t-Channel Unitarity and the πN σ Term*

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We unitarize the low-energy symmetric πN amplitude introduced by Altarelli *et al.* under the assumption that the 2π intermediate state in the t-channel unitarity condition is the dominant contribution. The low-energy $\pi\pi$ amplitude is described using a model of Graham and Johnson. The unitarity corrections at the current-algebra point are found to be small. Hence, considerations of t-channel analyticity probably do not seriously affect the calculation by Cheng and Dashen of the πN σ term.

I. INTRODUCTION

The σ terms in meson-baryon scattering have been the focus of considerable attention recently as they can provide important information about how chiral symmetry (and perhaps scale invariance) is broken. Of particular interest has been the surprisingly large value of $\sigma_{NN} \simeq 110$ MeV calculated by Cheng and Dashen[1] which seemed to make the symmetry-breaking model of Gell-Mann, Oakes, and Renner[2] (GMOR) less attractive. This value was soon shown[3,4] to be consistent with the GMOR model, however, provided that the chiral- and scale-invariance limits are realized simultaneously. In addition, there have been other calculations[5] of σ_{NN} which yield substantially lower values and, hence, do not require this connection between chiral and scale invariance. It thus becomes very important to estimate, if possible, the magnitude of corrections to the Cheng-Dashen result, among others.

To this end, two types of corrections have been considered. The first, which takes into account the effect of N^* resonances on the extrapolation from soft to hard pions, has been found[3,6] to be quite small $[\sim O(m_\pi^4/m_N^4)]$. The second source of uncertainty in the Cheng-Dashen result concerns the effects of t-channel analyticity. It is this latter type of correction which interests us here.

An initial calculation,[7] making use of analytic hard-pion methods,[8] found a tremendously large correction. This seemed to us a very surprising result.[9] We decided therefore, that it would be worthwhile to undertake an independent analysis of the effects of t-channel analyticity. Hence, in the following, we present an elementary calculation in which approximate unitarity is imposed on the relevant πN scattering amplitude in the t-channel, i.e., $\pi\pi \to N\bar{N}$. The assumption is made that the most important contribution to the unitarity relation comes from the two-pion state. This seems to be an eminently reasonable approximation, because the value of t at which the current-algebra result holds lies well below the $\pi\pi$ threshold.

The πN amplitude used in the calculation will be that given by Altarelli, Cabibbo, and Maiani.[3,10] The $\pi\pi$ amplitude is taken from a model for low-energy $\pi\pi$ scattering developed by one of the authors (RHG) in collaboration with Johnson.[11]

It will be shown that the change in the πN amplitude as a result of its unitarization in the t channel amounts to only a few percent, at the current-algebra point. Thus, the calculation of σ_{NN} by Cheng and Dashen is probably not seriously affected by questions of t-channel analyticity.

The paper is organized as follows: In Sec. II we sketch a derivation of the unitarity relation for helicity amplitudes from which is constructed a partially unitarized t-channel helicity amplitude for πN scattering in terms of "bare" πN and $\pi\pi$ helicity amplitudes; in Secs. III and IV, respectively, we discuss the models for $\pi\pi$ and πN scattering employed in the analysis; our results and conclusions are presented in Sec. V.

II. UNITARITY CONSTRAINTS IN THE t CHANNEL

For arbitrary initial and final states, i and f, the unitarity of the S matrix implies that the T matrix satisfies the well-known relation[12]

$$i(T_{fi} - T_{fi}^\dagger) = (2\pi)^4 \sum_n \delta^4(p_i - p_n) T_{fn}^\dagger T_{ni} \,. \qquad (2.1)$$

As we wish to apply Eq. (2.1) to the process $\pi\pi \to N\bar{N}$, we take

$$|i\rangle = |\pi(q)\pi(q')\rangle$$

and

$$|f\rangle = |N(p)\bar{N}(p')\rangle$$

with $q+q'=p+p'$ and $t=(q+q')^2$, $s=(q-p)^2$ and $u=(p-q')^2$. We make the approximation of keeping only the 2π state in the sum and thus $|n\rangle$

$=|\pi(k)\pi(k')\rangle$. We are led, in the t-channel center-of-mass system, to the relation

$$i(T_{N\pi}-T^{\dagger}_{N\pi})=(2\pi)^4\frac{k}{8W}\int d\Omega_k\, T^{\dagger}_{N\pi}T_{\pi\pi}\ , \qquad (2.2)$$

where $T_{N\pi}$ and $T_{\pi\pi}$ represent the amplitudes for $\pi\pi\to N\bar{N}$ and $\pi\pi\to\pi\pi$, respectively, while $W=\sqrt{t}$ and $4k^2=t-4\mu^2$ with μ the pion mass. We want now to use Eq. (2.2) to obtain a unitarity constraint on t-channel helicity amplitudes.

We denote a two-particle helicity state vector in the usual way by

$$|\,\theta,\phi,\mu_1,\mu_2,\gamma\rangle$$

where θ, ϕ give the relative orientation of the particles in their center-of-mass system, μ_1 and μ_2 are their respective helicities and γ represents any other quantum numbers necessary to describe the two-particle system.

The T matrix in this basis can be expressed in terms of amplitudes of definite total angular momentum J as[13]

$$\langle\theta',\phi',\mu_1',\mu_2',\gamma'|\,T\,|\,\theta,\phi,\mu_1,\mu_2,\gamma\rangle=\frac{1}{4\pi}\sum_{J,M}(2J+1)D^{J*}_{M\mu'}(\phi',\theta',-\phi')D^{J}_{M\mu}(\phi,\theta,-\phi)\,\langle\mu_1',\mu_2',\gamma'|\,T^J|\,\mu_1,\mu_2,\gamma\rangle\ , \qquad (2.3)$$

where the $D^J_{\mu\nu}$ are the familiar rotation matrices, $\mu=\mu_1-\mu_2$, $\mu'=\mu_1'-\mu_2'$, and M is the third component of the total angular momentum. If we use the representation (2.3) for $T_{N\pi}$ and $T_{\pi\pi}$ in Eq. (2.2), making use of the orthogonality of the rotation matrices, we obtain[14,15]

$$\mathrm{Im}\langle\mu_1\mu_2,N\bar{N}|\,T^J|\,\pi\pi\rangle=-(2\pi)^4\frac{k}{16W}\langle\mu_1\mu_2,N\bar{N}|\,T^J|\,\pi\pi\rangle^*\langle\pi\pi|\,T^J|\,\pi\pi\rangle\ . \qquad (2.4)$$

If we define

$$\langle\mu_1\mu_2,N\bar{N}|\,T^J|\,\pi\pi\rangle\equiv R^J_{\mu_1\mu_2;00}(t) \qquad (2.5)$$

and

$$\langle\pi\pi|\,T^J|\,\pi\pi\rangle\equiv G^J(t)\ , \qquad (2.6)$$

Eq. (2.4) takes the form

$$\mathrm{Im}R^J_{\mu_1\mu_2;00}(t)=-\tfrac{1}{2}\pi^4\rho(t)R^{J*}_{\mu_1\mu_2;00}(t)G^J(t) \qquad (2.7)$$

with

$$\rho(t)=\frac{2k}{W}\ . \qquad (2.8)$$

We could attempt to construct a complex amplitude by taking as its imaginary part Eq. (2.7) and as its real part the original (real) amplitude, R^J. However, this introduces an unwanted singularity at $t=0$. An analytic amplitude, real below the $\pi\pi$ threshold and having Eq. (2.7) as its imaginary part above, is given[11,16] by

$$\begin{aligned}R^{uJ}_{\mu_1\mu_2;00}(t)&=R^J_{\mu_1\mu_2;00}(t)+\tfrac{1}{2}\pi^3\rho(t)\ln\left(\frac{1+\rho(t)}{1-\rho(t)}\right)R^J_{\mu_1\mu_2;00}(t)G^J(t)-\tfrac{1}{2}i\pi^4\rho(t)R^J_{\mu_1\mu_2;00}(t)G^J(t)\\&=R^J_{\mu_1\mu_2;00}(t)+\tfrac{1}{2}\pi^3\rho(t)\ln\left(\frac{\rho(t)+1}{\rho(t)-1}\right)R^J_{\mu_1\mu_2;00}(t)G^J(t)\\&=R^J_{\mu_1\mu_2;00}(t)+\tfrac{1}{2}H(t)R^J_{\mu_1\mu_2;00}(t)G^J(t)\ ,\end{aligned} \qquad (2.9)$$

where we have defined

$$H(t)=\pi^3\rho(t)\ln\left(\frac{\rho(t)+1}{\rho(t)-1}\right)\ . \qquad (2.10)$$

We have used the superscript u to denote that the amplitude R^J has been approximately unitarized in the small-t region.

If we take the pion momenta along the z axis in the c.m. system, then we can write Eq. (2.3) in the form

$$T_{\mu_1\mu_2;00}=\frac{1}{4\pi}\sum_J(2J+1)D_{0\mu}(\phi,\theta,-\phi)T^J_{\mu_1\mu_2;00}\ . \tag{2.11}$$

Then,

$$R^u_{\mu_1\mu_2;00}(t,\theta,\phi)=R_{\mu_1\mu_2;00}(t,\theta,\phi)+\tfrac{1}{2}H(t)\sum_J\left(\frac{2J+1}{4\pi}\right)D^{J}_{0\mu}{}^*(\phi,\theta,-\phi)R^J_{\mu_1\mu_2;00}(t)G^J(t)\ . \tag{2.12}$$

It can be seen from the first line of Eq. (2.9) that the real part of R^{uJ} below the $\pi\pi$ threshold is not equal to R^J. It is the magnitude of the change which we will estimate in the following. We must first find a suitable $G^J(t)$.

III. MODEL FOR $\pi\pi$ SCATTERING

To determine $G^J(t)$ we will use a model[11] for low-energy $\pi\pi$ scattering which is crossing-symmetric and satisfies, in addition to the Adler condition, a set of rigorous constraints. The relevant t-channel amplitude with isospin zero is[17]

$$A_t^0(t,s)=\tfrac{3}{2}[F(t,s)+F(t,u)]-\tfrac{1}{2}F(s,u)\ , \tag{3.1}$$

where $F(x,y)$ is a quadratic polynomial symmetric in its arguments,

$$F(x,y)=A+B(x+y)+Cxy+D(x^2+y^2)\ . \tag{3.2}$$

In the t channel, t is the square of the center-of-mass energy, the scattering angle is

$$\cos\theta=1+\frac{2s}{t-4\mu^2}$$

and

$$s+t+u=4\mu^2\ .$$

The connection between the amplitude A_t^0 and the T matrix we are using here is given by

$$T_{\pi\pi}(t,s)=-\frac{1}{8\pi^5}A_t^0(t,s)\ . \tag{3.3}$$

The coefficients appearing in Eq. (3.2) have been determined[11] to be

$$A\cong-3\mu^2a_1\ , \tag{3.4}$$

$$B=\tfrac{3}{2}a_1-a_1(2+X)\frac{\mu^2}{k_\rho{}^2}\ , \tag{3.5}$$

$$C=\tfrac{1}{4}(2+X)\frac{a_1}{k_\rho{}^2}\ , \tag{3.6}$$

and

$$D=\tfrac{1}{4}(X+\tfrac{1}{2})\frac{a_1}{k_\rho{}^2}\ , \tag{3.7}$$

where

$$4k_\rho{}^2=m_\rho{}^2-4\mu^2 \tag{3.8}$$

and

$$a_1=\frac{1}{8}\frac{m_\rho{}^2\Gamma_\rho}{k_\rho{}^5}\ . \tag{3.9}$$

m_ρ and Γ_ρ represent, respectively, the mass and width of the ρ meson while X is a parameter whose preferred range[11] is $0\lesssim X\lesssim\frac{1}{2}$. Terms of order $\sim\mu^4$ have been dropped in Eq. (3.4).

Now it turns out in this model, and experimentally, that the D and higher partial waves are negligible, for small t, compared to the S waves. In what follows then, we will retain only the S-wave part of A_t^0. If we write

$$A_t^0(t,s)=\frac{1}{4\pi}\sum_J(2J+1)A_J(t)D^{J}_{00}{}^*(\phi,\theta,-\phi)$$
$$=\frac{1}{4\pi}\sum_J(2J+1)A_J(t)P_J(\cos\theta)\ , \tag{3.10}$$

then

$$A_J(t)=2\pi\int_{-1}^{1}A_t^0(t,s)P_J(\cos\theta)d(\cos\theta)\ . \tag{3.11}$$

We find, from Eqs. (3.1), (3.2), and (3.11),

$$A_0(t)=2\pi[5A+4B(t+2\mu^2)]\ , \tag{3.12}$$

where terms of $O(\mu^4)$ have been omitted.[18] Using Eqs. (3.4) and (3.5) in Eq. (3.12) we have

$$A_0(t)=6\pi a_1(2t-\mu^2)\ . \tag{3.13}$$

From the relative normalization given in Eq. (3.3) we then find that

$$G^0(t)=-\frac{3}{4\pi^4}a_1(2t-\mu^2)\ . \tag{3.14}$$

As discussed above, we assume that

$$G^J(t)=0\ ,\quad J\geq1\ . \tag{3.15}$$

IV. THE LOW-ENERGY PION-NUCLEON AMPLITUDE

Pion-nucleon scattering may be described in terms of the four invariant amplitudes,[19] $A^{(\pm)}(\nu,t)$ and $B^{(\pm)}(\nu,t)$ where

$$\nu=\frac{s-u}{4m} \tag{4.1}$$

and

$$s+t+u=2m^2+2\mu^2\ , \tag{4.2}$$

m being the nucleon mass; the plus and minus signs on A and B correspond to t-channel isospin zero and one, respectively.

Let us consider the crossing-even (under $\nu \to -\nu$) combination[20]

$$F^{(+)}(\nu, t) = A^{(+)}(\nu, t) + \nu B^{(+)}(\nu, t) . \tag{4.3}$$

We may explicitly remove the Born pole from $F^{(+)}(\nu, t)$ by forming[3,10]

$$\tilde{F}^{(+)}(\nu, t) = F^{(+)}(\nu, t) - \frac{g^2}{m} \frac{\nu_B^{\,2}}{\nu_B^{\,2} - \nu^2} , \tag{4.4}$$

where

$$\nu_B = \frac{t - 2\mu^2}{4m} \tag{4.5}$$

and g is the pion-nucleon coupling constant ($g^2/4\pi \simeq 14.6$). The amplitude $\tilde{F}$ is analytic in a neighborhood about $\nu = t = 0$ where it can be expanded in the form

$$\tilde{F}^{(+)}(\nu, t) \simeq a\mu^2 + bt + c\nu^2 . \tag{4.6}$$

The coefficients a, b, and c have been determined by Altarelli *et al.*[21] and their values will be used in Sec. V for our numerical estimate of the unitarity correction.

Since the amplitudes $A(\nu, t)$ and $B(\nu, t)$ are linearly related to the helicity amplitudes $R_{\frac{1}{2}\,\frac{1}{2};00}$ and $R_{\frac{1}{2}\,-\frac{1}{2};00}$, we have[22] from Eqs. (2.12) and (3.15)

$$F^{u}(\nu, t) = F(\nu, t) + \frac{1}{8\pi} H(t) F^0(t) G^0(t) , \tag{4.7}$$

where

$$F^0(t) = 2\pi \int_{-1}^{1} d(\cos\theta) F(\nu, t) . \tag{4.8}$$

The dependence of ν on θ is given by

$$\nu = \frac{|\vec{q}\,||\vec{p}\,|}{m} \cos\theta , \tag{4.9}$$

where

$$|\vec{q}\,| = \tfrac{1}{2}(t - 4\mu^2)^{1/2}$$

and

$$|\vec{p}\,| = \tfrac{1}{2}(t - 4m^2)^{1/2}$$

are the c.m. momenta of the pions and nucleons, respectively.

We can rewrite the correction to $F(\nu, t)$ in terms of $\tilde{F}^0(t)$ as

$$F^{u}(\nu, t) = F(\nu, t) + \frac{1}{8\pi} H(t) \tilde{F}^0(t) \tilde{G}^0(t) + f(t)\nu_B . \tag{4.10}$$

The last term in Eq. (4.10) arises from the S-wave projection of the last term in Eq.(4.4). Since we are interested in determining the effects of unitarity at the current-algebra point, $\nu = \nu_B = 0$, we will drop this term in the following [$f(t)$ is finite in the limit $\nu_B \to 0$].

Using Eqs. (4.6), (4.8), and (4.9), we find

$$\tilde{F}^0(t) = 4\pi \left(\mu^2 a + tb + \frac{1}{6m^2} |\vec{q}\,|^2 |\vec{p}\,|^2 c \right) . \tag{4.11}$$

V. RESULTS AND CONCLUSIONS

We have assembled enough information in the previous sections to enable us to estimate the unitarity correction to the result of Cheng and Dashen. To this end we will use Eq. (4.10) to calculate the difference between $F^u(\nu, t)$ and $F(\nu, t)$ at the current-algebra point $\nu = 0$ and $\nu_B = 0$ ($t = 2\mu^2$).

The $\pi\pi$ scattering contribution to this correction is determined from Eq. (3.14) to be

$$G^0(2\mu^2) = -\frac{9\mu^2}{4\pi^4} a_1 . \tag{5.1}$$

Taking $m_\rho = 765$ MeV and $\Gamma_\rho = 135$ MeV, we find

$$a_1 = 1.7 \times 10^{-6}\ \text{MeV}^{-2} , \tag{5.2}$$

and so

$$G^0(2\mu^2) = -8.3 \times 10^{-4} . \tag{5.3}$$

From Eq. (4.11) the contribution of $\tilde{F}^0$ at $t = 2\mu^2$ is

$$\tilde{F}^0(2\mu^2) = 4\pi(a\mu^2 + 2b\mu^2 + \tfrac{1}{8} c\mu^2) . \tag{5.4}$$

Using the values of a and c from the analysis of Altarelli *et al.*,[3,10] we get

$$\tilde{F}^0(2\mu^2) = 1.7 \times 10^{-1}\ \text{MeV}^{-1} . \tag{5.5}$$

Finally, since

$$H(2\mu^2) = \tfrac{1}{2}\pi^4 , \tag{5.6}$$

we obtain the fractional correction

$$\frac{F^u(0, 2\mu^2) - F(0, 2\mu^2)}{F(0, 2\mu^2)} = -2.1 \times 10^{-2} \tag{5.7}$$

corresponding to the Cheng-Dashen value of $\sigma_{NN} = 110$ MeV. As an indication of the sensitivity of our result on the value of σ_{NN}, the fractional correction is $\simeq -2.4 \times 10^{-2}$ for $\sigma_{NN} = 50$ MeV.

It is possible to carry out the above analysis with off-mass-shell pions in which case we would work with amplitudes $F(\nu, t, q^2, q'^2)$, etc. which depend explicitly on the pion four-momenta q^2 and q'^2. We find for the case of massless pions ($q = q'^2 = 0$) that

$$\frac{F^u(0, 0, 0, 0) - F(0, 0, 0, 0)}{F(0, 0, 0, 0)} = 1.8 \times 10^{-2} \tag{5.8}$$

independent of the values of σ_{NN}.

It can be seen from Eqs. (5.7) and (5.8) that corrections to $F(\nu, t)$, due to the imposition of approximate t-channel unitarity, amount to only a few percent, at $\nu = \nu_B = 0$. Thus, we would not expect corrections to the Cheng-Dashen method, arising from considerations of t-channel analyticity, to be greater than a few percent.

In passing, we note that our results can be applied to another problem – that of determining the difference between the σ term evaluated at $t = 2\mu^2$, $\sigma_{NN}(2\mu^2)$, and at $t = 0$, $\sigma_{NN}(0)$. The former quantity is the actual result of the Cheng-Dashen method (up to terms of order μ^4) while the latter is obtained in the limit of zero pion mass. We find[6]

$$\sigma_{NN}(2\mu^2) - \sigma_{NN}(0) = F_\pi^{\,2}[F^u(0, 2\mu^2) + F^u(0, 0, 0, 0) + O(\mu^2)] \simeq -4.1 \text{ MeV} \tag{5.9}$$

for the Cheng-Dashen value of $\sigma_{NN}(2\mu^2) = 110$ MeV. This result may be compared with a recent calculation[23] by Pagels and Pardee who determine $\sigma_{NN}(2\mu^2) - \sigma_{NN}(0)$ from a once-subtracted dispersion relation whose discontinuity is approximated by the 2π contribution. They find

$$\sigma_{NN}(2\mu^2) - \sigma_{NN}(0) = 13 \text{ MeV} \quad \text{(Pagels-Pardee)} \tag{5.10}$$

independent of the value of $\sigma_{NN}(2\mu^2)$.

The difference in magnitude and sign between Eqs. (5.9) and (5.10) arises for several reasons. First, our approach is designed mainly to give unitarity corrections at the current-algebra point and, because of the simple models used in the calculations, off-mass-shell extrapolations can only be estimated very roughly.

In the second place, the estimate of Pagels and Pardee is based on a leading term which is formally of order μ^3. Ours is of order $\sim\mu^4$ since, in the model of Altarelli *et al.* which we have used, the relation $\bar{F}(0, 2\mu^2) + \bar{F}(0, 0, 0, 0) = 0$ for the "bare" amplitudes holds good to within 1%. This relation could be violated by terms of $O(\mu^3)$ without appreciably affecting our main conclusions about unitarity corrections.

To conclude, we have carried out an independent investigation of the possible effects of t-channel analyticity on the calculation of Cheng and Dashen for the πN σ term. We impose approximate t-channel unitarity on a simple model of low-energy πN scattering and find a correction to the "bare" amplitude amounting to only a few percent. This indicates that considerations of t-channel analyticity should not greatly affect the procedure of Cheng and Dashen.

If our results are taken together with those of Brown *et al.*[6] and Altarelli *et al.*,[3,10] it seems that the technique of Cheng and Dashen should be taken rather seriously. Other types of calculations of σ_{NN} (such as those of von Hippel and Kim[5] or Ericson and Rho[5]) which find substantially lower values for σ_{NN}, may be subject to proportionately larger corrections.

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[13]For an excellent discussion of helicity amplitudes, see A. D. Martin and T. D. Spearman, *Elementary Particle Theory* (North-Holland, Amsterdam, 1970).

[14]For the process $\pi\pi \to N\bar{N}$ the pions have zero helicity, while for the nucleons $\mu_1, \mu_2 = \frac{1}{2}$ or $-\frac{1}{2}$.

[15]In arriving at Eq. (2.4) we make use of the relation

$$\langle \mu_1', \mu_2', \gamma' | T^J | \mu_1, \mu_2, \gamma \rangle = \langle \mu_1, \mu_2, \gamma | T^J | \mu_1', \mu_2', \gamma' \rangle,$$

which follows from the time-reversal invariance of the strong interactions [see Martin and Spearman (Ref. 13)].

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[18]In writing down Eq. (3.12) we have anticipated applications in a later section where the value of t at which the appropriate amplitudes are evaluated is of $O(\mu^2)$. This would make the terms in C and D of $O(\mu^4)$, and so they have been dropped.

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[20]This is the combination studied by Cheng and Dashen (Ref. 1). Their result is that $F(0, 2\mu^2) = F_\pi^{-2}\sigma_{NN} + O(\mu^4)$ with F_π the pion decay constant (=93 MeV) and the limit $t \to 2\mu^2$ taken before $\nu \to 0$.

[21]Actually, Altarelli *et al.* (Refs. 3 and 10) use an expansion which allows the pions to be off mass shell. Our coefficients, while not the same as theirs, are related to them.

[22]In arriving at Eq. (4.7) it is important to note that the coefficient of $R_{\frac{1}{2},\frac{1}{2};00}(\nu, t)$ depends only on t.

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Three-Particle Scattering with Three-Particle Interactions*

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The scattering formalism of Alt, Grassberger, and Sandhas is extended to include a possible three-particle interaction. This is then employed to find a set of scattering integral equations in which no unphysical auxiliary amplitudes related to the three-body force appear, as well as to develop a practical method for treating perturbatively the effect of a weak three-body force. The general K-matrix formalism and connected-kernel Heitler equations are also developed. This yields some indication of the structure that might be expected in a relativistic connected-kernel K-matrix formalism.

I. INTRODUCTION

Virtually all of the existing calculations of both the bound and the continuum states of the three-nucleon system (or any other three-particle system, such as that consisting of three pions) include only two-particle interactions. The role of a three-nucleon force is unclear at present,[1] and its possible nature is obscured by the uncertainties in the off-shell behavior of the two-nucleon transition amplitude. Given the validity of a nonrelativistic dynamics for the three-nucleon system, it is obvious that the questions of off-shell behavior and of the magnitude and character of the three-nucleon force are intimately correlated, assuming that the bound-state and on-shell scattering parameters of both the two- and the three-nucleon systems have been accounted for. These questions are also rather ill-defined if one adopts the stance of phenomenological potential scattering, and should they prove to be quantitatively significant a somewhat more fundamental approach to the entire problem may be in order.

Nonetheless, there are several reasons for examining how a three-particle force alters the scattering integral equations and the computational procedures derived from them. One of these is the opportunity to examine in a dynamically well-defined framework a situation somewhat related to the relativistic three-particle problem. Another is to use these equations to formulate modifications of some of the standard methods for calculating three-particle amplitudes so that three-particle forces can be introduced and their effects studied.

The rather straightforward modification of the three-particle scattering integral equations which is entailed when three-body forces are included in addition to the usual pair interactions was first pointed out by Newton.[2] Calculations using a separable three-nucleon force to simulate some of the noncentral and short-range features of the two-nucleon interaction were carried out by Phillips.[3] Both of the preceding authors employed, essentially, the Lovelace[4] version of the scattering integral equations.

Our objectives in this paper are twofold. First, we wish to embed the ideas of Newton and Phillips within the somewhat more practical form of the scattering integral equations due to Alt, Grass-