ϕ^+, the energy density T^{+-} or the Hamiltonian will be very complicated since the phase factor in (3.53) appears.

[15]The spectral representation for a nonconserved vector exists in the literature. See, for example, S.-J. Chang, H. T. Nieh, and T.-M. Yan, Nuovo Cimento 46, 364 (1966).

[16]See, for example, K. Johnson. Nucl. Phys. 25, 435 (1961).

[17]D. A. Dicus, R. Jackiw, and V. Teplitz, Phys. Rev. D 4, 1733 (1971).

[18]J. D. Bjorken, J. Kogut, and D. Soper, Phys. Rev. D 3, 1382 (1971).

[19]J. Cornwall and R. Jackiw, Phys. Rev. D 4, 367 (1971); R. A. Brandt, *ibid.* 1, 2808 (1970); H. Fritzsch and M. Gell-Mann, in *Broken Scale Invariance and the Light Cone*, 1971 Coral Gables Conference on Fundamental Interactions at High Energy, edited by M. Dal Cin, G. J. Iverson, and A. Perlmutter (Gordon and Breach, New York, 1971).

[20]S. Adler, Phys. Rev. 143, 1144 (1966).

[21]R. F. Dashen and M. Gell-Mann, in *Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1966*, edited by A. Perlmutter *et al*. (Freeman, San Francisco, 1966); S. Fubini, Nuovo Cimento 34A, 475 (1966).

[22]G. Calucci, R. Jengo, G. Furlan, and C. Rebbi, Phys. Letters 37B, 416 (1971).

[23]J. D. Bjorken, Phys. Rev. 148, 1469 (1966); K. Johnson and F. Low, Progr. Theoret. Phys. (Kyoto) Suppl. 37-38, 74 (1966). For application of this technique to light-front quantization, see Cornwall and Jackiw, Ref. 19.

[24]The derivation given here essentially follows the one given in Ref. 17 for a vector current. Our new contribution is to relate (6.26) to the p^+ distribution of the constituents of the hadron as given in (6.38).

[25]This assumption is known to be false in the perturbation calculation of a renormalizable field theory. See S. Adler and W.-K. Tung, Phys. Rev. Letters 22, 978 (1969); R. Jackiw and G. Preparata, *ibid.* 22, 975 (1969).

[26]Equation (6.39) can be regarded as the covariant version of the parton model as formulated by P. V. Landshoff, J. C. Polkinghorne, and R. D. Short [Nucl. Phys. B28, 225 (1971)]. The function $\Delta(x,0)$ describes the free propagation of a particle between x and 0. For more details of this interpretation, see R. Jackiw and R. E. Waltz, Phys. Rev. D 6, 702 (1972).

[27]See, for example, R. Jackiw, R. Van Royen, and G. B. West, Phys. Rev. D 2, 2473 (1970).

[28]R. L. Jaffe, Phys. Rev. D 6, 716 (1972).

[29]Application of the parton model to this process has been considered by, for example, S. D. Drell and T.-M. Yan [Phys. Rev. Letters 24, 181 (1970)].

[30]J. Schwinger, Phil. Mag. 44, 1171 (1953).

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Quantum Field Theories in the Infinite-Momentum Frame. IV. Scattering Matrix of Vector and Dirac Fields and Perturbation Theory*

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The scattering matrix of coupled spin-one and Dirac field theories formulated in light-front quantization of the preceding paper is studied. The scattering matrix of the vector-gluon model in this new formulation is shown to give the same predictions as in the equal-time formulation to all orders in perturbation theory. Renormalizability of this model in the new formulation is also established. A further test of the light-front quantization of spin-one fields is discussed by examples of fermion-fermion interaction and virtual as well as real Compton scattering in the axial-vector–gluon model in the lowest-order perturbation theory. A reduction formula for vector particles is derived and the Wick theorem is proved. Peculiarities in the perturbation theory of the light-front formulation are discussed. Finally, a partonlike model for scattering of two energetic particles is proposed which satisfies manifest s-channel unitarity.

I. INTRODUCTION

In this fourth and final paper in a program[1] devoted to the study of quantum field theories in light-front coordinates, we study the properties of the S matrix of the coupled spin-one field theories formulated in the preceding paper,[2] and certain general questions in perturbation theory in this new formulation.

The Hamiltonian and the propagators for Dirac and vector particles involve many noncovariant terms and are much more complicated than the corresponding expressions in the usual equal-time formulation. The problem is further complicated by the operator phase transformation on the Dirac field which is necessary in order to maintain simple commutation relations and a simple Hamiltonian. It is therefore crucial to study whether the S matrix in this new formulation gives the same physical predictions as in the ordinary formulation. This question is analyzed in two steps, and we find that the answer is affirmative.

We first prove by Schwinger's functional technique[3,4] that the new S matrix indeed is formally identical to the conventional one to all orders in perturbation theory. The noncovariant terms in the Hamiltonian are canceled by the corresponding ones in the propagators of Dirac and vector particles. This formal identity of the S matrix does not yet establish the complete equivalence of the two formulations since the procedures of evaluating a momentum integral differ in the two theories. One employs the light-front decomposition of a four-momentum vector into $p^{\pm}$ and $\vec{p}$; the other employs the usual space-time decomposition in terms of p^0, p^1, p^2, and p^3. Although these two sets of variables are related by a simple transformation, it turns out that sufficient care must be taken to establish the equivalence between the two procedures of evaluating formally identical invariant-momentum integrals. This result implies in particular that both the light-front formulation and the conventional formulation lead to the same parametric integral representation for an unrenormalized (but properly regularized) Feynman amplitude. Appelquist's[5] prescription for renormalization in terms of parametric integral representations of Feynman amplitudes enables us to conclude that the two formulations also give the same renormalized amplitudes.

Other topics discussed in this paper include the following. A reduction formula for vector particles is given. The Wick theorem for Dirac and vector fields is derived rather than assumed as we did in paper II. Simple examples are discussed in the axial-vector-coupling model in order to illustrate how the noncovariant Hamiltonian and the noncovariant propagators of a spin-one particle conspire to reproduce, for example, the covariant propagator of a spin-one particle with the correct gradient term. Finally a model for high-energy scattering of two energetic particles is proposed. It satisfies manifest S-channel unitarity and is a physical realization of Feynman's parton model.[6]

II. REDUCTION FORMULA

In this section some simple properties of a free spin-one field will be first discussed. The reduction formula for spin-one particles will then be derived. Just as in the earlier derivation of reduction formulas for scalar and Dirac particles, surface terms will simply be ignored without detailed justification.

We begin with the expansion of a free Hermitian vector field B_μ in terms of creation and destruction operators

$$B_\mu(x) = \int \frac{dk^+ d^2k}{[2k^+(2\pi)^3]^{1/2}}\, \theta(k^+) \times \sum_{\lambda=1}^{3} \epsilon_\mu(k,\lambda)[a(k,\lambda)e^{-ik\cdot x} + a^\dagger(k,\lambda)e^{ik\cdot x}], \tag{2.1}$$

where k_μ is an on-shell four-momentum vector

$$k^2 = k^+k^- - \vec{k}^2 = m^2, \tag{2.2}$$

with m being the boson mass; the polarization vectors $\epsilon_\mu(k,\lambda)$ satisfy the properties

$$\begin{aligned} &k^\mu \epsilon_\mu(k,\lambda) = 0, \\ &\epsilon_\mu(k,\lambda)\epsilon^\mu(k,\lambda') = -\delta_{\lambda\lambda'}, \\ &\sum_{\lambda=1}^{3} \epsilon_\mu(k,\lambda)\epsilon_\nu(k,\lambda) = -\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right). \end{aligned} \tag{2.3}$$

The creation and destruction operators $a^\dagger$ and a obey the commutation relations

$$\begin{aligned} &[a(k,\lambda), a^\dagger(k',\lambda')] = \delta_{\lambda\lambda'}\delta(k^+ - k'^+)\delta^2(k-k'), \\ &[a,a] = [a^\dagger, a^\dagger] = 0. \end{aligned} \tag{2.4}$$

The single-particle states defined by

$$|k,\lambda\rangle = a^\dagger(k,\lambda)|0\rangle \tag{2.5}$$

are normalized to

$$\langle k,\lambda|k',\lambda'\rangle = \delta_{\lambda\lambda'}\delta(k^+ - k'^+)\delta^2(k-k'). \tag{2.6}$$

Equations (2.1), (2.3), and (2.4) imply the following representation for the vacuum expectation value of the operator product:

$$\langle 0|B_\mu(x)B_\nu(0)|0\rangle = -\left(g_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{m^2}\right)\Delta^{(+)}(x,m^2), \tag{2.7}$$

where $\Delta^{(+)}$ is the invariant function defined by

$$\Delta^{(\pm)}(x,m^2) = \int \frac{d^4k}{(2\pi)^3}\,\theta(k^+)\delta(k^2-m^2)e^{\mp ik\cdot x}. \tag{2.8}$$

In the light-front formulation the vector field which appears naturally in the interactions is given by

$$\bar{B}_\mu(x) = B_\mu(x) - \partial_\mu\Lambda(x), \tag{2.9}$$

with

$$\Lambda(x) = \tfrac{1}{4}\int dy^-\epsilon(x^- - y^-)B^+(y), \tag{2.10}$$

where y^μ is specified by

$$y^+=x^+,\ \vec{y}=\vec{x},\ \text{and}\ y^- . \tag{2.11}$$

In particular, $\overline{B}^k$ satisfies the equal-x^+ commutation relation

$$x^+=y^+:\quad [\overline{B}^k(x),\overline{B}^l(y)]=-\tfrac{1}{4}i\delta^{kl}\epsilon(x^- - y^-)\delta^2(x-y). \tag{2.12}$$

From (2.1), (2.9), and (2.10) we get

$$\overline{B}_\mu(x)=\int\frac{dk^+d^2k}{[2k^+(2\pi)^3]^{1/2}}\,\theta(k^+)\times\sum_{\lambda=1}^{3}\bar{\epsilon}_\mu(k,\lambda)[a(k,\lambda)e^{-ik\cdot x}+a^\dagger(k,\lambda)e^{ik\cdot x}]. \tag{2.13}$$

The new "polarization vectors" are given by

$$\bar{\epsilon}^\mu(k,\lambda)=\epsilon^\mu(k,\lambda)-\frac{k^\mu}{k^+}\epsilon^+(k,\lambda), \tag{2.14}$$

with the properties

$$\begin{aligned}
&\bar{\epsilon}^+\equiv 0,\\
&k_\mu\bar{\epsilon}^\mu(k,\lambda)=-\frac{m^2}{k^+}\epsilon^+(k,\lambda),\\
&\bar{\epsilon}_\mu(k,\lambda)\bar{\epsilon}^\mu(k,\lambda')=-\delta_{\lambda\lambda'}+\frac{m^2}{(k^+)^2}\epsilon^+(k,\lambda)\epsilon^+(k,\lambda'),\\
&\sum_{\lambda=1}^{3}\bar{\epsilon}^\mu(k,\lambda)\bar{\epsilon}^\nu(k,\lambda)=-g^{\mu\nu}+g^{+\mu}\frac{k^\nu}{k^+}+g^{+\nu}\frac{k^\mu}{k^+}.
\end{aligned} \tag{2.15}$$

The analog of (2.7) for $\overline{B}_\mu$ is

$$\langle 0|\overline{B}^\mu(x)\overline{B}^\nu(0)|0\rangle=-\left(g^{\mu\nu}-g^{+\mu}\frac{\partial^\nu}{\partial^+}-g^{+\nu}\frac{\partial^\mu}{\partial^+}\right)\Delta^{(+)}(x,m^2), \tag{2.16}$$

where $1/\partial^+$ is the integral operator

$$\frac{1}{\partial^+}\Delta^{(+)}(x,m^2)=\tfrac{1}{4}\int dy^-\,\epsilon(x^- - y^-)\Delta^{(+)}(y,m^2). \tag{2.17}$$

It now follows from (2.16) that

$$\begin{aligned}
\langle 0|T^+(\overline{B}^\mu(x)\overline{B}^\nu(0))|0\rangle
&=-\left(g^{\mu\nu}-g^{+\mu}\frac{\partial^\nu}{\partial^+}-g^{+\nu}\frac{\partial^\mu}{\partial^+}\right)i\Delta_F(x,m^2)\\
&\quad+g^{+\mu}g^{+\nu}\tfrac{1}{4}i|x^-|\delta(x^+)\delta^2(x),
\end{aligned} \tag{2.18}$$

where T^+ stands for x^+-ordering and

$$\Delta_F(x,m^2)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\frac{1}{k^2-m^2+i\epsilon}. \tag{2.19}$$

This completes the brief discussion on a free vector field. We now turn to the derivation of the reduction formula for vector particles.[7] Introduce the wave function for a vector particle

$$B^\mu_{k,\lambda}(x)=\frac{1}{[2k^+(2\pi)^3]^{1/2}}\epsilon^\mu(k,\lambda)e^{-ik\cdot x}. \tag{2.20}$$

Equation (2.13) can be inverted to give

$$\begin{aligned}
a(k,\lambda)&=-\tfrac{1}{2}i\int dx^-d^2x\,B^\mu_{k,\lambda}(x)^*\overleftrightarrow{\partial}^+\overline{B}_\mu(x)\\
&=i\int dx^-d^2x\,\partial^+B^\mu_{k,\lambda}(x)^*\overline{B}_\mu(x).
\end{aligned} \tag{2.21}$$

The second step follows since $\partial^+=2\partial/\partial x^-$ is a "space" rather than a "time" derivative. Similarly

$$a^\dagger(k,\lambda)=-i\int dx^-d^2x\,\partial^+B^\mu_{k,\lambda}(x)\overline{B}_\mu(x). \tag{2.22}$$

Equations (2.21) and (2.22) apply to both the in- and out-field associated with a vector particle.

For the reduction of a vector particle of momentum k, polarization λ in the in-state for the reaction $\alpha+(k,\lambda)\to\beta$, we follow the usual procedure

$$\begin{aligned}
\langle\beta\,\text{out}|\alpha,(k,\lambda)\,\text{in}\rangle&=\langle\beta\,\text{out}|a^\dagger_{\text{in}}(k,\lambda)|\alpha\,\text{in}\rangle\\
&=\langle\beta-(k,\lambda)\,\text{out}|\alpha\,\text{in}\rangle\\
&\quad+\langle\beta\,\text{out}|[a^\dagger_{\text{in}}(k,\lambda)-a^\dagger_{\text{out}}(k,\lambda)]|\alpha\,\text{in}\rangle.
\end{aligned} \tag{2.23}$$

Making use of (2.22) for in- and out-fields, we have

$$\langle\beta\,\text{out}|\alpha,(k,\lambda)\,\text{in}\rangle=\langle\beta-(k,\lambda)\,\text{out}|\alpha\,\text{in}\rangle-i\int dx^-d^2x\,\partial^+B^\mu_{k,\lambda}(x)\langle\beta\,\text{out}|[\overline{B}_{\text{in}\,\mu}(x)-\overline{B}_{\text{out}\,\mu}(x)]|\alpha\,\text{in}\rangle.$$

The weak asymptotic conditions[8]

$$\overline{B}_\mu(x)\xrightarrow[x^+\to+\infty]{}\sqrt{Z_3}\,\overline{B}_{\text{out}\,\mu}(x),\quad \overline{B}_\mu(x)\xrightarrow[x^+\to-\infty]{}\sqrt{Z_3}\,\overline{B}_{\text{in}\,\mu}(x) \tag{2.25}$$

then imply

$$\langle\beta\,\text{out}|\alpha,(k,\lambda)\,\text{in}\rangle=\langle\beta-(k,\lambda)\,\text{out}|\alpha\,\text{in}\rangle+\frac{i}{\sqrt{Z_3}}\left(\lim_{x^+\to+\infty}-\lim_{x^+\to-\infty}\right)\int dx^-d^2x\,\partial^+B^\mu_{k,\lambda}(x)\langle\beta\,\text{out}|\overline{B}_\mu(x)|\alpha\,\text{in}\rangle \tag{2.26}$$

or

$$\langle \beta \text{out} | \alpha, (k,\lambda) \text{in} \rangle = \langle \beta - (k,\lambda) \text{out} | \alpha \text{ in} \rangle + \frac{i}{\sqrt{Z_3}} \int d^4x\, \partial^- [\partial^+ B^{\mu}_{k,\lambda}(x) \langle \beta \text{out} | \overline{B}_{\mu}(x) | \alpha \text{ in} \rangle] . \tag{2.27}$$

Since $B^{\mu}_{k,\lambda}(x)$ satisfies the Klein-Gordon equation,

$$(\partial^2 + m^2) B^{\mu}_{k,\lambda}(x) = (\partial^+\partial^- - \vec{\nabla}^2 + m^2) B^{\mu}_{k,\lambda}(x) = 0 , \tag{2.28}$$

(2.27) can be cast in the form

$$\langle \beta \text{out} | \alpha, (k,\lambda) \text{in} \rangle = \langle \beta - (k,\lambda) \text{out} | \alpha \text{ in} \rangle - \frac{i}{\sqrt{Z_3}} \int d^4x\, B^{\mu}_{k,\lambda}(x) (\partial^2 + m^2) \langle \beta \text{out} | \overline{B}_{\mu}(x) | \alpha \text{ in} \rangle . \tag{2.29}$$

This is the desired result.

Formulas for removing two or more vector particles from the in-state can be worked out similarly. Corresponding formulas for removing vector particles from the out-state can also be derived in similar fashion. The reduction procedure can be continued until all the particles are removed from the state vectors. For a given process involving Dirac and vector particles, the connected contribution is simply given by the product of individual one-particle factors (wave function and Klein-Gordon or Dirac differential operator) operating on the vacuum expectation value of Heisenberg fields

$$\langle 0 | T^+ (\overline{B}_{\mu_1}(x_1) \cdots \overline{B}_{\mu_n}(x_n) \psi(y_1) \cdots \psi(y_m) \overline{\psi}(z_1) \cdots \overline{\psi}(z_l)) | 0 \rangle .$$

For practical calculations a transition to the interaction picture is performed. The operators in the two pictures are related by a unitary transformation

$$O(x) = U^{-1}(x^+) O_{\text{in}}(x) U(x^+) , \tag{2.30}$$

with

$$U(x^+) = T^+ \exp\left[-i \int_{-\infty}^{x^+} d^4x\, \mathcal{H}_I(x) \right] , \tag{2.31}$$

where operators in U are in the interaction picture. Thus we are led to calculate the vacuum expectation values of the x^+-ordered products of free fields. This calculation is facilitated by the Wick theorem to be proved in Sec. III.

The Wick theorem states that the vacuum expectation value of an x^+-ordered product of a free vector field $\overline{B}^{\mu}$ is given by the sum of all possible contractions between pairs of vector fields with the following substitution:

$$\begin{aligned} \overline{B}^{\mu}(x)^{\bullet} \overline{B}^{\nu}(y)^{\bullet} &= \langle 0 | T^+ (\overline{B}^{\mu}(x) \overline{B}^{\nu}(y)) | 0 \rangle \\ &= -\left(g^{\mu\nu} - g^{+\mu} \frac{\partial^{\nu}}{\partial^+} - g^{+\nu} \frac{\partial^{\mu}}{\partial^+} \right) i \Delta_F(x-y, m^2) \\ &\quad + g^{+\mu} g^{+\nu} \tfrac{1}{4} i \, | x^- - y^- | \, \delta(x^+ - y^-) \delta^2(x-y) . \end{aligned} \tag{2.32}$$

Because of the Klein-Gordon operator $(\partial^2 + m^2)$ in the reduction formula (2.29) and its generalization to more particles, the last term in (2.32) which occurs in the final contractions with the external lines does not contribute to the S matrix since this term has no pole at $k^2 = m^2$ to cancel the Klein-Gordon operator $(k^2 - m^2)$ in momentum space. The gradient terms in (2.32) associated with external-line contractions do not contribute either since

$$\begin{aligned} k_{\mu} B^{\mu}_{k,\lambda\nu}(x) &= \partial_{\mu} B^{\mu}_{k,\lambda}(x) \\ &= 0 . \end{aligned} \tag{2.33}$$

Because of (2.33) either $B_{\mu}(x)$ or $\overline{B}_{\mu}(x)$ can be used in the reduction formula (2.29). The additional terms produced by moving the gradient operator ∂_{μ} through the x^+-ordering sign have no poles at $k^2 = m^2$.

III. WICK THEOREM

In paper II the Wick theorem for scalar and Dirac fields was assumed. The Wick theorem states that the vacuum expectation value of an x^+-ordered product of free fields $\phi(x)$, $\overline{\psi}(x)$, and $\psi(x)$ for scalar and Dirac particles is given by the sum of all possible contractions between pairs of operators with the following substitution

$$\begin{aligned} \phi(x)^{\bullet} \phi(0)^{\bullet} &= T^+ \langle 0 | \phi(x) \phi(0) | 0 \rangle \\ &= i \Delta_F(x) , \\ \psi(x)^{\bullet} \overline{\psi}(0)^{\bullet} &= T^+ \langle 0 | \psi(x) \overline{\psi}(0) | 0 \rangle \\ &\equiv i \overline{S}_F(x) \\ &= i S_F(x) - \tfrac{1}{4} \gamma^+ \epsilon(x^-) \delta(x^+) \delta^2(x) . \end{aligned} \tag{3.1}$$

In this section we will derive this rule (3.1) for a Dirac field and the corresponding rule for a vector field. The scalar-field case is so trivial that it will not be discussed. To derive the theorem the external-source technique of Schwinger[3] will be employed. In this derivation how the various noncovariant terms arise in the contractions will be made explicit.

A. Wick Theorem for Dirac Field

Consider the Lagrange function for a Dirac field coupled to an anticommuting c-number source η:

$$\mathcal{L} = \bar{\psi}(\gamma^\mu i\partial_\mu - M)\psi + \bar{\psi}\eta + \bar{\eta}\psi . \tag{3.2}$$

The field equations implied by (3.2) are

$$(\gamma^\mu i\partial_\mu - M)\psi = -\eta \tag{3.3}$$

and its Hermitian conjugate, or

$$\begin{aligned} i\partial^- \psi^{(+)} &= \gamma^0[(i\gamma_k\partial_k + M)\psi^{(-)} - \eta^{(-)}] ,\\ i\partial^+ \psi^{(-)} &= \gamma^0[(i\gamma_k\partial_k + M)\psi^{(+)} - \eta^{(+)}] ,\end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \eta^{(\pm)} &= \Lambda^{(\pm)}\eta ,\\ \Lambda^{(\pm)} &= \tfrac{1}{2}(1 \pm \gamma^0\gamma^3) .\end{aligned} \tag{3.5}$$

The second equation in (3.4) can be integrated to give

$$\psi^{(-)}(x) = -\tfrac{1}{4}i \int dy^- \,\epsilon(x^- - y^-)\gamma^0 \times[(i\gamma_k\partial_k + M)\psi^{(+)}(y) - \eta^{(+)}(y)] . \tag{3.6}$$

The response of the vacuum-to-vacuum amplitude $\langle 0\,\text{out}|0\,\text{in}\rangle$ to the variation of the external source is, according to Schwinger's action principle,

$$\begin{aligned} \delta_{\bar\eta}\langle 0\,\text{out}|0\,\text{in}\rangle &= i\int d^4x \langle 0\,\text{out}|\delta\mathcal{L}(x)|0\,\text{in}\rangle \\ &= i\int d^4x\, \delta\bar\eta(x)\langle 0\,\text{out}|\psi(x)|0\,\text{in}\rangle \end{aligned} \tag{3.7}$$

or

$$\frac{\delta_l}{\delta\bar\eta(x)}\langle 0\,\text{out}|0\,\text{in}\rangle = i\langle 0\,\text{out}|\psi(x)|0\,\text{in}\rangle , \tag{3.8}$$

where the subscript l refers to the "left derivative."[9] Application of the field equation (3.3) then gives

$$\begin{aligned} (\gamma^\mu i\partial_\mu - M)\frac{\delta_l}{\delta\bar\eta(x)}\langle 0\,\text{out}|0\,\text{in}\rangle &= i\langle 0\,\text{out}|(\gamma^\mu i\partial_\mu - M)\psi(x)|0\,\text{in}\rangle \\ &= -i\eta(x)\langle 0\,\text{out}|0\,\text{in}\rangle . \end{aligned} \tag{3.9}$$

The space-time integration can be done with the aid of the Green's function

$$(\gamma^\mu i\partial_\mu - M)S_F(x-y) = \delta^4(x-y) , \tag{3.10}$$

which satisfies the appropriate outgoing-wave boundary conditions. The result is

$$\frac{\delta_l}{\delta\bar\eta(x)}\langle 0\,\text{out}|0\,\text{in}\rangle = -i\int d^4y\, S_F(x-y)\eta(y)\langle 0\,\text{out}|0\,\text{in}\rangle . \tag{3.11}$$

The $\bar\eta$ integration then gives

$$\langle 0\,\text{out}|0\,\text{in}\rangle = \exp\left[-i\int d^4x d^4y\, \bar\eta(x)S_F(x-y)\eta(y)\right] . \tag{3.12}$$

The multiplicative constant is determined by the condition

$$\langle 0\,\text{out}|0\,\text{in}\rangle|_{\eta,\bar\eta=0} = 1 . \tag{3.13}$$

Another representation for $\langle 0\,\text{out}|0\,\text{in}\rangle$ in terms of operator products is obtained by calculating the higher derivatives of $\langle 0\,\text{out}|0\,\text{in}\rangle$ with respect to the external source.

To calculate a second variation we rewrite (3.7) as

$$\delta_{\bar\eta}\langle 0\,\text{out}|0\,\text{in}\rangle = i\int d^4x\, \delta\bar\eta(x)\langle 0\,\text{out}|x^+\rangle X\langle x^+|\psi(x)|x^+\rangle X\langle x^+|0\,\text{in}\rangle , \tag{3.14}$$

where $|x^+\rangle X\langle x^+|$ symbolizes a summation over a complete set of states specified at x^+. A second variation applied to (3.14) involves the responses

$$\begin{aligned} \delta_\eta\langle 0\,\text{out}|x^+\rangle &= i\int d^4y\langle 0\,\text{out}|\bar\psi(y)|x^+\rangle\delta\eta(y) ,\\ \delta_\eta\langle x^+|0\,\text{in}\rangle &= -i\int d^4y\,\delta\eta(y)\langle x^+|\bar\psi(y)|0\,\text{in}\rangle , \end{aligned} \tag{3.15}$$

and the response of $\psi(x)$ as a result of its explicit dependence on η [Eq. (3.4)],

$$\delta'_\eta \psi(x) = +i \int d^4y \, \tfrac{1}{4}\gamma^+ \epsilon(x^- - y^-)\delta(x^+ - y^+)\delta^2(x-y)\delta\eta(y), \tag{3.16}$$

where δ'_η refers to response due to explicit dependence. Equations (3.14)–(3.16) combine to yield

$$\delta_\eta \delta_{\bar\eta} \langle 0\,\text{out}|0\,\text{in}\rangle = i^2 \int d^4x\, d^4y\, \delta\bar\eta(x) \langle 0\,\text{out}|[T^+(\psi(x)\bar\psi(y)) + \tfrac{1}{4}\gamma^+\epsilon(x^- - y^-)\delta(x^+ - y^+)\delta^2(x-y)]|0\,\text{in}\rangle\, \delta\eta(y) \tag{3.17}$$

or

$$\frac{\delta_r}{\delta\eta(y)} \frac{\delta_l}{\delta\bar\eta(x)} \langle 0\,\text{out}|0\,\text{in}\rangle = i^2 \langle 0\,\text{out}|[T^+(\psi(x)\bar\psi(y)) + \tfrac{1}{4}\gamma^+\epsilon(x^- - y^-)\delta(x^+ - y^+)\delta^2(x-y)]|0\,\text{in}\rangle, \tag{3.18}$$

where left and right derivatives[9] are distinguished by the subscripts l and r. The explicit dependence of $\psi^{(-)}$ on $\eta^{(-)}$ gives rise to the noncovariant terms. These terms can be consistently and completely removed by introducing $\langle 0\,\text{out}|0\,\text{in}\rangle'$ defined by

$$\langle 0\,\text{out}|0\,\text{in}\rangle = \langle 0\,\text{out}|0\,\text{in}\rangle' \exp\left[i \int d^4x d^4y \, \tfrac{1}{4} i \epsilon(x^- - y^-)\delta(x^+ - y^+)\delta^2(x-y)\bar\eta(x)\gamma^+\eta(y)\right] . \tag{3.19}$$

Then

$$\frac{\delta_r}{\delta\eta(y_1)} \cdots \frac{\delta_r}{\delta\eta(y_n)} \frac{\delta_l}{\delta\bar\eta(x_m)} \cdots \frac{\delta_l}{\delta\bar\eta(x_1)} \langle 0\,\text{out}|0\,\text{in}\rangle' \Big|_{\eta,\bar\eta=0} = i^{m+n} \langle 0| T^+(\psi(x)\cdots\psi(x_m)\bar\psi(y_1)\cdots\bar\psi(y_n))|0\rangle . \tag{3.20}$$

The Taylor expansion for $\langle 0\,\text{out}|0\,\text{in}\rangle'$ around $\eta, \bar\eta = 0$ is therefore

$$\langle 0\,\text{out}|0\,\text{in}\rangle' = \left\langle 0 \left| T^+ \exp\left\{ i \int d^4x [\bar\psi(x)\eta(x) + \bar\eta(x)\psi(x)]\right\} \right| 0 \right\rangle \tag{3.21}$$

or

$$\langle 0\,\text{out}|0\,\text{in}\rangle = \left\langle 0 \right| T^+ \exp\left\{ i \int d^4x \left[\bar\psi(x)\eta(x) + \bar\eta(x)\psi(x) + \tfrac{1}{4} i \int d^4y\, \epsilon(x^- - y^-)\delta(x^+ - y^+)\delta^2(x-y)\bar\eta(x)\gamma^+\eta(y)\right]\right\} \left| 0 \right\rangle . \tag{3.22}$$

Equating (3.12) and (3.22) we obtain

$$\exp\left\{ -i \int d^4x d^4y\, \bar\eta(x) S_F(x-y)\eta(y)\right\}$$
$$= \left\langle 0 \right| T^+ \exp\left\{ i \int d^4x \left[\bar\psi(x)\eta(x) + \bar\eta(x)\psi(x) + \tfrac{1}{4} i \int d^4y\, \epsilon(x^- - y^-)\delta(x^+ - y^+)\delta^2(x-y)\bar\eta(x)\gamma^+\eta(y)\right]\right\} \left| 0 \right\rangle , \tag{3.23}$$

which is the content of the Wick's theorem for the Dirac field. In particular, it gives the rule stated in (3.1).

Equation (3.22) is a simple example of Dyson's formula for the S matrix. The energy operator associated with this system is given by

$$P^- = \int dx^- d^2x [\psi^{(+)\dagger}\gamma^0(i\gamma_k\partial_k + M)\psi^{(-)} - \psi^{(+)\dagger}\gamma^0\eta^{(-)} - \eta^{(+)*}\gamma^0\psi^{(-)} - \eta^{(-)*}\gamma^0\psi^{(+)}], \tag{3.24}$$

where $\psi^{(-)}$ is given by (3.6). In terms of the free-field operators in the absence of external source with the help of (3.6), the interaction Hamiltonian density is then

$$P^-(\eta,\bar\eta) - P^-(\eta = \bar\eta = 0) = \int dx^- d^2x\, \mathcal{H}_I ,$$
$$\mathcal{H}_I(x) = -\bar\psi(x)\eta(x) - \bar\eta(x)\psi(x) - \tfrac{1}{8} i \int dy^- \, \epsilon(x^- - y^-)\bar\eta(x)\gamma^+\eta(y) . \tag{3.25}$$

The Dyson formula for the S matrix

$$S=T^{+}\exp\left[-i\int d^4x\,\mathcal{H}_I(x)\right] \tag{3.26}$$

reproduces (3.22). For simplicity we have used the same notations ψ and $\overline{\psi}$ in (3.25) for the free field operators which refer to the noninteracting situation.

B. Wick Theorem for Vector Field

Consider a vector field B_μ coupled to a nonconserved commuting c-number external current J^μ. The Lagrange function is

$$\mathcal{L}=-\tfrac{1}{2}B^{\mu\nu}(\partial_\mu B_\nu-\partial_\nu B_\mu)+\tfrac{1}{4}B^{\mu\nu}B_{\mu\nu}+\tfrac{1}{2}m^2B^\mu B_\mu-J_\mu B^\mu\,. \tag{3.27}$$

The vacuum-to-vacuum transition amplitude $\langle 0\,\text{out}|0\,\text{in}\rangle$ for this system can be solved exactly. Following a similar procedure used for a Dirac field, we obtain the well-known structure

$$\langle 0\,\text{out}|0\,\text{in}\rangle=\exp\left[\tfrac{1}{2}i\int d^4x d^4y\,J^\mu(x)\left(g_{\mu\nu}+\frac{\partial_\mu\partial_\nu}{m^2}\right)\Delta_F(x-y)J^\nu(y)\right]\,. \tag{3.28}$$

To derive an alternative expression for $\langle 0\,\text{out}|0\,\text{in}\rangle$ in terms of operator products it is convenient to rearrange the coupling term as

$$J^\mu B_\mu=J^\mu(\overline{B}_\mu+\partial_\mu\Lambda)\to J^\mu\overline{B}_\mu-K\Lambda\,, \tag{3.29}$$

where

$$K(x)=\partial_\mu J^\mu(x) \tag{3.30}$$

and

$$\Lambda(x)=\tfrac{1}{4}\int dy^-\epsilon(x^--y^-)B^+(y)\,. \tag{3.31}$$

One can derive the analog of (3.22) for a vector field from a similar procedure used before in terms of higher variations. Since the result is anticipated to be identical to a direct application of Dyson's formula, we need only calculate the energy operator. Standard procedure then gives

$$P^-=\int dx^-d^2x\left[\tfrac{1}{8}(B^{+-})^2+\tfrac{1}{4}(B^{kl})^2+\tfrac{1}{2}m^2(B^k)^2-J_k\overline{B}_k-K\Lambda\right]. \tag{3.32}$$

In terms of the independent dynamical variables B^{+k} and B^+, we have

$$\begin{aligned}
B^{+-}(x)&=\tfrac{1}{2}\int dy^-\,\epsilon(x^--y^-)\left[\partial^kB^{+k}(y)+m^2B^+(y)-J^+(y)\right],\\
B^k(x)&=\tfrac{1}{4}\int dy^-\,\epsilon(x^--y^-)\left[B^{+k}(y)+\partial^kB^+(y)\right],\\
B^{kl}&=\partial^kB^l-\partial^lB^k\,.
\end{aligned} \tag{3.33}$$

The interaction Hamiltonian density in the interaction picture implied by (3.32) and (3.33) is

$$\mathcal{H}_I(x)=J_\mu\overline{B}^\mu-K\Lambda-\tfrac{1}{8}\int d^4y\,|x^--y^-|\,\delta(x^+-y^+)\delta^2(x-y)J^+(x)J^+(y)\,. \tag{3.34}$$

Dyson's formula for the S matrix,

$$S=T^+\exp\left[-i\int d^4x\mathcal{H}_I(x)\right]\,,$$

then provides alternatively

$$\langle 0\,\text{out}|0\,\text{in}\rangle=\left\langle 0\left|T^+\exp\left[-i\int d^4x\,\mathcal{H}_I(x)\right]\right|0\right\rangle\,. \tag{3.35}$$

Equations (3.28), (3.35), and (3.34) supply the relation[10]

$$\left\langle 0\right| T^+\exp\left\{-i\int d^4x\left[J^\mu\overline{B}_\mu - K\Lambda - \tfrac{1}{8}\int d^4y|x^- - y^-|\delta(x^+-y^+)\delta^2(x-y)J^+(x)J^+(y)\right]\right\}\left|0\right\rangle$$
$$= \exp\left\{\tfrac{1}{2}\int d^4x d^4y\left[-J^\mu(x)i\Delta'_{\mu\nu}(x-y)J^\nu(y) + g^{+\mu}J_\mu(x)\frac{1}{\partial^+}i\Delta_F(x-y)K(y)\right.\right.$$
$$\left.\left. -K(x)\frac{1}{\partial^+}i\Delta_F(x-y)g^{+\nu}J_\nu(y) - iK(x)\frac{1}{m^2}\Delta_F(x-y)K(y)\right]\right\}, \tag{3.36}$$

where

$$\Delta'^{\mu\nu}(x-y) \equiv -\left(g^{\mu\nu} - g^{+\mu}\frac{\partial^\nu}{\partial^+} - g^{+\nu}\frac{\partial^\mu}{\partial^+}\right)\Delta_F(x-y). \tag{3.37}$$

The $J^+(x)J^+(y)$ term in (3.36) arises from the explicit dependence of B^{+-} on J^+ as given by (3.33). Equation (3.36) is the Wick theorem for a vector field. It leads to the following rules for contractions of two field operators in a vacuum expectation value of an x^+-ordered product:

$$\overline{B}^\mu(x)^\bullet\overline{B}^\nu(0)^\bullet = \langle 0|T^+(\overline{B}^\mu(x)\overline{B}^\nu(0))|0\rangle$$
$$\equiv i\overline{\Delta}^{\mu\nu}(x)$$
$$= i\Delta'^{\mu\nu}(x) + \tfrac{1}{4}i g^{+\mu}g^{+\nu}|x^-|\delta(x^+)\delta^2(x), \tag{3.38}$$

$$\overline{B}^\mu(x)^\bullet\Lambda(0)^\bullet = \langle 0|T^+(\overline{B}^\mu(x)\Lambda(0))|0\rangle$$
$$= +g^{+\mu}\frac{1}{\partial^+}i\Delta_F(x), \tag{3.39}$$

$$\Lambda(x)^\bullet\Lambda(0)^\bullet = \langle 0|T^+(\Lambda(x)\Lambda(0))|0\rangle$$
$$= \frac{i}{m^2}\Delta_F(x). \tag{3.40}$$

This decomposition of a vector field B^μ into $\overline{B}^\mu$ and Λ is particularly useful since these field operators naturally occur in the interaction Hamiltonians in the light-front formulation of field theories of vector particles. For a vector field coupled to a conserved current only $\overline{B}^\mu$ appears. Equations (3.39) and (3.40) are needed only when the coupled current is not conserved, as in the case of the axial-vector gluon model considered in paper III.

IV. THE S MATRIX

Before we give a formal proof in Sec. V that the vector-gluon model quantized in paper III leads to the same S matrix as in the conventional formalism, we will first derive the S matrix in the interaction picture in this section. At the end of the section we will also calculate some low-order diagrams in the axial-vector-coupling model to demonstrate in these simple cases that complete cancellation occurs between the noncovariant terms in the propagators and those in the interaction Hamiltonian; the resultant S matrix is covariant and identical to the well-known results.

A. S Matrix in the Vector-Gluon Model

The energy operator for the vector-gluon model described in Sec. III B of paper III is

$$P^- = \tfrac{1}{2}\int dx^- d^2x\{\tfrac{1}{4}(B^{+-})^2 + \tfrac{1}{2}(B^{kl})^2 + m^2(B^k)^2 + 2\psi^{(+)\dagger}\gamma^0[\gamma_k(\tfrac{1}{2}i\overleftrightarrow{\partial}_k - g\overline{B}_k) + M]\psi^{(-)}\}. \tag{4.1}$$

All field operators in (4.1) can be expressed in terms of the independent variables B^{+k}, B^+, and $\psi^{(+)}$ with the aid of the constraint equations[2]

$$B^k(x)=\tfrac{1}{4}\int dy^-\,\epsilon(x^--y^-)[B^{+k}(y)+\partial^kB^+(y)],$$

$$B^{kl}=\partial^kB^l-\partial^lB^k,$$

$$\bar{B}^k(x)=\tfrac{1}{4}\int dy^-\,\epsilon(x^--y^-)B^{+k}(y), \tag{4.2}$$

$$B^{+-}(x)=\tfrac{1}{2}\int dy^-\,\epsilon(x^--y^-)[\partial^kB^{+k}(y)+m^2B^+(y)-j^+(y)],$$

$$\psi^{(-)}(x)=-\tfrac{1}{4}i\int dy^-\,\epsilon(x^--y^-)\gamma^0\{\gamma_k[i\partial_k-g\bar{B}_k(y)]+M\}\psi^{(+)}(y).$$

Let us denote the interaction-independent part of a field operator by a subscript 0; then

$$B^k=B_0^k,\quad \bar{B}^k=\bar{B}_0^k,$$

$$B^{+k}=B_0^{+k},\quad B^+=B_0^+,\quad \psi^{(+)}=\psi_0^{(+)},$$

$$B^{+-}(x)=B_0^{+-}(x)-\tfrac{1}{2}\int dy^-\,\epsilon(x^--y^-)j_0^+(y), \tag{4.3}$$

$$\psi^{(-)}(x)=\psi_0^{(-)}(x)+\tfrac{1}{4}ig\int dy^-\,\epsilon(x^--y^-)\gamma^0\gamma_k\bar{B}_{k0}^{(y)}\psi_0^{(+)}(y),$$

where

$$B_0^{+-}(x)=\tfrac{1}{2}\int dy^-\epsilon(x^--y^-)[\partial^kB^{+k}(y)+m^2B^+(y)],$$

$$\psi_0^{(-)}(x)=-\tfrac{1}{4}i\int dy^-\epsilon(x^--y^-)\gamma^0[\gamma_ki\partial_k+M]\psi^{(+)}(y). \tag{4.4}$$

Collecting terms with and without dependence on the coupling constant g separately, we obtain from (4.1)–(4.4)

$$P^-=P_0^-+P_I^-, \tag{4.5}$$

where P_0^- is identical in form to (4.1), by setting $g=0$, and

$$P_I^-=\int dx^-d^2x\,\mathcal{H}_I(x), \tag{4.6}$$

with

$$\mathcal{H}_I(x)=g\bar{\psi}(x)\gamma^\mu\psi(x)\bar{B}_\mu(x)+\tfrac{1}{32}g^2\int dy_1^-dy_2^-\epsilon(x^--y_1^-)\epsilon(x^--y_2^-)\bar{\psi}(y_1)\gamma^+\psi(y_1)\bar{\psi}(y_2)\gamma^+\psi(y_2)$$
$$-\tfrac{1}{8}ig^2\int dy^-\epsilon(x^--y^-)\bar{B}^\mu(x)\bar{\psi}(x)\gamma_\mu\gamma^+\gamma_\nu\psi(y)\bar{B}^\nu(y), \tag{4.7}$$

where y^μ, y_1^μ, y_2^μ differ from x^μ only in their "minus" components. We have deliberately omitted the subscript 0 from the field operators in (4.7). But it should be understood that all field operators in (4.7) are the interaction-independent parts as defined in (4.3) and (4.4). Notice that the last term in (4.7) can be simplified since

$$\bar{B}^\mu(x)\bar{\psi}(x)\gamma_\mu\gamma^+\gamma_\nu\psi(y)\bar{B}^\nu(y)=\bar{B}_k(x)\bar{\psi}(x)\gamma_k\gamma^+\gamma_l\psi(y)\bar{B}_l(y), \tag{4.8}$$

as a result of

$$\bar{B}^+\equiv 0,\quad (\gamma^+)^2=0. \tag{4.9}$$

But we prefer the covariant-looking expression (4.7).

When transformed to the interaction picture, the interaction Hamiltonian $\mathcal{H}_I$ is precisely given by (4.7) with all field operators defined in interaction picture, i.e., they are free field operators. Notice that the interaction Hamiltonian bears no simple relation to the interaction Lagrangian

$$\mathcal{L}_I=-g\bar{\psi}\gamma_\mu\psi\bar{B}^\mu. \tag{4.10}$$

In Sec. V it will be shown that to all orders in perturbation the second and third term in (4.7) are canceled by the noncovariant terms in the propagators for vector and Dirac particles, respectively. The resultant S matrix is then identical to the one found in the conventional formalism.

B. S Matrix in Axial-Vector Coupling

The energy operator in this model of an axial-vector field coupled to a nonconserved axial-vector current is given in paper III. Following a similar procedure employed in Sec. IV A, the interaction Hamiltonian for this axial-vector-gluon model is

$$\begin{aligned}\mathcal{H}_I(x) = {} & g\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)\bar{a}_\mu(x) - \tfrac{1}{16}g^2\int dy^-|x^- - y^-|\bar{\psi}(x)\gamma^+\psi(x)\bar{\psi}(y)\gamma^+\psi(y)\\ & -\tfrac{1}{8}ig^2\int dy^-\,\epsilon(x^- - y^-)\bar{a}^\mu(x)\bar{\psi}(x)\gamma_\mu\gamma^+\gamma_\nu\psi(y)\bar{a}^\nu(y) + M\bar{\psi}(x)[I(x)-1]\psi(x)\\ & -\tfrac{1}{8}igM\int dy^-\,\epsilon(x^- - y^-)\bar{a}_\mu(x)\bar{\psi}(x)\gamma^+\gamma^\mu\gamma_5[I(y)-1]\psi(y)\\ & -\tfrac{1}{8}igM\int dy^-\,\epsilon(x^- - y^-)\bar{\psi}(x)\gamma^+[I^{-1}(x)-1]\gamma^\mu\gamma_5\psi(y)\bar{a}_\mu(y)\\ & -\tfrac{1}{8}iM^2\int dy^-\,\epsilon(x^- - y^-)\bar{\psi}(x)\gamma^+[I^{-1}(x)-1][I(y)-1]\psi(y),\end{aligned} \tag{4.11}$$

where $I(x)$ is defined by

$$I(x) = e^{-2ig\gamma_5\lambda(x)}, \tag{4.12}$$

$$\lambda(x) = \tfrac{1}{4}\int dy^-\,\epsilon(x^- - y^-)a^+(y). \tag{4.13}$$

Since this theory is not renormalizable it seems pointless to discuss the whole S matrix associated with the $\mathcal{H}_I$ given by (4.11). Therefore we will content ourselves with application of (4.11) to second-order diagrams, one without and one with a closed loop. We shall specifically discuss the fermion-fermion interaction, and virtual as well as real "Compton scattering." The purpose is to illustrate how such a noncovariant looking $\mathcal{H}_I$ of (4.11) and the noncovariant propagators (3.38)–(3.40) conspire to recover the well-known covariant result.

We first expand $\mathcal{H}_I$ to the second order as desired. To this order we have

$$I(x) - 1 = -2gi\gamma_5\lambda(x) - 2g^2[\lambda(x)]^2 \tag{4.14}$$

and

$$\begin{aligned}\mathcal{H}_I(x) = {} & g\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x)\bar{a}_\mu(x) - 2Mg\bar{\psi}(x)i\gamma_5\psi(x)\lambda(x)\\ & -\tfrac{1}{16}g^2\int dy^-|x^- - y^-|\bar{\psi}(x)\gamma^+\gamma_5\psi(x)\bar{\psi}(y)\gamma^+\gamma_5\psi(y) - \tfrac{1}{8}g^2\int dy^-\epsilon(x^- - y^-)\bar{a}^\mu(x)\bar{\psi}(x)\gamma_\mu\gamma^+\gamma_\nu\psi(y)\bar{a}^\nu(y)\\ & -2Mg^2\bar{\psi}(x)\psi(x)[\lambda(x)]^2 - \tfrac{1}{4}Mg^2\int dy^-\epsilon(x^- - y^-)\bar{a}_\mu(x)\bar{\psi}(x)\gamma^+\gamma^\mu\psi(y)\lambda(y)\\ & -\tfrac{1}{4}Mg^2\int dy^-\epsilon(x^- - y^-)\lambda(x)\bar{\psi}(x)\gamma^+\gamma^\mu\psi(y)\bar{a}_\mu(y) - \tfrac{1}{2}iM^2g^2\int dy^-\epsilon(x^- - y^-)\lambda(x)\bar{\psi}(x)\gamma^+\psi(y)\lambda(y).\end{aligned} \tag{4.15}$$

The fermion-fermion interaction will be discussed first. The S matrix for this process can be obtained from (4.15) by the Dyson formula

$$S = T^+\exp\left[-i\int d^4x\,\mathcal{H}_I(x)\right]. \tag{4.16}$$

It is

$$S_{f-f} = g^2T^+ \int d^4x d^4y \{(-\tfrac{1}{2})[\bar\psi(x)\gamma^\mu\gamma_5\psi(x)\bar a_\mu(x)\bar a_\nu(y)\bar\psi(y)\gamma^\nu\gamma_5\psi(y) - 4M\bar\psi(x)\gamma^\mu\gamma_5\psi(x)\bar a_\mu(x)\lambda(y)\bar\psi(y)i\gamma_5\psi(y) + 4M^2\bar\psi(x)i\gamma_5\psi(x)\lambda(x)\lambda(y)\bar\psi(y)i\gamma_5\psi(y)] + \tfrac{1}{8}ig^2\delta^2(x-y)\delta(x^+-y^+)|x^- - y^-|\bar\psi(x)\gamma^+\gamma_5\psi(x)\bar\psi(y)\gamma^+\gamma_5\psi(y)\}. \tag{4.17}$$

We apply the Wick theorem (3.38)–(3.40) to the axial-vector field and obtain

$$S_{f-f} = T^+g^2 \int d^4x d^4y \Big\{ \tfrac{1}{2}i\bar\psi(x)\gamma^\mu\gamma_5\psi(x)g_{\mu\nu}\Delta_F(x-y, m^2)\bar\psi(y)\gamma^\nu\gamma_5\psi(y) + \tfrac{1}{2}\bar\psi(x)\gamma_\mu\gamma_5\psi(x)\bar\psi(y)\gamma_\nu\gamma_5\psi(y)\Big[g^{+\mu}\frac{\partial^\nu}{\partial^+}i\Delta_F(x-y, m^2) + g^{+\nu}\frac{\partial^\mu}{\partial^+}i\Delta_F(x-y, m^2)\Big] + 2Mg^2\bar\psi(x)i\gamma_5\psi(x)\bar\psi(y)\gamma^+\gamma_5\psi(y)\frac{1}{\partial^+}i\Delta_F(x-y, m^2) - 2M^2g^2\frac{1}{m^2}\bar\psi(x)i\gamma_5\psi(x)i\Delta_F(x-y, m^2)\bar\psi(y)i\gamma_5\psi(y)\Big\}. \tag{4.18}$$

It will be assumed that all the fermions are on the mass shell. We can then integrate by parts freely and make use of the relation

$$\partial_\mu(\bar\psi\gamma^\mu\gamma_5\psi) = 2M\bar\psi i\gamma_5\psi, \tag{4.19}$$

which is correct to the order required, to obtain

$$S_{f-f} = \tfrac{1}{2}ig^2T^+ \int d^4x d^4y \Big[\bar\psi(x)\gamma^\mu\gamma_5\psi(x)g_{\mu\nu}\Delta_F(x-y, m^2)\bar\psi(y)\gamma^\nu\gamma_5\psi(y) - \partial_\mu(\bar\psi(x)\gamma^\mu\gamma_5\psi(x))\frac{1}{m^2}\Delta_F(x-y, m^2)\partial_\nu(\bar\psi(y)\gamma^\nu\gamma_5\psi(y))\Big] \tag{4.20}$$

or

$$S_{f-f} = \tfrac{1}{2}ig^2T^+ \int d^4x d^4y\, \bar\psi(x)\gamma^\mu\gamma_5\psi(x)\left(g_{\mu\nu} + \frac{\partial_\mu\partial_\nu}{m^2}\right)\Delta_F(x-y, m^2)\bar\psi(y)\gamma^\nu\gamma_5\psi(y), \tag{4.21}$$

which is the desired well-known covariant result. For particles on the mass shell, T^+ is equivalent to T^*, the covariant T product. The reemergence of the $\partial_\mu\partial_\nu\Delta_F$ term in the vector propagator should be noted. This term would be lost if the coupled current were conserved. The result (4.21) can be used to discuss the fermion-fermion scattering, fermion-antifermion scattering, and the fermion-antifermion annihilation.

We now turn to the Compton scattering, both real and virtual. Expand the S matrix (4.16) to order g^2, and apply the Wick theorem to the Dirac field to contract the internal fermion line; we get

$$S_C = g^2T^+ \int d^4x d^4y \{-i\bar\psi(x)\gamma^\mu\gamma_5 S_F(x-y)\gamma^\nu\gamma_5\psi(y)\bar a_\mu(x)\bar a_\nu(y) - \tfrac{1}{4}|x^- - y^-|\delta(x^+-y^+)\delta^2(x-y)\bar\psi(x)\gamma^+\gamma_5 S_F(x-y)\gamma^+\gamma_5\psi(y) + 2Mi[\bar\psi(x)\gamma^\mu\gamma_5 S_F(x-y)i\gamma_5\psi(y)\bar a_\mu(x)\lambda(y) + \bar\psi(x)i\gamma_5 S_F(x-y)\gamma^\mu\gamma_5\psi(y)\lambda(x)\bar a_\mu(y)] - 4M^2 i\bar\psi(x)i\gamma_5 S_F(x-y)i\gamma_5\psi(y)\lambda(x)\lambda(y) + 2Mi\bar\psi(x)\psi(x)[\lambda(x)]^2\}. \tag{4.22}$$

We will assume that the fermions are on the mass shell. We then have

$$(i\gamma^\mu\partial_\mu - M)\psi = 0, \quad \bar\psi(i\gamma^\mu\overleftarrow{\partial}_\mu + M) = 0, \quad \partial^\mu(\bar\psi\gamma_\mu\psi) = 0. \tag{4.23}$$

In addition, $S_F(x-y)$ satisfies the differential equation

$$(i\gamma^\mu\partial_\mu - M)S_F(x-y) = \delta^4(x-y), \quad S_F(x-y)(i\gamma_\mu\overleftarrow{\partial}^\mu_y + M) = -\delta^4(x-y). \tag{4.24}$$

Equations (4.23) and (4.24) imply

$$\partial_\mu[\bar{\psi}(x)\gamma^\mu\gamma_5 S_F(x-y)] = 2Mi\bar{\psi}(x)\gamma_5 S_F(x-y) + i\delta^4(x-y)\bar{\psi}(x)\gamma_5\,,$$
$$\partial_y^\mu[S_F(x-y)\gamma_\mu\psi(y)] = 2MiS_F(x-y)\gamma_5\psi(y) + i\delta^4(x-y)\gamma_5\psi(y)\,. \tag{4.25}$$

Consider first the real Compton scattering where the axial-vector particles as well as the fermions are on the mass shell. We can integrate by parts freely and make use of (4.23) and (4.25) to get

$$S_C^{\text{real}} = -ig^2T^+\int d^4x d^4y\,\bar{\psi}(x)\gamma^\mu\gamma_5 S_F(x-y) \times\gamma^\nu\gamma_5\psi(y)a_\mu(x)a_\nu(y)\,, \tag{4.26}$$

which agrees with the familiar result. Again T^+ can be replaced by the covariant T product since all the particles are on the mass shell.

We now proceed to the virtual Compton scattering which appears in the self-energy correction to the fermion. We return to (4.22) and apply the Wick theorem to the axial-vector field. The resultant expressions can be further simplified by (4.23), (4.25), and the relations

$$\frac{1}{\partial^+}\Delta_F(x)\bigg|_{x=0} = \partial_\mu\Delta_F(x)|_{x=0} = 0\,, \tag{4.27}$$

which hold since $\Delta_F(x)$ is an even function in x. The final result is

$$S_C^{\text{virtual}} = -ig^2T^+\int d^4x d^4y\,\bar{\psi}(x)\gamma^\mu\gamma_5 S_F(x-y,M)\gamma^\nu\gamma_5\psi(y) \times(-i)\left(g_{\mu\nu}+\frac{\partial_\mu\partial_\nu}{m^2}\right)\Delta_F(x-y,m^2)\,. \tag{4.28}$$

Equations (4.26) and (4.28) together imply that the well-known result for real and virtual Compton scattering in this axial-vector coupling model is reproduced in the light-front formulation. In particular it means that (4.26) applies to both virtual and real Compton scattering if the axial-vector fields are contracted covariantly,

$$a_\mu(x)^\cdot a_\nu(y)^\cdot = -i\left(g_{\mu\nu}+\frac{\partial_\mu\partial_\nu}{m^2}\right)\Delta_F(x-y,m^2)\,. \tag{4.29}$$

V. THE EQUIVALENCE THEOREM

In this section a formal proof will be given that the S matrix associated with (4.7) for a vector-gluon model leads to the same predictions for all possible physical processes as the S matrix in the conventional formalism. The proof is based on the functional-derivative technique of Schwinger[3,4] already employed in paper II. The proof here is more complicated than the one studied in paper II where only scalar and Dirac fields are involved. In the present case the propagators for both the Dirac and vector particles have noncovariant contributions. The proof will proceed in two steps. The internal contractions of the vector field will be considered first, and then the internal contractions of the Dirac field.

We begin with (4.7) and define

$$\mathcal{H}_I = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3\,, \tag{5.1}$$

where

$$\mathcal{H}_1 = g\bar{\psi}\gamma_\mu\psi\bar{B}^\mu\,, \tag{5.2}$$

$$\mathcal{H}_2(x) = -\tfrac{1}{4}ig^2\int d^4y\,\bar{\psi}(x)\gamma^\mu\gamma^+\gamma^\nu\psi(y)\bar{B}_\mu(x)\bar{B}_\nu(y) \times\epsilon(x^- - y^-)\,\delta(x^+-y^+)\delta^2(x-y)\,, \tag{5.3}$$

$$\mathcal{H}_3(x) = -\tfrac{1}{8}g^2\int d^4y|x^- - y^-|\,\delta(x^+-y^+)\delta^2(x-y) \times\bar{\psi}(x)\gamma^+\psi(x)\bar{\psi}(y)\gamma^+\psi(y)\,. \tag{5.4}$$

Consider first all possible internal contractions of the vector field. We are only interested in the effects of the term proportional to $g^{+\mu}g^{+\nu}$ in the vector propagator (3.38). This term is nonvanishing only if $\mu=\nu=-$. As a result of the identity

$$(\gamma^+)^2 = 0\,, \tag{5.5}$$

this term proportional to $g^{+\mu}g^{+\nu}$ is ineffective when the vector field in $\mathcal{H}_2$ is contracted with another vector field. Thus $\mathcal{H}_2$ need not concern us for our purpose. We only have to consider all possible internal contractions among the vector fields in the expression

$$S_1 = T^+\exp\left[-i\int d^4x\,\mathcal{H}_1(x)\right]\,. \tag{5.6}$$

The desired contractions can be achieved by the substitution in (5.6)

$$\bar{B}^\mu(x)\to\bar{B}^\mu(x) + \int d^4y\,i\bar{\Delta}^{\mu\nu}(x-y)\frac{\delta}{\delta\bar{B}^\nu(y)}\,, \tag{5.7}$$

and treating $\bar{B}^\mu$ on the right-hand side of (5.7) as c-number functions. The function $\bar{\Delta}^{\mu\nu}$ is given by (3.38). The contractions in Dirac field will not be considered for the moment. Matrix notation will be employed with space-time coordinates as well as tensor indices as labels for columns and rows. The usual matrix product is now represented by the space-time integrations and summation over repeated tensor indices. Thus we write

$$S_1 = T^+e^{-ij\bar{B}}\,, \tag{5.8}$$

where

$$j^{\mu}=g\overline{\psi}\gamma^{\mu}\psi. \tag{5.9}$$

In this matrix notation (5.7) becomes

$$\overline{B}\to\overline{B}+i\overline{\Delta}\frac{\delta}{\delta\overline{B}}. \tag{5.10}$$

After this substitution (5.8) becomes

$$S_1=\exp\left[-ij\left(\overline{B}+i\overline{\Delta}\frac{\delta}{\delta\overline{B}}\right)\right]. \tag{5.11}$$

It leads to the differential equation

$$\frac{\delta S_1}{\delta\overline{B}}=-ijS_1, \tag{5.12}$$

which can be readily integrated to give

$$S_1=C(j)e^{-ij\overline{B}}. \tag{5.13}$$

The constant of integration can depend only on j. To determine $C(j)$ compare $\delta S_1/\delta j|_{\overline{B}=0}$ obtained from (5.11) and (5.13). From (5.13) we have

$$\left.\frac{\delta S_1}{\delta j}\right|_{\overline{B}=0}=\frac{\delta C}{\delta j}, \tag{5.14}$$

but from (5.11) we also have

$$\left.\frac{\delta S_1}{\delta j}\right|_{\overline{B}=0}=\overline{\Delta}\frac{\delta}{\delta\overline{B}}S_1\bigg|_{\overline{B}=0}$$
$$=(-i)\overline{\Delta}jC. \tag{5.15}$$

So

$$\frac{\delta C}{\delta j}=-i\overline{\Delta}jC. \tag{5.16}$$

Integrating, we get

$$C(j)=e^{-ij\overline{\Delta}j/2}, \tag{5.17}$$

where the integration constant is fixed by the condition

$$S_1=1 \text{ at } j=0. \tag{5.18}$$

Thus

$$S_1=e^{-ij\overline{\Delta}j/2-ij\overline{B}}. \tag{5.19}$$

Now in (5.19) we substitute (3.38):

$$\overline{\Delta}^{\mu\nu}(x-y)=\Delta'^{\mu\nu}(x-y)+\tfrac{1}{4}g^{+\mu}g^{+\nu}|x^--y^-|\delta(x^+-y^+)\delta^2(x-y). \tag{5.20}$$

It is seen that $\mathcal{H}_3$ is exactly canceled by the second term in (5.20). The complete S matrix can now be written as

$$S=T^+\exp\left\{-i\int d^4x\, g\overline{\psi}(x)\gamma^{\mu}\psi(x)\left(\overline{B}+i\Delta'\frac{\delta}{\delta\overline{B}}\right)^{\mu}(x)\right.$$
$$\left.-\int d^4xd^4y\,\tfrac{1}{4}g^2\epsilon(x^--y^-)\delta(x^+-y^+)\delta^2(x-y)\overline{\psi}(x)\gamma^{\mu}\gamma^+\gamma^{\nu}\psi(y)\left(\overline{B}+i\Delta'\frac{\delta}{\delta\overline{B}}\right)_{\mu}(x)\left(\overline{B}+i\Delta'\frac{\delta}{\delta\overline{B}}\right)_{\nu}(y)\right\}, \tag{5.21}$$

where

$$\left(\overline{B}+i\Delta'\frac{\delta}{\delta\overline{B}}\right)^{\mu}(x)=\overline{B}^{\mu}(x)+\int d^4y\, i\Delta'^{\mu\nu}(x-y)\frac{\delta}{\delta\overline{B}^{\nu}(y)} \tag{5.22}$$

and

$$\Delta'^{\mu\nu}(x-y)=-\left(g^{\mu\nu}-g^{+\mu}\frac{\partial^{\nu}}{\partial^+}-g^{+\nu}\frac{\partial^{\mu}}{\partial^+}\right)\Delta_F(x-y). \tag{5.23}$$

The x^+-ordering T^+ in (5.21) applies to the Dirac field but the vector field $\overline{B}^{\mu}(x)$ is temporarily regarded as a c-number function. Consider now the internal contractions among the Dirac field. Observe that

$$\left[\left(\overline{B}+i\Delta'\frac{\delta}{\delta\overline{B}}\right)^{\mu}(x),\left(\overline{B}+i\Delta'\frac{\delta}{\delta\overline{B}}\right)^{\nu}(y)\right]$$
$$=i\Delta'^{\mu\nu}(x-y)-i\Delta'^{\nu\mu}(y-x)$$
$$=0, \tag{5.24}$$

since

$$\Delta'^{\mu\nu}(x-y)=\Delta'^{\nu\mu}(y-x). \tag{5.25}$$

Thus the combination

$$\phi=\overline{B}+i\Delta'\frac{\delta}{\delta\overline{B}} \tag{5.26}$$

is totally commutative with itself. We can therefore regard ϕ as ordinary c numbers rather than functional differential operators when we discuss the internal contractions of the Dirac field.

We rewrite S in the form

$$S=T^+\exp\left[-i\int d^4xd^4y\,\overline{\psi}(x)H(x,y)\psi(y)\right], \tag{5.27}$$

where

$$H(x,y)=H_1(x,y)+H_2(x,y), \tag{5.28a}$$

$$H_1(x,y)=g\gamma_{\mu}\phi^{\mu}(x)\delta^4(x-y), \tag{5.28b}$$

$$H_2(x,y)=-\tfrac{1}{4}ig^2\epsilon(x^--y^-)\delta(x^+-y^+)\times\delta^2(x-y)\gamma^{\mu}\gamma^+\gamma^{\nu}\phi_{\mu}(x)\phi_{\nu}(y). \tag{5.28c}$$

The internal contractions of the Dirac field can be achieved by the substitution in (5.27)

$$\psi_\alpha(x) \to \psi_\alpha(x) + \int d^4y\, i\bar{S}_{F\alpha\beta}(x-y)\frac{\delta}{\delta\bar{\psi}_\beta(y)}\,,$$
$$\bar{\psi}_\alpha(x) \to \bar{\psi}_\alpha(x) - \int d^4y\, \frac{\delta}{\delta\psi_\beta(y)}\, i\bar{S}_{F\beta\alpha}(y-x)\,, \tag{5.29}$$

treating ψ and $\bar{\psi}$ in the resulting expression as anticommuting c numbers. The propagator function $\bar{S}_F(x-y)$ is given by (3.1). In matrix notations we have

$$\psi \to \psi + i\bar{S}_F\frac{\delta}{\delta\bar{\psi}}\,,$$
$$\bar{\psi} \to \bar{\psi} - \frac{\delta}{\delta\psi}\, i\bar{S}_F\,, \tag{5.30}$$

and

$$S = \exp\left[-i\left(\bar{\psi} - \frac{\delta}{\delta\psi}\, i\bar{S}_F\right) H\left(\psi + i\bar{S}_F\frac{\delta}{\delta\bar{\psi}}\right)\right]\,. \tag{5.31}$$

From (5.28) one notes the property

$$H_2 = H_1 H'\,, \tag{5.32}$$

where

$$H'(x,y) = -\tfrac{1}{4}ig\gamma^+\gamma^\nu\,\phi_\nu(y)\epsilon(x^- - y^-)\delta(x^+ - y^+)\delta^2(x-y)\,. \tag{5.33}$$

Therefore

$$H = H_1 + H_2$$
$$= H_1(1 + H')\,. \tag{5.34}$$

We find by differentiation that

$$\frac{\delta_l S}{\delta\psi} = i\bar{\psi}HS + \frac{\delta_l S}{\delta\psi}\bar{S}_F H \tag{5.35}$$

or

$$\frac{\delta_l}{\delta\psi}S(1 - \bar{S}_F H) = i\bar{\psi}HS\,, \tag{5.36}$$

which integrates to give

$$S = C(H)e^{-i\bar{\psi}H(1-\bar{S}_F H)^{-1}\psi}\,. \tag{5.37}$$

To determine $C(H)$ we evaluate and compare $\delta S/\delta H|_{\psi=0}$ from (5.31) and (5.37). The result is the equation

$$\frac{\delta C}{\delta H} = -[(1-\bar{S}_F H)^{-1}\bar{S}_F]^T C(H)\,, \tag{5.38}$$

where the superscript T refers to the operation of transposition of a matrix. Integrating (5.38) we obtain

$$C(H) = \exp[\mathrm{Tr}\ln(1-\bar{S}_F H)]\,. \tag{5.39}$$

Consequently

$$S = \exp[\mathrm{Tr}\ln(1-\bar{S}_F H)]\exp[-i\bar{\psi}H(1-\bar{S}_F H)^{-1}\psi]\,. \tag{5.40}$$

We now write

$$\bar{S}_F = S_F + S'\,, \tag{5.41}$$

where S' is the noncovariant part in $\bar{S}_F$,

$$S'(x-y) = \tfrac{1}{4}i\gamma^+\epsilon(x^- - y^-)\delta(x^+ - y^+)\delta^2(x-y)\,. \tag{5.42}$$

It has the properties

$$S'H_2 = H_2 S'$$
$$= 0\,,$$
$$H' = -S'H_1\,. \tag{5.43}$$

Thus

$$1 - \bar{S}_F H = 1 - (S_F + S')(H_1 + H_2)$$
$$= (1 - S_F H_1)(1 - S'H_1)\,, \tag{5.44}$$

where use has been made of (5.32)–(5.34) and (5.43). Similarly,

$$H = H_1(1 - S'H_1)\,. \tag{5.45}$$

Equations (5.44) and (5.45) imply

$$H(1-\bar{S}_F H)^{-1} = H_1(1 - S_F H_1)^{-1} \tag{5.46}$$

and

$$\mathrm{Tr}\ln(1-\bar{S}_F H) = \mathrm{Tr}\ln(1 - S_F H_1) + \mathrm{Tr}\ln(1 - S'H_1)\,. \tag{5.47}$$

Now

$$[S'H_1](x-y) = -\tfrac{1}{8}ig\epsilon(x^- - y^-)\delta(x^+ - y^+)$$
$$\times\,\delta^2(x-y)\gamma^+\gamma_k\phi_k(y)\,, \tag{5.48}$$

so that

$$[S'H_1]^2 = 0 \tag{5.49}$$

as a consequence of (5.5). Furthermore

$$\mathrm{Tr}[S'H_1] = 0\,. \tag{5.50}$$

Thus

$$\mathrm{Tr}\ln(1 - S'H_1) = 0\,. \tag{5.51}$$

Finally

$$S = \exp[\mathrm{Tr}\ln(1 - S_F H_1)]\exp[-i\bar{\psi}H_1(1 - S_F H_1)^{-1}\psi]\,. \tag{5.52}$$

We see that S' and H_2 disappear completely from the S matrix. Moreover, (5.52) is identical in form to (5.40) if we make the correspondence

$$H \to H_1\,,$$
$$\bar{S}_F \to S_F\,. \tag{5.53}$$

In other words, if we ignore the noncovariant term $\mathcal{H}_2$ in the interaction Hamiltonian (5.1) and simultaneously ignore the noncovariant term S' in the Dirac propagator (5.41), the S matrix remains unchanged. Together with the conclusion about the internal contraction of the vector field reached earlier, this implies that the S matrix remains unchanged if the substitution is made,

$$\begin{aligned} &\mathcal{H}_I \to \mathcal{H}_1 ,\\ &\bar{S}_F \to S_F ,\\ &\bar{\Delta}^{\mu\nu} \to \Delta'^{\mu\nu} . \end{aligned} \tag{5.54}$$

But the gradient terms in $\Delta'^{\mu\nu}$ can be shown to give no contribution to the S matrix as a result of current conservation.[11] Finally we conclude that the S matrix for the vector-gluon model can be expressed as

$$S = T^* \exp\left[-i \int d^4x\, g\bar{\psi}\gamma_\mu\psi\bar{B}^\mu\right] , \tag{5.55}$$

where T^* denotes the covariant T product where the rules for internal contractions are

$$\begin{aligned} &\psi(x)^\cdot\bar{\psi}(y)^\cdot \to iS_F(x-y) ,\\ &\bar{B}_\mu(x)^\cdot\bar{B}_\nu(y)^\cdot \to -ig_{\mu\nu}\Delta_F(x-y) . \end{aligned} \tag{5.56}$$

These rules (5.55) and (5.56) are precisely the same rules used in the conventional formalism of the gluon model. The formal equivalence of the light-front formulation of the gluon model to the conventional formulation is now established.

VI. COVARIANT PERTURBATION THEORY

In the previous section we have shown that the S matrix of the vector gluon model in the light-front formulation is formally identical to the S matrix in the conventional equal-time formulation. However, there remains a subtle difference between the two S matrices. It is related to how the momentum integrals are actually carried out. Consider the following integral which frequently occurs in a perturbation calculation involving closed loops:

$$I = \int d^4p \, \frac{1}{(p^2 - M^2 + i\epsilon)^3} . \tag{6.1}$$

Although the same expression will appear in the light-front formulation and the equal-time formulation, its interpretations in both formulations are quite different. In the light-front formulation we have

$$\begin{aligned} &d^4p = \tfrac{1}{2}dp^+dp^-d^2p ,\\ &p^2 = p^+p^- - \vec{p}^{\,2} . \end{aligned} \tag{6.2}$$

But in the conventional treatment we have

$$\begin{aligned} &d^4p = dp^0dp^1dp^2dp^3 ,\\ &p^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 . \end{aligned} \tag{6.3}$$

Although (6.2) and (6.3) are related by a simple change of variables

$$p^\pm = p^0 \pm p^3 , \tag{6.4}$$

it turns out that it is not a trivial matter to establish the equivalence of the two procedures. The purpose of this section[12] is to show that with sufficient care the two procedures indeed lead to identical results. The complete equivalence of the light-front formulation and the conventional formulation will then be established.

In the conventional interpretation (6.3) of the integral (6.1) it can be evaluated by a Wick rotation and we have

$$\begin{aligned} I_1 &= \int dp^0dp^1dp^2dp^3 \, \frac{1}{(p^2 - M^2 + i\epsilon)^3} \\ &= \frac{\pi^2}{2iM^2} . \end{aligned} \tag{6.5}$$

In the light-front interpretation (6.2) of the same integral, we have

$$I_2 = \tfrac{1}{2} \int dp^+dp^-d^2p \, \frac{1}{(p^+p^- - \vec{p}^{\,2} - M^2 + i\epsilon)^3} . \tag{6.6}$$

The question is whether I_2 will also give the same result as (6.5). To dramatize the situation consider the following seemingly correct arguments. The p^1, p^2 integrations give

$$I_2 = -\tfrac{1}{4}\pi \int dp^+dp^- \frac{1}{(p^+p^- - M^2 + i\epsilon)^2} . \tag{6.7}$$

For any fixed value of p^+, there is one double pole for p^- at

$$p^- = \frac{M^2 - i\epsilon}{p^+} . \tag{6.8}$$

To carry out the p^- integration, if the pole (6.8) is in the upper half of the complex p^- plane (when $p^+ < 0$) we can close the contour from below; if it is in the lower-half plane (when $p^+ > 0$), we can close the contour from above. The singularity can always be avoided and we are led to conclude[13]

$$I_2 \equiv 0 \quad \text{(false)}, \tag{6.9}$$

which is certainly incorrect. Clearly, there is a loophole in the above arguments. The arguments indeed break down when $p^+ = 0$, since the pole (6.8) is then at infinity and it cannot be avoided by closing the contour either from above or below. This indicates that the p^- integration should yield a result proportional to $\delta(p^+)$. In fact this is exactly what happens. To see this, we define our integral as

$$I_2=-\tfrac{1}{4}\pi\int_{-\infty}^{+\infty}dp^+\lim_{\Lambda,\Lambda'\to\infty}\int_{-\Lambda}^{\Lambda'}dp^-\frac{1}{(p^+p^--M^2+i\epsilon)^2}\,. \tag{6.10}$$

The p^- integration gives

$$I_2=\tfrac{1}{4}\pi\int_{-\infty}^{\infty}\frac{dp^+}{p^+}\lim_{\Lambda,\Lambda'\to\infty}\left(\frac{1}{p^+\Lambda'-M^2+i\epsilon}-\frac{1}{-p^+\Lambda-M^2+i\epsilon}\right). \tag{6.11}$$

Making use of the algebraic identity

$$\frac{1}{p^+}\left(\frac{1}{p^+\Lambda'-M^2+i\epsilon}-\frac{1}{-p^+\Lambda-M^2+i\epsilon}\right)=\frac{1}{M^2}\left(\frac{\Lambda'}{p^+\Lambda'-M^2+i\epsilon}-\frac{\Lambda}{p^+\Lambda+M^2-i\epsilon}\right), \tag{6.12}$$

we obtain in the limit $\Lambda,\Lambda'\to\infty$

$$\begin{aligned}I_2&=\frac{\pi}{4M^2}\int_{-\infty}^{\infty}dp^+\left(\frac{1}{p^++i\epsilon}-\frac{1}{p^+-i\epsilon}\right)\\&=\frac{\pi}{4M^2}\int_{-\infty}^{\infty}dp^+[-2\pi i\,\delta(p^+)]\\&=+\frac{\pi^2}{2iM^2}\,,\end{aligned} \tag{6.13}$$

which now agrees with (6.5). Equations (6.12) and (6.13) make it explicit that indeed the p^- integration results in a δ function $\delta(p^+)$. The same result can also be obtained by exponentiating the denominator

$$\frac{1}{p^+p^--\vec{p}-M^2+i\epsilon}=-i\int_0^{\infty}d\alpha\,e^{i\alpha(p^+p^--\vec{p}^2-M^2+i\epsilon)}\,. \tag{6.14}$$

In terms of (6.14) I_2 becomes

$$I_2=-\tfrac{1}{2}i\left(\frac{d^2}{dM^2}\right)^2\int_0^{\infty}d\alpha\,e^{-i\alpha(M^2-i\epsilon)}F(\alpha)\,, \tag{6.15}$$

where

$$\begin{aligned}F(\alpha)&=\int d^4p\,e^{i\alpha p^2}\\&=\tfrac{1}{2}\int dp^+dp^-d^2p\,e^{i(p^+p^--\vec{p}^2)}\,.\end{aligned} \tag{6.16}$$

Now

$$\int d^2p\,e^{-i\alpha\vec{p}^2}=\frac{\pi}{i\alpha} \tag{6.17}$$

and

$$\int dp^-\,e^{+i\alpha p^+p^-}=\frac{2\pi}{\alpha}\,\delta(p^+)\,. \tag{6.18}$$

Consequently

$$F(\alpha)=\frac{\pi^2}{i\alpha^2} \tag{6.19}$$

and

$$\begin{aligned}I_2&=\tfrac{1}{2}\pi^2\int_0^{\infty}d\alpha\,e^{-i\alpha(M^2-i\epsilon)}\\&=\frac{\pi^2}{2iM^2}\,,\end{aligned} \tag{6.20}$$

which again agrees with (6.5). It should be noted that the above calculations of I_2 in the light-front formulation involve no Wick rotation at all.

In two ways we have shown that with sufficient care the integral I_2 defined in terms of light-front coordinates gives the same value as I_1 defined in terms of conventional space-time coordinates. For integrals involving higher powers of the denominator, the answer can be obtained by repeated differentiation of I_1 and I_2 with respect to M^2. It is clear that the two procedures, light-front coordinate and space-time coordinate evaluation, always yield identical results.

For field theories involving particles with spin, there will be in general momentum factors in the numerator of an integral similar to (6.1). One infers from symmetry consideration and Lorentz covariance that

$$\begin{aligned}&\int d^4p\,p^\mu f(p^2)=0\,,\\&\int d^4p\,p^\mu p^\nu f(p^2)=\tfrac{1}{4}g^{\mu\nu}\int d^4p\,p^2f(p^2)\,,\end{aligned} \tag{6.21}$$

etc.

In the light-front formulation, d^4p and p^2 are given by (6.2). If $f(p^2)$ is of the form $(p^2-M^2+i\epsilon)^{-n}$ or $e^{-i\alpha p^2}$, the results (6.21) can be readily verified if integration is performed symmetrically in positive and negative values; in particular, one sets $\Lambda=\Lambda'$ in (6.11).

Now, a common practice in a perturbation calculation of Feynman diagrams with closed loops is either to combine the propagator denominators using Feynman parameters or to exponentiate each denominator according to (6.14). After rearrangements including the simplification (6.21) all momentum integrals are reduced to the type (6.1) or its derivatives with respect to M^2. In general, M^2 is a combination of squares of sums of other momenta and masses. Since it has been demonstrated that the light-front evaluation of (6.1) reproduces the correct result, the light-front formulation and the conventional space-time formulation lead to the same parametric integral representation for any (unrenormalized and properly regulated) Feynman amplitude. A renormalization procedure based on the parametric integral representation of unrenormalized amplitudes has appeared in the literature.[14] In particular, we refer to the work of Appelquist.[5] His starting points are the parametric integral representation (6.14) for the propagator and the basic integrals (6.16) and (6.19). His renormalization prescription does not rely directly on Weinberg's power-counting theorem.[15] Moreover, Appelquist has shown that his prescription is formally equivalent to the usual recursive subtraction formula for writing renormalized amplitudes. Using Appelquist's prescription and our discussion above, we conclude that renormalization of a renormalizable field theory in the light-front formulation can be carried out consistently to all orders in perturbation theory and the renormalized amplitudes agree with the conventional formulation.

Proof of renormalizability of a field theory in the light-front formalism based directly on Weinberg's power-counting theorem will be more difficult. This is because the mere change of variables (6.4) alters the convergence property of an integral. For instance, for fixed values of p^+ and $\vec{p}$, the integrand of (6.6) behaves like $(p^-)^{-3}$ as $p^- \to \infty$ if $p^+ \neq 0$, and worse if $p^+ = 0$. On the other hand, in the space-time formulation, for any fixed p^1, p^2, and p^3, the integrand in (6.5) always behaves like $(p^0)^{-6}$ as $p^0 \to \infty$.

VII. NONCOVARIANT PERTURBATION THEORY AND APPLICATION TO HIGH-ENERGY SCATTERING

As already mentioned in paper II for scalar and Dirac fields, instead of using the covariant perturbation theory developed in Secs. V and VI, one can also employ the S matrix as given by the Dyson formula

$$S = T^+ \exp\left[-i \int d^4x\, \mathcal{H}_I(x)\right] \qquad (7.1)$$

and expand the x^+-ordered product directly. Or equivalently, one can integrate over all p^-'s first in the covariant perturbation series. The result is similar to the "old-fashioned" time-ordered perturbation expansion. Conservation of p^+'s at each vertex in this x^+-ordered perturbation expansion implies that each intermediate particle can only have a non-negative value for p^+ less than the total value of p^+ for the initial state. These are essentially Weinberg's rules for "old-fashioned" perturbation theory in the infinite-momentum limit.[16] In simple cases these new rules greatly simplify the calculations. Their application to the second-order renormalization of the pseudoscalar coupling for a fermion and a spinless boson has been given in paper II. The corresponding second-order calculation for (massive) QED has also appeared in the literature.[17,18] It is not greatly different from the case discussed in paper II. Therefore such calculations for (massive) QED will not be duplicated here.

A major difficulty in the noncovariant perturbation theory in the light-front formulation has also been mentioned in paper II. It is related to the singularities in diagrams with closed loops as $p^+ \to 0$. These singularities arise from the noncovariant terms in the Hamiltonian and the noncovariant terms in the propagators for the Dirac and vector particles. If covariant perturbation theory is employed these singularities explicitly cancel in pairs and no difficulty arises. On the other hand, in noncovariant perturbation theory the cancellation of these singularities is no longer transparent and the resulting integrals are sometimes not well defined. In such cases their meaning must be defined in terms of covariant Feynman diagrams. This is a caution one must keep in mind.

In this section two points will be discussed: vacuum diagrams and application of the x^+-ordered noncovariant perturbation theory to high-energy scattering.

A. Vacuum Diagrams

The above rules for the noncovariant perturbation theory in the light-front formulation seem to imply that vacuum diagrams should vanish since all the p^+ of each line in a vacuum diagram cannot be negative and the sum of all p^+ must be zero. This is possible only if all p^+ vanish. If this one point in phase space could be ignored, the above conclusion would be correct. This would be, however, a contradiction to the claimed equivalence of the light-front formulation to the ordinary formulation in which the vacuum diagrams are known not to vanish (in fact they are generally infinite).

This question has been studied by Chang and Ma in a ϕ^3 theory.[17] They show that the contribution from the point $p^+ = 0$ is indeed nonvanishing. We will reexamine the same question on the basis of the results we obtained in Sec. VI. We will demonstrate that it is related to the apparent paradox in regard to the integral (6.7) discussed there. Consider the lowest-order vacuum diagram in a ϕ^3 theory. Apart from some irrelevant constant factors, this diagram is proportional to the integral

$$I_v = \int d^4p_1 d^4p_2 \frac{1}{(p_1^2-\mu^2+i\epsilon)(p_2^2-\mu^2+i\epsilon)[(p_1+p_2)^2-\mu^2+i\epsilon]}, \tag{7.2}$$

where μ is the meson mass. For fixed p_1 the p_2 integration can be carried out. The result is a spectral representation[19]

$$\int d^4p_2 \frac{1}{(p_2^2-\mu^2+i\epsilon)[(p_1+p_2)^2-\mu^2+i\epsilon]} = \int_{4\mu^2}^{\infty} dM^2 \frac{\rho(M^2)}{p_1^2-M^2+i\epsilon} \tag{7.3}$$

and

$$I_v = \int dM^2 \rho(M^2) \int d^4p_1 \frac{1}{(p_1^2-\mu^2+i\epsilon)(p_1^2-M^2+i\epsilon)}. \tag{7.4}$$

If we integrate over p_1^- first, there are two simple poles for p_1^- at

$$p_1^- = \frac{\vec{p}_1^{\,2}+\mu^2-i\epsilon}{p_1^+}, \qquad p_1^- = \frac{\vec{p}_1^{\,2}+M^2-i\epsilon}{p_1^+}. \tag{7.5}$$

These two poles of (7.5) are always on the same side of the real axis in the complex p_1^- plane regardless of the sign of p_1^+. Thus, it seems that we can close the contour from the other side to avoid these poles and conclude

$$I_v = 0 \quad \text{(false)}. \tag{7.6}$$

This argument is wrong since it neglects the point $p_1^+=0$. In order to borrow the result in Sec. VI, observe the identity

$$\frac{1}{(p_1^2-\mu^2+i\epsilon)(p_1^2-M^2+i\epsilon)} = \int_{\mu^2}^{M^2} d\lambda^2 \frac{1}{(p_1^2-\lambda^2+i\epsilon)^2}\frac{1}{M^2-\mu^2}. \tag{7.7}$$

Following the procedure given in Sec. VI, we get

$$\lim_{\Lambda,\Lambda'\to\infty} \int_{-\Lambda'}^{\Lambda} dp_1^- \frac{1}{(p_1^+p_1^- - \vec{p}_1^{\,2}-\lambda^2+i\epsilon)^2} = \frac{1}{(\vec{p}_1^{\,2}+\lambda^2)}(2\pi i)\delta(p_1^+), \tag{7.8}$$

which clearly shows that the single point $p_1^+=0$ gives the whole contribution. Finally

$$I_v = \pi i \int dM^2 \rho(M^2) \int d^2p_1 \frac{1}{M^2-\mu^2} \ln\frac{\vec{p}_1^{\,2}+M^2}{\vec{p}_1^{\,2}+\mu^2} \neq 0. \tag{7.9}$$

If one follows the rules of the x^+-ordered perturbation expansion, I_v is proportional to

$$I_v \propto \int \frac{dp_1^+d^2p_1}{p_1^+}\frac{dp_2^+d^2p_2}{p_2^+}\frac{dp_3^+}{p_3^+}\theta(p_1^+)\theta(p_2^+)\theta(p_3^+)\frac{\delta(p_1^++p_2^++p_3^+)\delta^2(p_1+p_2+p_3)}{(\vec{p}_1^{\,2}+\mu^2)/p_1^+ - (\vec{p}_2^{\,2}+\mu^2)/p_2^+ - (\vec{p}_3^{\,2}+\mu^2)/p_3^+}, \tag{7.10}$$

which is not unambiguous due to the singularity of the integrand at $p_1^+=p_2^+=p_3^+=0$. This singularity is a reflection of the fact that I_v is not zero. If the singularity were absent, I_v would indeed vanish. It also reminds us of the caution called for in the beginning of the section.

B. Application to High-Energy Scattering

The light-front formulation of quantum field theory is particularly suited for application in two areas. One concerns the lepton-induced processes which have been extensively studied in the literature[20] and are briefly described in papers I and III. Another area of application is the scattering of a high-energy particle in an external field. The former relies mainly on the operator structure of the current commutation relations on a light front. The latter makes use of the x^+-ordered noncovariant perturbation theory in this formulation.

Bjorken, Kogut, and Soper[18,21] have applied the light-front formulation of QED to the scattering of energetic electrons and photons off an external field. The essential result of their investigation can be summarized by the compact equation for the scattering matrix element at high energies.

$$S_{ba} = \langle Vb|F|Ua\rangle, \tag{7.11}$$

where $|Ua\rangle = U|a\rangle$, $|Vb\rangle = V|b\rangle$, and

$$U=U(0,-\infty)\,,\qquad V=U(\infty,0)\,, \tag{7.12}$$

with

$$U(x_1^+,x_2^+)=T^+\exp\left[-i\int_{x_2^+}^{x_1^+}d^4x\,\mathcal{H}_I(x)\right] \tag{7.13}$$

and

$$F=\exp\left[-i\tfrac{1}{4}\int dx^+dx^-d^2x \times\phi(x^+,\vec{x},x^-=0)\,j^+(x^+=0,\vec{x},x^-)\right], \tag{7.14}$$

where

$$j^+=\bar\psi\gamma^+\psi \tag{7.15}$$

and $\phi(x)$ is the external potential. Notice that all the field operators in F are evaluated at $x^+=0$, so that the x^+-ordering in F can be ignored. A physical picture emerges from (7.11) which proves to be a realization of Feynman's "parton ideas."[6] In this picture the incoming electron is composed of bare constituents, which, at high energies, interact slowly with one another. Each constituent is scattered from the external field in a simple way. And then the constituents again interact among themselves to form the final state. These three steps are described by $|\,Ua\rangle$, F, and $\langle\,Vb\,|$, respectively.

Based on this result of Bjorken, Kogut, and Soper,[18] and Lee,[21] we would like to propose a model for scattering between two energetic particles. In this model one of the two particles serves as the external field for the other and vice versa. We will assume that the scattering takes place in the center-of-mass system. The model can be characterized by the following proposed expression for the scattering amplitude of the scattering between a right-moving particle a and a left-moving particle b which produces a right-moving group of particles n_a and a left-moving group of particles n_b:

$$S_{fi}=\langle\,V_+n_a|\langle\,V_-n_b|\,F_{ab}|U_+a\rangle|U_-b\rangle\,, \tag{7.16}$$

where

$$U_+=U_+(0,-\infty)\,,\quad V_+=U_+(\infty,0)\,,\quad U_-=U_-(0,-\infty)\,,\quad V_-=U_-(\infty,0)\,, \tag{7.17}$$

with

$$U_+(x_1^+,x_2^+)=T^+\exp\left[-i\int_{x_2^+}^{x_1^+}d^4x\,\mathcal{H}_I(x)\right],\qquad U_-(x_1^-,x_2^-)=T^-\exp\left[-i\int_{x_2^-}^{x_1^-}d^4x\,\mathcal{H}_I(x)\right], \tag{7.18}$$

ordered according to x^+ and x^-, respectively. The operators U_+ and V_+ act on the right-moving particles, and U_- and V_- act on the left-moving particles. The operator F_{ab} describes the interactions between the two groups of partons associated with particles a and b, and has the general structure

$$F_{ab}=\exp\left[-i\int d^2xd^2y\,\sigma_+(\vec{x})\phi(\vec{x}-\vec{y})\sigma_-(\vec{y})\right], \tag{7.19}$$

where

$$\sigma_+(\vec{x})=\tfrac{1}{2}\int dx^-j_a^+(x^+=0,\vec{x},x^-)\,,\qquad \sigma_-(\vec{x})=\tfrac{1}{2}\int dx^+j_b^-(x^+,\vec{x},x^-=0)\,, \tag{7.20}$$

and $\phi(\vec{x}-\vec{y})$ is a real potential. The current j_a^+ acts on the right-moving particles and j_b^- on the left-moving ones.

In this model the right-moving particles are described by a quantum field theory quantized on equal-x^+ surfaces, and the left-moving particles are described by another quantum field theory quantized on equal-x^- surfaces. It is not clear whether one can derive such a formula (7.18) from quantum field theory since it involves quantum fields quantized on different surfaces.

Nevertheless, the model defined by (7.18) reproduces the leading contribution in each order of perturbation in massive QED with the choice

$$\phi(\vec{x})=\int\frac{d^2q}{(2\pi)^2}\,e^{i\vec{q}\cdot\vec{x}}\,\frac{1}{\vec{q}^{\,2}+\mu^2}\,. \tag{7.21}$$

A covariant version of (7.18) for massive QED has been derived by Chang[22] in this leading-order approximation. The noncovariant model (7.18) has the advantage that as long as the potential $\phi(\vec{x})$ is real the model exhibits manifest unitarity in the s channel. This is readily seen as a result of the unitarity of the operators $U_\pm$ and $V_\pm$

$$U_\pm^\dagger=U_\pm^{-1}\,,\qquad V_\pm^\dagger=V_\pm^{-1}\,, \tag{7.22}$$

and the property of F_{ab}

$$F_{ab}^\dagger=F_{ab}^{-1}\,, \tag{7.23}$$

which follows from the reality of the potential $\phi(x)$. Equation (7.22) implies

$$\sum_{n_a} V_+|n_a\rangle\langle n_a|V_+^{-1}=\sum_{n_a}|n_a\rangle\langle n_a| =1\,,$$
$$\sum_{n_b} V_-|n_b\rangle\langle n_b|V_-^{-1}=\sum_{n_b}|n_b\rangle\langle n_b| =1\,. \tag{7.24}$$

Consequently, in the application of the optical theorem to calculate the total cross section from the forward elastic amplitude, the summation over a complete set of intermediate physical states can be replaced by a summation over the corresponding complete set of bare states. The consideration of the recombination of bare constituents into physical final states is necessary only when the scattering into specific channels is interested. This is a situation very similar to the case in deep-inelastic electron scattering.[23] This discussion clarifies why the absorptive part of the T matrix in the forward scattering amplitude

$$S_{ab\to ab}=\langle V_+a|\langle V_-b|F_{ab}|U_+a\rangle|U_-b\rangle \tag{7.25}$$

is capable of giving the total cross section even though only the bare constituents enter the sum of intermediate states when unitarity cut is made.

The model proposed here is an impulse approximation. On the one hand, because of time dilation, each of the two energetic colliding particles, when they are far apart, is regarded as a collection of almost free constituents. On the other hand, because of Lorentz contraction, the colliding constituents have almost instantaneous interactions as they pass through each other. The instantaneous interactions are described by the eikonal potential F_{ab} in (7.19). If the wave functions $|U_+a\rangle$, $|U_-b\rangle$, etc. are proposed to be energy-independent, and satisfy the normalization conditions

$$\langle U_+a|U_+a'\rangle=\delta(p_a^+-p_{a'})\delta^2(p_a-p_{a'}), \tag{7.26}$$

etc., then the model will lead to a constant total cross section. Whether (7.16) is consistent with Regge behavior as well as other questions can only be answered by detailed exploration of the model.[24]

VIII. CONCLUSION

In this series of articles an attempt has been made to provide a better theoretical understanding of the light-front quantization which finds interesting applications to various physical problems such as the lepton-induced processes and high-energy scattering. The most important questions to which we address ourselves are a general framework for a consistent quantization of all popular renormalizable field theories and the equivalence of the light-front formulation to the conventional equal-time formulation.

Both questions have been investigated in some detail by Schwinger's action principle and his functional-derivative technique. Our study leads to an affirmative answer to both questions. Thus, light-front quantization is an alternative to the canonical equal-time formulation. Other topics in quantum field theory have been also discussed, such as reduction formula, Wick theorem, and spectral sum rules, etc. Applications to current-algebra sum rules and lepton-induced processes are only briefly mentioned since they have already appeared in the literature.[20] Feynman's parton model[6] for deep-inelastic electron scattering is derived in paper III. This derivation utilizes both the singularity structure of the current commutator near the light cone and the nonrelativistic wave-function-like interpretation of the noncovariant perturbation expansion of the U operator in this formulation.

Light-front quantization appears to be the first attempt at quantizing field theories on a nonspacelike surface. The advantage of using a light front as a quantization surface is obvious. It contains a line in common with the light cone. By Lorentz covariance and causality the information about a current commutator on the light cone can be inferred from its behavior on the line in common with the light cone provided by the canonical equal-x^+ commutation relations. Among other things this information is valuable to determine the structure functions associated with the so-called deep-inelastic lepton processes.

Furthermore, the trajectory of a fast particle moving in the positive x^3 axis is very close to the normal of a light front $x^+=x^0+x^3=\text{constant}$. Therefore light-front coordinates are particularly suited for describing the motion of fast particles moving in more or less the same direction. An intriguing question is: What are the natural coordinates for quantization which are suited for describing scattering of two energetic particles moving in opposite directions such as the hadron collision in the center-of-mass system? The model proposed in Sec. VII for scattering of two energetic particles represents an initial response to such a question. It is not clear how such a model can be derived or how it would be modified in a complete treatment.

The advantage of the light-front quantization is not gained without a price paid for working on a nonspacelike surface. The price is the singular behavior of individual terms near $p^+=0$ in the x^+-ordered perturbation expansion of the S matrix. As a result, for example, the renormalization program in the noncovariant perturbation approach is extremely difficult to carry out beyond the second-order calculation,[1,17,18,25] and the vacuum diagram discussed in (7.10) is not without ambiguity. Nonetheless, the light-front formulation could still be useful in structure analysis, where only general properties of the formalism are required such as unitarity of the S matrix and the singularity structure of the current commutators near the light cone.

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[1]The earlier papers in this series are S.-J. Chang, R. Root, and T.-M. Yan, Phys. Rev. D 7, 1133 (1973); S.-J. Chang and T.-M. Yan, *ibid.* 7, 1147 (1973). These two papers will be referred to as papers I and II, respectively.

[2]T.-M. Yan, preceding paper, Phys. Rev. D 7, 1760 (1973). This paper will be referred to as paper III.

[3]J. Schwinger, Proc. Nat. Acad. Sci. U. S. 37, 452 (1951).

[4]This technique has been recently applied to chiral loops by I. S. Gerstein, R. Jackiw, B. W. Lee, and S. Weinberg [Phys. Rev. D 3, 2486 (1971)]. See also T. D. Lee and C. N. Yang, Phys. Rev. 128, 885 (1962).

[5]T. Appelquist, Ann. Phys. (N.Y.) 54, 27 (1969).

[6]R. P. Feynman, Phys. Rev. Letters 23, 1415 (1969); and in *High Energy Collisions*, edited by C. N. Yang *et al.* (Gordon and Breach, New York, 1969).

[7]The following derivation is inspired by the work (paper II) in collaboration with S.-J. Chang. The investigation of the reduction formula in paper II is in turn stimulated by B. Hasslacher.

[8]For a discussion of weak asymptotic condition see J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[9]Left and right derivatives are defined by

$$\delta S = \delta\bar{\eta}\,\frac{\delta_l}{\delta\bar{\eta}}\,S = \frac{\delta_r}{\delta\bar{\eta}}\,S\,\delta\bar{\eta}\ .$$

[10]The right-hand side of (3.36) is a simple rearrangement of the right-hand side of (3.28).

[11]For a proof see R. P. Feynman, Phys. Rev. 76, 769 (1949); Bjorken and Drell, Ref. 8, Chap. 17.

[12]Investigation in this section is stimulated by a conversation with D. Gross and D. Soper, to whom the author is grateful.

[13]This paradox was first presented to the author by D. Gross.

[14]References in this subject can be found in Ref. 5.

[15]S. Weinberg, Phys. Rev. 118, 838 (1960).

[16]S. Weinberg, Phys. Rev. 150, 1313 (1966).

[17]S.-J. Chang and S. K. Ma, Phys. Rev. 180, 1506 (1969).

[18]J. D. Bjorken, J. Kogut, and D. Soper, Phys. Rev. D 3, 1382 (1971).

[19]The integral must be properly regulated, i.e., $\rho(M^2)$ in general depends on a cutoff parameter.

[20]See for example, J. Cornwall and R. Jackiw, Phys. Rev. D 4, 367 (1971); D. A. Dicus, R. Jackiw, and V. Teplitz, Phys. Rev. D 4, 1733 (1971).

[21]See also B. W. Lee, Phys. Rev. D 1, 2361 (1970); J. Kogut, *ibid.* 4, 3101 (1971).

[22]S.-J. Chang, Phys. Rev. D 2, 2886 (1970). See also L. N. Chang and N. P. Chang, *ibid.* 4, 1856 (1971).

[23]See the discussion by S. D. Drell and T.-M. Yan, Phys. Rev. Letters 24, 855 (1970).

[24]The basic ideas of the model proposed here were developed when the author was collaborating with S. D. Drell on the parton model for deep-inelastic electron scattering. The author thanks Professor Drell for encouragement and stimulating conversations.

[25]Recently S. Brodsky and R. Roskies [SLAC report, 1972 (unpublished)] have applied renormalization in the infinite momentum frame to reproduce the electron magnetic moment in fourth order, and to calculate representative contributions to the sixth order.