\alpha}(z) = \bar{\delta}_{\beta\alpha}G_0(z)^{-1} + V_4 + \sum_{\gamma=1}^{3}\bar{\delta}_{\beta\gamma}t_\gamma(z)G_0(z)U_{\gamma\alpha}(z) + V_4G_0(z)[1 + t_\alpha(z)G_0(z)]U_{\alpha\alpha}(z)\,, \tag{2.24}$$

where $\beta, \alpha \neq 4$. In place of Eqs. (2.13) we then obtain

$$F_{\beta\alpha}(z) = \bar{\delta}_{\beta\alpha}G_0(z) + G_0(z)V_4G_0(z) + G_0(z)\sum_{\gamma=1}^{3}\bar{\delta}_{\beta\gamma}t_\gamma(z)F_{\gamma\alpha}(z) + G_0(z)V_4[1 + G_0(z)t_\alpha(z)]F_{\alpha\alpha}(z)\,. \tag{2.25}$$

Equations (2.24) and (2.25) have the apparent advantage that one does not have to solve the subsidiary three-body problem for $t_4(z)$. However, in contrast to Eqs. (2.12) and (2.13), one cannot obtain from them a set of equations involving only the vertex-state matrix elements of $F_{\beta\alpha}(z)$ if the $t_\alpha(z)$ for $\alpha \neq 4$ happen to be of finite rank. On the other hand, because of the connected and localized structure of V_4 the calculation of $t_4(z)$ may not be all that difficult, and if this is the case Eqs. (2.13) offer some computational advantage, particularly in the interesting circumstance of finite-rank pair interactions.

A V_4-explicit form of perturbation theory can also be derived from Eqs. (2.14), which in component form become

$$U_{\beta\alpha}(z) = U^{(0)}_{\beta\alpha}(z) + U^{(0)}_{\beta4}(z)G_0(z)t_4(z)G_0(z)U_{4\alpha}(z)\,. \tag{2.26}$$

Then using (2.23) we obtain ($\beta, \alpha \neq 4$)

$$U_{\beta\alpha}(z) = U^{(0)}_{\beta\alpha}(z) + [\delta_{\beta0} + U^{(0)}_{\beta\alpha}(z)G_0(z)]V_4 \times\{1 + G_0(z)[1 + t_\alpha(z)G_0(z)]U_{\alpha\alpha}(z)\}\,. \tag{2.27a}$$

Gregorio and Avalos[16] derived Eq. (2.27a) in the case $\beta = \alpha = 0$. Equation (2.27a) can be rewritten in the somewhat more symmetrical form

$$U_{\beta\alpha}(z)=U^{(0)}_{\beta\alpha}(z)+\{1+U^{(0)}_{\beta\beta}(z)[G_0(z)t_\beta(z)+1]G_0(z)\}V_4\times\{1+G_0(z)[1+t_\alpha(z)G_0(z)]U_{\alpha\alpha}(z)\}. \quad (2.27b)$$

We note by way of comparison that (2.26) becomes, using Eqs. (2.6) to define the index-four operators,

$$U_{\beta\alpha}(z)=U^{(0)}_{\beta\alpha}(z)+\left[G_0(z)^{-1}+\sum_{\lambda=1}^{3}U^{(0)}_{\beta\lambda}(z)G_0(z)t_\lambda(z)\right]\times G_0(z)t_4(z)G_0(z)\left[G_0(z)^{-1}+\sum_{\gamma=1}^{3}t_\gamma(z)G_0(z)U_{\gamma\alpha}(z)\right], \quad (2.28)$$

where we suppose that $\beta,\ \alpha\neq 4$.

Clearly, the V_4-explicit equations (2.27) can be used to define a perturbation theory. In fact, the lowest-order correction term generated by (2.27) can be shown to be identical to that contained in Eqs. (2.19)–(2.22), as one might expect. However, referring back to the convenience of equations which involve only vertex-state matrix elements of $F_{\beta\alpha}$ or of $F^{(0)}_{\beta\alpha}$, we see that Eq. (2.22), for example, is definitely preferable to the form of $F^{(1)}_{\beta\alpha}(z)$ generated directly by Eq. (2.27b).

Our entire development of including the three-body force in the three-particle equations is committed at the outset to a generalization of the Alt *et al.*[5] off-shell extension of the scattering operators. This extension achieves its preferred status by virtue of Eq. (2.3) as well as the fact that

$$G(z)=G_\beta(z)\delta_{\beta\alpha}+G_\beta(z)U_{\beta\alpha}(z)G_\alpha(z), \quad (2.29)$$

for $\alpha,\beta=0,1,2,3,4$, where

$$G_\alpha=(z-H_0-V_\alpha)^{-1}.$$

The identities (2.29) in particular would seem to suggest that the $U_{\beta\alpha}(z)$ form the most natural off-shell extension when one is considering a three-particle subsystem of a multiparticle problem. Other off-shell extensions of the multichannel operators (cf. Ref. 4), either with or without a three-body force, have relatively few attributes to distinguish them in any way, nor do the integral equations which they satisfy possess any special features which would lead one to prefer them over Eqs. (2.6).

III. UNITARITY AND THE K MATRIX

In this section we generalize the work of Refs. 6 and 11. There is little in the way of a surprise here except for the structure of the connected-kernel Heitler equations.[6] The latter, we find, differ slightly in form from the special case without a three-body force.

We expect that the discontinuity relations (off-shell unitarity) for $U(z)$ will be the same as in the case without the three-particle interactions, at least for those components which are related to physical scattering processes. This turns out to be the case, and in point of fact we find from Eqs. (2.6) using standard methods[11] that

$$(\Delta U)_{\beta\alpha}=-2i\sum_{\gamma=0}^{3}U_{\beta\gamma}(\pm)D_\gamma U_{\gamma\alpha}(\mp)+2i\{U_{\beta 0}(\pm)D_0G_0(\mp)^{-1}\bar\delta_{0\alpha}+\bar\delta_{\beta 0}G_0(\pm)^{-1}D_0U_{0\alpha}(\mp)+(\bar\delta_{\beta\alpha}-\bar\delta_{\beta 0}\bar\delta_{0\alpha})G_0(\pm)^{-1}D_0G_0(\mp)^{-1}\}. \quad (3.1)$$

We note that the submatrix of (3.1) with $\beta,\ \alpha\neq 4$ is entirely decoupled from the unphysical components $U_{4\lambda}$ and $U_{\lambda 4}$. In (3.1) we have introduced[6,11]

$$\Delta U\equiv U(+)-U(-)=U(+)-U(+)^\dagger,$$

$$U(\pm)\equiv[U(z)]_{z=E\pm i0},$$

$$D_\gamma\equiv\pi\sum_{E'_\gamma,\eta_\gamma}|\phi_\gamma(\eta_\gamma,E'_\gamma)\rangle\delta(E-E')\langle\phi_\gamma(\eta_\gamma,E'_\gamma)|,$$

and the η_γ are any other labels which are needed to specify the asymptotic configurations. As pointed out previously,[11] all terms within the curly brackets on the right-hand side of (3.1) vanish when the appropriate on-shell matrix elements of this equation are taken. Henceforth, we will be concerned with only this on-shell version of (3.1),

$$\Delta U_{\beta\alpha}=-2i\sum_{\gamma=0}^{3}U_{\beta\gamma}(\pm)D_\gamma U_{\gamma\alpha}(\mp), \quad (3.2)$$

and only when $\beta,\ \alpha\neq 4$. Equations (3.2) imply the usual statement of on-shell unitarity of the scattering operator.

Next we consider the so-called reduced K-matrix formalism.[6] As in Ref. 6, we decompose $t(z)$ into the sum

$$t(z)=\bar t(z)+t_b(z), \quad (3.3)$$

where $[t_b(z)]_\alpha$, $\alpha=1,2,3$, is that part of the two-body bound-state pole contained in $t_\alpha(z)$ which gives rise to a Dirac δ function when $z=E\pm i0$. For $\alpha=0$ or 4, $[t_b(z)]_\alpha=0$. We then see from Eqs.

(2.7)–(2.9) that

$$U_{\beta\alpha}(\pm)=\bar{U}_{\beta\alpha}(\pm)\mp i\sum_{\gamma=1}^{3}\bar{U}_{\beta\gamma}(\pm)D_\gamma U_{\gamma\alpha}(\pm)$$
$$=\bar{U}_{\beta\alpha}(\pm)\mp i\sum_{\gamma=1}^{3}U_{\beta\gamma}(\pm)D_\gamma\bar{U}_{\gamma\alpha}(\pm)\,, \qquad (3.4)$$

where $\bar{U}(z)$ satisfies Eqs. (2.9) with $t^{(2)}=\bar{t}$. Again we need consider Eqs. (3.4) for only $\beta,\alpha\neq 4$, and there is no coupling with the index-four operators. $\bar{U}_{\beta\alpha}(+)$ with $\beta,\alpha\neq 4$ is an off-shell extension of the reduced K matrix $(K_p)_{\beta\alpha}$ and satisfies the on-shell discontinuity equation

$$(\Delta\bar{U})_{\beta\alpha}=-2i\bar{U}_{\beta 0}(\pm)D_0\bar{U}_{0\alpha}(\mp)\,. \qquad (3.5)$$

Only the on-shell forms of Eqs. (3.4) and the corresponding discontinuity relations (3.2) and (3.5) are needed for the exploitation of this formalism, which in this general case has precisely the same structure as when no three-body interaction is present.[6]

Now we will investigate the complete K-matrix formalism. This is a vehicle for generating classes of fully unitary approximate three-particle scattering amplitudes in a manner described in Ref. 6. We find, in contrast to the discussion in this section up to this point, some differences between the cases with and without the three-body force.

As in Ref. 6 let us define an operator $\Gamma(z)$ by

$$\bar{U}(z)=\bar{\delta}G_0(z)^{-1}+\bar{\delta}\Gamma(z)\bar{\delta}\,, \qquad (3.6)$$

which implies that

$$\Gamma(z)=\bar{t}(z)+\bar{t}(z)\bar{\delta}G_0(z)\Gamma(z)$$
$$=\bar{t}(z)+\Gamma(z)G_0(z)\bar{\delta}\bar{t}(z)\,. \qquad (3.7)$$

It is easy to show that

$$\Gamma(\pm)=\kappa\mp i\kappa D_0(1+\bar{\delta})\Gamma(\pm)$$
$$=\kappa\mp i\Gamma(\pm)D_0(1+\bar{\delta})\kappa\,, \qquad (3.8)$$

where

$$\kappa=k+k\bar{\delta}G\kappa$$
$$=k+\kappa\bar{\delta}Gk\,, \qquad (3.9)$$
$$G=G_0(+)+iD_0\,,$$

and the operators k are defined as solutions of the Heitler equations

$$k=\bar{t}(\pm)\pm i\bar{t}(\pm)D_0k$$
$$=\bar{t}(\pm)\pm ikD_0\bar{t}(\pm)\,. \qquad (3.10)$$

When $\alpha=1,2,3$, k_α is just the two-particle K matrix defined on the three-particle space. k_4 is the (connected) three-particle K matrix corresponding to t_4. Evidently, $k_0=0$.

From the first of Eqs. (3.8) we see that, for example,

$$\bar{U}_{\beta\alpha}(+)=K_{\beta\alpha}-iK_{\beta 0}D_0\bar{U}_{0\alpha}(+)\,, \qquad (3.11)$$

where

$$K_{\beta\alpha}\equiv\bar{\delta}_{\beta\alpha}G_0(+)^{-1}+[\bar{\delta}\kappa\bar{\delta}]_{\beta\alpha}$$

are the components of the complete three-particle K matrix. Again we note that the Heitler equations (3.11) for $\beta,\alpha\neq 4$ have no coupling with any index-four operators.

Now neither one of the standard Heitler-type equations (3.8) and (3.11) possesses connected kernels, which renders them useless for practical calculations.[6] It will be as a consequence of our efforts to obtain connected-kernel counterparts of these equations that we will obtain some new features due to the presence of a three-body interaction.

Let us first concentrate on Eqs. (3.8). We separate κ into its disconnected and connected parts:

$$\kappa=\bar{k}+\kappa^c\,,$$

where

$$\bar{k}_\alpha=k_\alpha\,,\quad \alpha=1,2,3$$
$$=0\,,\quad \alpha=0$$

so that

$$\kappa^c_{\beta\alpha}=k_4\delta_{\beta 4}\delta_{4\alpha}+(k\bar{\delta}G\kappa)_{\beta\alpha}\,.$$

Also, let

$$\hat{t}_\alpha=\bar{t}_\alpha\,,\quad \alpha=1,2,3$$
$$=0\,,\quad \alpha=0,4.$$

Now if, for example, we multiply the first of Eqs. (3.8) on the left by $[1\mp i\hat{t}(\pm)D_0]$ we obtain

$$\Gamma(\pm)=\hat{t}(\pm)+\kappa^c_L(\pm)\mp i\hat{t}(\pm)\bar{\delta}D_0\Gamma(\pm)\mp i\kappa^c_L(\pm)\bar{D}\,\Gamma(\pm)\,, \qquad (3.12a)$$

where

$$\kappa^c_L(\pm)\equiv[1\mp i\hat{t}(\pm)D_0]\kappa^c\,,$$
$$\bar{D}\equiv D_0(1+\bar{\delta})\,.$$

Equation (3.12) is a connected-kernel equation. Given any κ^c such that

$$(\kappa^c)^\dagger=\kappa^c\,, \qquad (3.13)$$

we will obtain from (3.12) solutions $\Gamma(\pm)$ which satisfy

$$\Delta\Gamma=-2i\Gamma(\pm)\bar{D}\Gamma(\mp)\,, \qquad (3.14)$$

and therefore reduced K matrices $\bar{U}(\pm)$ which satisfy (3.5) and consequently via (3.4) a fully unitary set of physical scattering amplitudes. We also see that

$$\Gamma(\pm) = \hat{t}(\pm) + \kappa_R^c(\pm) \mp i\Gamma(\pm)\bar{\delta}D_0\hat{t}(\pm) \mp i\Gamma(\pm)\bar{D}\kappa_R^c(\pm)\,, \tag{3.12b}$$

where

$$\kappa_R^c(\pm) \equiv \kappa^c[1 \mp iD_0\hat{t}(\pm)]\,.$$

It is much more convenient for practical calculations to work with the operators

$$\zeta \equiv \bar{\delta}\Gamma\bar{\delta} \tag{3.15}$$

rather than the auxiliary operators Γ.[6] We see from Eq. (3.6) that ζ is the reduced K matrix except for a simple exchange term. From Eqs. (3.12) and (3.15) and the observations that

$$\Gamma_{0\alpha} = \Gamma_{\alpha 0} = 0 \tag{3.16a}$$

and

$$\kappa_{0\alpha} = \kappa_{\alpha 0} = 0 \tag{3.16b}$$

for all α we find that

$$\begin{aligned}\zeta_{\beta\alpha}(\pm) &= \{\bar{\delta}[\hat{t}(\pm) + \kappa_L^c(\pm)]\bar{\delta}\}_{\beta\alpha} \\ &\quad \mp i \sum_{\gamma=0}^{3} \{\bar{\delta}_{\beta\gamma}\hat{t}_\gamma(\pm) + [\bar{\delta}\kappa_L^c(\pm)\bar{\delta}]_{\beta\gamma}\delta_{\gamma 0}\} D_0 \zeta_{\gamma\alpha}(\pm) \\ &= \{\bar{\delta}[\hat{t}(\pm) + \kappa_R^c(\pm)]\bar{\delta}\}_{\beta\alpha} \\ &\quad \mp i \sum_{\gamma=0}^{3} \zeta_{\beta\gamma}(\pm) D_0 \{\bar{\delta}_{\gamma\alpha}\hat{t}_\gamma(\pm) + \delta_{\gamma 0}[\bar{\delta}\kappa_R^c(\pm)\bar{\delta}]_{\gamma\alpha}\}\,,\end{aligned} \tag{3.17}$$

which are also connected-kernel equations. We note that the submatrix of (3.17) for $\beta, \alpha \neq 4$, which is all we require for the execution of the unitary program, is decoupled from the index-four ζ operators. Again, if κ^c satisfies (3.13) the solution of (3.17) satisfies

$$(\Delta\zeta)_{\beta\alpha} = -2i\zeta_{\beta 0}(\pm)D_0\zeta_{0\alpha}(\mp)$$

and therefore defines a reduced K matrix with the correct discontinuity relations. Methods for solving on-shell equations of the form (3.17) were discussed in Ref. 6.

Equations (3.17) for $\beta, \alpha \neq 4$ have the same form and the same number of components of $\zeta_{\beta\alpha}$ entering into them as in the case without the three-body force. This similarity is, however, illusory because the input, in the form of κ^c, is somewhat more complicated than in the latter case. Namely, various (on-shell) index-four components of $\kappa^c_{\lambda\gamma}$ enter in and must be included in order to represent the effects of the three-body force. This complication can be appreciated by rewriting Eqs. (3.17) in terms of the connected part of the complete K matrix, which we define as

$$K^c \equiv \bar{\delta}\kappa^c\bar{\delta}\,. \tag{3.18}$$

One finds, in contrast to the situation in Ref. 6 and to the unconnected-kernel Heitler equation (3.11), that *all* components of (3.17) contain the hitherto unnecessary index-four operators $K^c_{4\lambda}$ and $K^c_{\lambda 4}$.

This state of affairs is a consequence of our introduction of a separate unphysical channel ($\alpha = 4$) corresponding to the three-body force. This is out of place in a general K-matrix formalism. However, we show next that it is possible to circumvent this index-four problem by effectively exploiting the redundancy implied by Eqs. (3.16).

Let us define for $\beta, \alpha \neq 4$

$$\bar{\Gamma}_{\beta\alpha} \equiv \Gamma_{\beta\alpha} + \tfrac{1}{3}\Gamma_{4\alpha} + \tfrac{1}{3}\Gamma_{\beta 4} + \tfrac{1}{9}\Gamma_{44}\,. \tag{3.19}$$

Then one finds that

$$\begin{aligned}\zeta_{\beta\alpha} &= \sum_{\lambda,\gamma=0}^{4} \bar{\delta}_{\beta\lambda}\Gamma_{\lambda\gamma}\bar{\delta}_{\gamma\alpha} \\ &= \sum_{\lambda,\gamma=0}^{3} \bar{\delta}_{\beta\lambda}\bar{\Gamma}_{\lambda\gamma}\bar{\delta}_{\lambda\alpha}\end{aligned} \tag{3.20}$$

for $\beta, \alpha \neq 4$. We also define for $\beta, \alpha \neq 4$

$$\bar{\kappa}^c_{\beta\alpha} \equiv \kappa^c_{\beta\alpha} + \tfrac{1}{3}\kappa^c_{4\alpha} + \tfrac{1}{3}\kappa^c_{\beta 4} + \tfrac{1}{9}\kappa^c_{44}\,. \tag{3.21}$$

Then we deduce from Eq. (3.18) that the connected part of the K matrix is given by

$$K^c_{\beta\alpha} = \sum_{\lambda,\gamma=0}^{3} \bar{\delta}_{\beta\lambda}\bar{\kappa}^c_{\lambda\gamma}\bar{\delta}_{\gamma\alpha}\,. \tag{3.22}$$

Integral equations for $\bar{\Gamma}_{\beta\alpha}$, with $\beta, \alpha \neq 4$, can be derived from Eq. (3.12a), using the definition (3.19), as follows:

$$\begin{aligned}\bar{\Gamma}_{\beta\alpha} &= \hat{t}_\beta\delta_{\beta\alpha} + [1 \mp i\hat{t}_\beta D_0]\bar{\kappa}^c_{\beta\alpha} \mp i\hat{t}_\beta D_0 \sum_{\gamma=0}^{3} \bar{\delta}_{\beta\gamma}\bar{\delta}_{0\gamma}\bar{\Gamma}_{\gamma\alpha} \\ &\quad \mp i[1 \mp i\hat{t}_\beta D_0]\left\{\sum_{\lambda=0}^{3} \bar{\kappa}^c_{\beta\lambda}\bar{\delta}_{\lambda 0}\right\} D_0 \left\{\sum_{\gamma=0}^{3} \bar{\delta}_{0\gamma}\bar{\Gamma}_{\gamma\alpha}\right\},\end{aligned} \tag{3.23}$$

where we have suppressed the ($\pm$) arguments for the sake of notational simplicity. Equation (3.23), which is a connected-kernel equation, is virtually identical in form to the case without a three-body force. The basic difference is that in contrast with Eq. (3.16) the various zero-index components of $\bar{\Gamma}_{\beta\alpha}$ and $\bar{\kappa}^c_{\beta\alpha}$ do not necessarily vanish unless there are only pair interactions. The counterpart of Eq. (3.23) corresponding to Eq. (3.12b) is easily deduced.

Equations (3.20), (3.22), and (3.23) imply that $\zeta_{\beta\alpha}$, for $\beta, \alpha \neq 4$, satisfies

$$\zeta_{\beta\alpha}=[\bar{\delta}\hat{t}\bar{\delta}]_{\beta\alpha}+\sum_{\lambda=0}^{3}\tau_{\beta\lambda}K^{c}_{\lambda\alpha}\mp i\sum_{\lambda=0}^{3}\bar{\delta}_{\beta\lambda}\hat{t}_{\lambda}D_0\left\{\zeta_{\lambda\alpha}-\tfrac{1}{3}\left[\sum_{\eta=0}^{3}\bar{\delta}_{0\eta}\zeta_{\eta\alpha}-2\zeta_{0\alpha}\right]\right\}\mp i\sum_{\lambda=0}^{3}\tau_{\beta\lambda}K^{c}_{\lambda 0}D_0\zeta_{0\alpha}, \tag{3.24}$$

as well as an equation corresponding to the second of Eqs. (3.17), where

$$\tau_{\beta\alpha}(\pm)\equiv\{\bar{\delta}[1\mp i\hat{t}(\pm)D_0]\bar{\delta}^{-1}\}_{\beta\alpha}$$

and (in four dimensions)

$$\bar{\delta}^{-1}=\tfrac{1}{3}(\bar{\delta}-2I).$$

Equations (3.24) are connected-kernel equations with no reference to index-four operators. In particular the K-matrix input consists only of the operators $K^{c}_{\beta\alpha}$ for β, $\alpha\neq 4$, just as in the (nonconnected-kernel) Heitler equation, which need only satisfy

$$(K^{c}_{\beta\alpha})^{\dagger}=K^{c}_{\alpha\beta}$$

in order to generate a unitary theory.

The difference between (3.24) and its counterpart in the case with only pair interactions consists in the presence of the terms

$$\pm\tfrac{1}{3}i\left(\sum_{\lambda=0}^{3}\bar{\delta}_{\beta\lambda}\hat{t}_{\lambda}D_0\right)\left(\sum_{\eta=0}^{3}\bar{\delta}_{0\eta}\zeta_{\eta\alpha}-2\zeta_{0\alpha}\right).$$

One easily verifies that in the absence of three-body forces

$$\sum_{\eta=0}^{3}\bar{\delta}_{0\eta}\zeta_{\eta\alpha}=2\zeta_{0\alpha} \tag{3.25}$$

and so the preceding term vanishes. Also,

$$\sum_{\eta=0}^{3}\bar{\delta}_{0\eta}K^{c}_{\eta\alpha}=2K^{c}_{0\alpha} \tag{3.26}$$

when there are only pair interactions. One can show using Eqs. (3.24) that (3.26) implies (3.25), although the converse may not necessarily be true. However, if one's input in the form of $K^{c}_{\beta\alpha}$ violates (3.26) then necessarily three-body forces are included in this input.

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